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THE POSSIBLE FUNDAMENTAL EQUATIONS OF THE CONTINUUM MECHANICS

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Abstract

The possible fundamental equations are looked for in cases of infinitesimal and finite strain based on the investigation of the acceleration wave. It is shown that also the selection of the kinematical equations has an important role besides the constitutive equation.

Keywords: acceleration wave, constitutive equation, kinematical equation, objective time derivative, infinitesimal strain, finite strain.

1. Introduction

The fundamental equations of the continuum mechanics cannot be generally treated, because there is no mathematical model, a system of differential equations known describing the continuum mechanics using mathematics. The most important reason of it is the absence of a system of equations expressing the properties of the material of the bodies called the constitutive equations. However, the literature suggests a lot of equations, unfortunately, these can describe only the material behaviour under some given conditions. They are not only material properties but being connected to the whole motion or phenomenon, can be called the law of phenomenon. These have of course a great importance, but in applications always occurs the question whether they can really be applied to the problem under consideration. Obviously, a clear constitutive equation would be better. The question is whether it exists. In the following, the existence of such a constitutive equation is assumed and its properties are looked for. Knowing that the constitutive equation exists and knowing also its basic properties, one can construct it having done appropriate experiments and calculations. The investigations are restricted to solid continua. The main idea is that the physical changes in the continuum do not happen in the same time everywhere, they propagate from a starting point. A constitutive equation should contain this general experience. Such a changing property is considered by the kinematical and dynamical compatibility conditions of wave dynamics [1]. The investigations are restricted to acceleration wave.

If $\varphi(x^{\widehat{p}})=0$, $(\widehat{p}=1, 2, 3, 4)$, the boundary of the part of the continuum where the change has already occurred at a given time $x^4=t$, knowing the acceleration wave surface $\varphi(x^{\widehat{p}})=0$

$c=-\dfrac{\varphi_4}{\sqrt{g^{pq}\varphi_p\varphi_q}}$	the wave propagation velocity,
$n_i=-\dfrac{\varphi_i}{\sqrt{g^{pq}\varphi_p\varphi_q}}$	the normal unit vector of the wave front,
v^i	the velocity of the element,
$C=c-v^i n_i$	the relative wave propagation velocity are obtained.

The index $\widehat{p}$ denotes the derivative according to $x^{\widehat{p}}$ and g^{pq} is the contravariant metric tensor. The corresponding upper and lower indices mean a summation.

The propagation of any physical changes in the continuum are connected to positive and negative c functions. Its number is at most four. The fundamental equations of a continuum consist of the first and second Cauchy equations of motion, the kinematical equations and a system of appropriate equations being the constitutive equation. From the number of the unknown functions in the equations of motion and kinematical equations one should have six material equations.

Introducing the Cauchy stress tensor t^{kl}, volumetric force density q^k, mass density ρ, the fundamental equations are:

$$t^{kl}{}_{;l}+q^k=\rho\dot{v}^k\,, \tag{1}$$

$$t^{kl}=t^{lk}\,, \tag{2}$$

the kinematical equations, (3)

$$f_\alpha(\ldots\ldots)=0 \qquad (\alpha=1,\,2,\,\ldots,\,6)\,, \tag{4}$$

the constitutive equation. $\dot{v}^k$ denotes the acceleration of an element in (1). In the following, keeping the upper and lower indices the calculations are formed in a right angular Descartes coordinate system.

2. The Fundamental Equations of a Continuum Having Infinitesimal Strain

Denoting the infinitesimal strain tensor by ε^i_j *Eq.* (3) is

$$2\dot{\varepsilon}^i_j = v^i{}_{;j} + v^j{}_{;i}\,. \tag{3a}$$

Let (4) be the function of $t^{kl}{}_{,\widehat{p}}$ and $\varepsilon^{ij}{}_{,\widehat{k}}$, t^{kl}, ε^{ij} and $x^{\widehat{p}}$.

For infinitesimal strain

$$\dot{\varepsilon}^{ij} = \varepsilon^{ij}{}_{4} = \frac{\partial \varepsilon^{ij}}{\partial t}\,.$$

The constitutive equation is

$$f_\alpha(t^{kl}{}_{,\widehat{p}},\, \varepsilon^{ij}{}_{,\widehat{k}},\, t^{kl},\, \varepsilon^{ij},\, x^{\widehat{p}}) = 0\,. \tag{4a}$$

Using (1), (2), (3a) and (4a) and applying a lemma of Hadamard [1], the kinematical and dynamical material compatibility conditions can be formulated for the acceleration wave [2]. These conditions mean such a system of partial differential equations for the fuction φ which contains no φ but its first derivative $\varphi_{\widehat{k}}$. This system of differential equations is compatible if the equations

$$(F_\alpha,\, F_\beta) \equiv \frac{\partial F_\alpha}{\partial \varphi_{\widehat{k}}} \cdot \frac{\partial F_\beta}{\partial x_{\widehat{k}}} - \frac{\partial F_\alpha}{\partial x_{\widehat{k}}} \cdot \frac{\partial F_\alpha}{\partial \varphi_{\widehat{k}}} = 0\,, \qquad (\alpha = 1, \ldots, 6) \tag{5}$$

are satisfied expressing that the Poisson brackets are equal to zero. In (5) $F_\alpha \equiv f_\alpha - \underset{\circ}{f}_\alpha$ is the difference of the values of f_α after and before the wave surface.

(5) is satisfied if [3]

$$\frac{\partial F_\alpha}{\partial t^{\vartheta\widehat{p}}} - L_{\vartheta\gamma\widehat{pq}} \cdot \frac{\partial F_\alpha}{\partial \varepsilon^{\gamma\widehat{q}}} = 0\,, \tag{6}$$

where ϑ and γ are the functions of k, l and i, j

$$\vartheta(k\,l) = \begin{cases} k, & \text{if} \quad k = l \\ k + l + 1, & \text{if} \quad k \neq l \end{cases}.$$

Function $L_{\vartheta\gamma\widehat{pq}}$ is a function connecting the strain derivative κ^ϑ and the wave amplitude of the stress derivative μ^γ [3].

If $L_{\vartheta\gamma\widehat{pq}}$ does not contain $t^{kl}{}_{,\widehat{p}}$ and $\varepsilon^{ij}{}_{,\widehat{k}}$, moreover $L_{\vartheta\gamma\widehat{pq}} = L_{\vartheta\gamma}\delta_{\widehat{pq}}$, the constitutive equation for infinitesimal strain from (6) is [4]

$$L_{\vartheta\gamma}\mathrm{d}t^{\gamma} + \mathrm{d}\varepsilon_{\vartheta} = B_{\vartheta}\mathrm{d}t\,, \tag{7}$$

where t^{γ} are the coordinates of the stress tensor, t is the time.

The computing methods make use of function matrices $L_{\vartheta\gamma}$ and B_{γ} in algebraic form possible. Having an appropriate number of problems solved and possessing the necessary number of experimental data, the elements of $L_{\vartheta\gamma}$ and B_{γ} can be obtained. In cases when the elements of $L_{\vartheta\gamma}$ and B_{γ} are the same, (7) is a law of phenomenon. If the elements of $L_{\vartheta\gamma}$ and B_{γ} are the same for all possible examples, (7) is a constitutive equation. Thus, the fundamental equations of a continuum having infinitesimal strain are the system of differential equations (1), (2), (3a) and (7)

3. The Fundamental Equations of a Continuum Having Finite Strain

The aim of the present investigation is to find the fundamental equations of a continuum having finite strain. One should start from the material equation (4). f_{α} should be a function of only physically objective quantities. Thus, the stress and strain tensors t^{ij}, a_{ij} and their physically objective $\overset{\circ}{t}{}^{ij}$, $\overset{\circ}{a}{}_{ij}$ velocities or flux and the coordinates $x^{\widehat{p}}$ can be taken into consideration as variables. The equation (4) is

$$f_{\alpha}(\overset{\circ}{t}{}^{ij},\, \overset{\circ}{a}{}_{ij},\, t^{ij},\, a_{ij},\, x^{\widehat{p}}) = 0\,, \tag{4a}$$

where $\alpha = 1, \ldots, 6$, $i, j = 1, 2, 3$, $\widehat{p} = 1, \ldots, 4$.

Let $\overset{\circ}{t}{}^{ij}$ and $\overset{\circ}{a}{}_{ij}$ be the Lie derivatives on the tensor under consideration on the velocity field v^k [5], that is,

$$\mathcal{L}_v(t)^{ij} = \dot{t}^{ij} - t^{kj}v^{i}{}_{;k} - t^{ij}v^{j}{}_{;k}$$

and

$$\mathcal{L}_v(a)_{ij} = \dot{a}_{ij} + a_{kj}v^{k}{}_{;i} + a_{ij}v^{k}{}_{;j}\,. \tag{8}$$

Let a_{ij} be the Euler strain tensor

$$a_{ij} = g_{ij} - G_{KL}X^{K}_{,\,i}X^{L}_{,\,j}\,,$$

then, taking the Lie derivative of that equation

$$v_{ij} = \dot{a}_{ij} + a_{kj}v^{k}{}_{;i} + a_{ik}v^{k}{}_{;j} \tag{3b}$$

is the kinematical equation. In (3b) v_{ij} is the rate of deformation. The material derivative in *Eqs.* (8) and (3b) is for example

$$\dot{a}_{ij} = a_{ij,4} + v^{p}a_{ij,p}$$

in case of the strain tensor. Let us introduce notations

$$S_{\alpha ij} \equiv \frac{\partial f_{\alpha}}{\partial \overset{\circ}{t}{}^{ij}} \qquad \text{and} \qquad E_{\alpha}{}^{ij} \equiv \frac{\partial f_{\alpha}}{\partial \overset{\circ}{a}_{ij}} . \tag{9}$$

Moreover, let ν^{i}, μ^{ij} and α_{ij} be the amplitudes of the acceleration, the stress derivative and the strain waves. Thus, if for example the strain derivative before the wave front $\varphi=0$ is $\underset{\circ}{a}_{ij,\hat{p}}$ then, after it is $\underset{\circ}{a}_{ij\hat{p}}+\alpha_{ij}\varphi_{\hat{p}}$. Using the dynamical compatibility condition

$$\nu^{i} = -\frac{\mu^{ij}n_{j}}{\rho C} , \tag{10}$$

the kinematical compatibility condition

$$\alpha_{ij} = \frac{1}{2\rho C^{2}} \cdot \mu^{pq}n_{q}\Big[(g_{ip} - 2a_{ip})n_{j} + (g_{ip} - 2a_{pj})n_{i}\Big] . \tag{11}$$

Taking into account (10) and (11) from the material compatibility condition, the wave equation

$$\Big\{\rho S_{\alpha kp}C^{2} + n_{p} \cdot \Big[S_{\alpha ij}(t^{qj}g_{k}^{i} + t^{iq}g_{k}^{j})n_{q}$$

$$+\frac{1}{2} \cdot E_{\alpha}{}^{ij} \cdot (g_{ij}n_{j} + g_{jk}n_{i})\Big]\Big\} \cdot \mu^{kp} = 0 \tag{12}$$

is obtained [6].

Using index function $\vartheta(kp)$ instead of the kp index of the multiplying matrix μ^{kp}, the determinant of the 6×6 matrix having the indices $\alpha\vartheta$ is zero. Thus, the relative wave propagation velocity C can be obtained. C, as it can easily be shown from (12), is a function of the stress tensor t^{ij}.

The condition of the existence of a wave is given by the system of equations (5) as in part 2. It can be written from *Eqs.* (3b) and (4b) and consists of two parts in its present form [6]. One of them is multiplied by φ_{p}, the other one is not. The equation should be satisfied for all φ_{p}, thus, both parts are zero. The first one is

$$\Big\{S_{\alpha ij}\Big[\mu^{ij}(g^{4\widehat{q}}+v^r g_r{}^{\widehat{q}})-(t^{qj}\nu^i+t^{iq}\nu^j)\cdot g_q{}^{\widehat{q}}$$

$$+E_\alpha{}^{ij}\Big[\alpha_{ij}(g^{4\widehat{q}}+v^r g_r{}^{\widehat{q}})+(a_{kj}g_i{}^{\widehat{q}}+a_{ik}g_j{}^{\widehat{q}})\cdot\nu^k\Big\}$$

$$\cdot\Big\{S_{\beta kl}\Big[\mu^{kl}(g^{4\widehat{p}}+v^r g_r{}^{\widehat{p}})-(t^{pl}\nu^k+t^{kp}\nu^l)\cdot g_p{}^{\widehat{p}}\Big]_{,\widehat{q}}$$

$$+E_\beta{}^{kl}\Big[\alpha_{kl}(g^{4\widehat{p}}+v^r g_r{}^{\widehat{p}})+(a_{rl}g_k{}^{\widehat{p}}+a_{kr}g_l{}^{\widehat{p}})\cdot\nu^r\Big]_{,\widehat{q}}\Big\}$$

$$+\ldots=0. \tag{13}$$

(13) is zero, if

$$S_{\alpha ij}-L_{ijkl}E_\alpha{}^{kl}=0 \tag{14}$$

and then

$$\alpha_{kl}+L_{ijkl}\mu^{ij}=0\,, \tag{15}$$

and

$$S_{\alpha ij}(t^{qj}\nu^i+t^{iq}\nu^j)=E_\alpha{}^{rs}(a_{ks}g_r{}^q+a_{rk}g_s{}^q)\nu^k\,. \tag{16}$$

From (14) and (15)

$$S_{\alpha ij}\mu^{ij}+E_\alpha{}^{kl}\alpha_{kl}=0\,. \tag{17}$$

Substituting (11) into (17) after simplification and with the aid of (16), the wave equation (12) is obtained.

Expressions (16) and (17) can also be found in another way. The constitutive equation (4b) contains in $\overset{\circ}{t}{}^{ij}$ and $\overset{\circ}{a}_{ij}$ the velocity v^p. Let

$$\frac{\partial f_\alpha}{\partial v^p}=0\,,$$

that is

$$S_{\alpha ij}t^{ij}{}_{,p}+E_\alpha{}^{kl}a_{kl,p}=0\,. \tag{18}$$

Taking (18) after and before the wave front and subtracting the second one from the first

$$(S_{\alpha ij}-\underset{\circ}{S}_{\alpha ij})\,\underset{\circ}{t}{}^{ij}{}_{,p}+(E_\alpha{}^{kl}-\underset{\circ}{E}_\alpha{}^{kl})\,\underset{\circ}{a}_{kl}$$

$$+(S_{\alpha ij}\mu^{ij} + E_{\alpha}{}^{kl}\alpha_{kl})\varphi_p = 0\,. \tag{19}$$

In (19) the lower zero denotes the values before the wave front. If S and E are continuous on the wave surface, then from (19) concludes

$$S_{\alpha ij}\mu^{ij} + E_{\alpha}{}^{kl}\alpha_{kl} = 0\,. \tag{20}$$

It is the same as the *Eq.* (17). Using (11), an equation can be obtained being similar to (12) concerning C^2. The condition of this equation being the same as (12) is (16). Thus, (18) means that the constitutive equation does not depend on v^p.

Returning to *Eqs.* (14) and (6), one can find that from the mathematical point of view these equations are the same. If in this case L_{ijpq} does not also depend on $\mathring{t}^{ij}$ and $\mathring{a}_{pq}$, then, according to [4], a constitutive equation in the form

$$L_{ijpq}\mathrm{d}\mathring{t}^{ij} + \mathrm{d}\,\mathring{a}_{pq} = B_{pq}\mathrm{d}t \tag{21}$$

can be taken into account, or also from [4] the form

$$L_{ijpq}\mathrm{d}t^{ij} + \mathrm{d}a_{pq} = B_{ijpq}\mathrm{d}t^{ij} + C_{ijpq}\mathrm{d}a^{pq} + D_{\widehat{k}pq}\mathrm{d}x^{\widehat{k}} \tag{22}$$

can be considered, too.

In (22) the restriction to L_{ijpq} in case of (21) is not valid and the other coefficient matrices are

$$B_{ijpq}\cdot\frac{\partial f_\alpha}{\partial\,\mathring{a}_{pq}} = \frac{\partial f_\alpha}{\partial t^{ij}}\,,\qquad C_{ijpq}\cdot\frac{\partial f_\alpha}{\partial\,\mathring{a}_{ij}} = \frac{\partial f_\alpha}{\partial a_{pq}}\,,$$

$$D_{\widehat{k}pq}\cdot\frac{\partial f_\alpha}{\partial\,\mathring{a}_{pq}} = \frac{\partial f_\alpha}{\partial x^{\widehat{k}}}\,.$$

The question whether (21) and (22) are constitutive equation or law of phenomenon can be answered by the investigation described at the end of part 2, thus *Eqs.* (1), (2), (3b) and (21) or (22) form the fundamental equations of the continuum mechanics.

4. Summary

The reason why the possible fundamental equations of the continuum mechanics cannot be treated is not only the absence of the constitutive equation, but also the several possible forms of the kinematic equation. Thus, on selecting a constitutive equation, one should also consider the selection of the kinematic equation.

If the results of the calculations by some symbolic manipulator program and the experimental data correspond, the selected law of phenomenon can be a constitutive equation. Then, the law of phenomenon is the same system of equations for all motions and belongs to the possible constitutive equations.

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