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NONLINEAR DYNAMICS OF JOY-STICK CONTROLLED MACHINES

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Abstract

In practice, joy-sticks are often used when a human operator controls the motion of a machine. The control system provides a kinematical constraint for the controlled machine, i.e. the position of the joy-stick determines the velocity of the machine with a certain gain. The human operator, of course, tries to control the joy-stick position in a way that the machine will reach the desired position. The nonlinear mathematical model of this system is analyzed in the paper which also involves the human operator's reflex delay.

Keywords: time delay, neutral functional differential equations, joy-stick control, stability, chaos.

Introduction

In case of joy-stick control, the position of the joy-stick determines the velocity of the controlled machine with a certain gain. The human operator tries to control the joy-stick position in a way that the machine will reach the desired position. The human operator's behaviour can often be described by proportional and derivative terms (PD). However, the delay of the human operator's reflex has a central role in system stability. If the nonlinear spring supporting the joy-stick is not chosen properly, the operator will not be able to work: robustness or even stability may be adversed, vibrations occur, etc.

This paper presents a nonlinear scalar, first order, neutral functional differential equation to model the above described phenomena. Stability charts are given and compared in cases of different weight functions with respect to the past, and the existence of supercritical Hopf bifurcation is proved for some special cases. A parameter domain is also presented where the existence of chaos is very likely in the system.

Mathematical Model

Fig. 1 shows the mechanical system in question where q is the position of the joy-stick and u is that of the one-degree-of-freedom controlled machine. The velocity $\dot{u}$ is determined by the position q of the joy-stick, $K > 0$ is the constant gain of the control. The spring at the joy-stick has nonlinear characteristics described by $Q = f(q)$, where f is odd and analytically in the neighbourhood of the origin. Thus, $s_1 = f'(0) > 0$ is the stiffness of the spring and $f(q) = s_1 q + s_3 q^3 + \dots$. The control force Q applied by the human operator is modelled by

$$Q(t) = -\int_{-\infty}^{0} \Big(Pu(t+\theta) + D\dot{u}(t+\theta)\Big) \mathrm{d}\eta(\theta) ,$$

where the scalar P and D are the gains at the operator, the dot represents the right-hand derivative with respect to the time t, and η is a scalar function of bounded variation mapping the interval $(-\infty, 0]$ to the non-negative reals, and where

$$\int_{-\infty}^{0} \mathrm{d}\eta = 1 , \qquad \int_{-\infty}^{0} \mathrm{e}^{-\nu\theta} |\mathrm{d}\eta(\theta)| < +\infty \quad \text{for some} \qquad \nu > 0 . \tag{1}$$

By means of the function η we may consider different weights of the past states of the system. In this way, the delay of the human operator's reflex is also involved in the model.

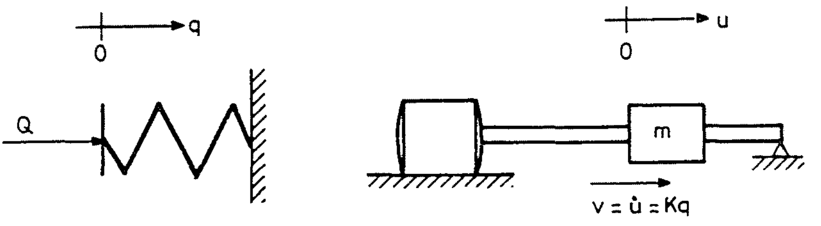

Fig. 1. Mechanical model

Summarizing these elements of the model, we get the scalar equations

$$\dot{u}(t) = Kq(t) ,$$

$$s_1q(t)+s_3q^3(t)+\ldots=-\int_{-\infty}^{0}\Big(Pu(t+\theta)+D\dot{u}(t+\theta)\Big)\mathrm{d}\eta(\theta)$$

for the functions q and u. Using a 3-jet approximation only, we can rearrange these equations into a scalar neutral functional differential equation (NFDE) with respect to u as follows:

$$\dot{u}(t)=-\int_{-\infty}^{0}\Big(\mu_1u(t+\theta)+\mu_2\dot{u}(t+\theta)\Big)\mathrm{d}\eta(\theta)$$
$$+\epsilon\Big[\int_{-\infty}^{0}\Big(\mu_1u(t+\theta)+\mu_2\dot{u}(t+\theta)\Big)\mathrm{d}\eta(\theta)\Big]^3, \qquad (2)$$

where

$$\mu_1=\frac{PK}{s_1}, \quad \mu_2=\frac{DK}{s_1}, \quad \epsilon=\frac{s_3}{s_1K^2}. \qquad (3)$$

In the following section, the asymptotic stability of the zero solution of the linear part of (2) will be investigated. In this way, it is not trivial to get any conclusion for the asymptotic stability of the zero solution of the nonlinear NFDE (2) [1]. However, we shall use special functions η in (2) which enable us to investigate not only the stability but also the Hopf bifurcation of the zero solution of (2).

The functions η we use are

$$\eta_0(\theta)=\mathrm{e}^{\theta/\tau}, \quad \eta_1(\theta)=\Big(1-\frac{\theta}{\tau}\Big)\mathrm{e}^{\theta/\tau}, \quad \eta_2(\theta)=\Big(1-2\frac{\theta}{\tau}+2\frac{\theta^2}{\tau^2}\Big)\mathrm{e}^{2\theta/\tau},$$

$$\eta_\infty(\theta)=\begin{cases}0, & \theta\in(-\infty,-\tau]\\ 1, & \theta\in(-\tau,0]\end{cases}. \qquad (4)$$

All these functions satisfy the conditions (1). There is a new parameter $\tau>0$ here which can be considered as the measure of the influence of the past. The weight functions $w=\eta'$ are

$$w_0(\theta)=\frac{1}{\tau}\mathrm{e}^{\theta/\tau}, \quad w_1(\theta)=-\frac{\theta}{\tau^2}\mathrm{e}^{\theta/\tau}, \quad w_2(\theta)=\frac{4\theta^2}{\tau^3}\mathrm{e}^{2\theta/\tau},$$

$$w_\infty(\theta) = \delta_{\text{Dirac}}(\theta + \tau)\,, \tag{5}$$

respectively, where w_1, w_2 and w_∞ have their maximum values at $\theta = -\tau$. In engineering, the human operator's reflex delay is usually considered as a discrete one, that is η_∞ is used in the model (2). However, the NFDE (2) can be transformed into a finite dimensional system of ordinary differential equations if η_j, $j=0$, 1, 2 are in use, and it remains infinite dimensional if η_∞ is applied.

We refer to the literature [1, 2, 3] regarding the problem of choosing the appropriate space of the initial functions of the NFDE (2).

Stability Analysis

The characteristic function related to the linear part of the NFDE (2) has the form

$$D(\lambda) = \lambda + \int_{-\infty}^{0} (\mu_1 + \mu_2\lambda) e^{\lambda\theta} d\eta(\theta)\,. \tag{6}$$

It gives polynomials if $\eta = \eta_j$, $j=0$, 1, 2, and D remains transcendental if $\eta = \eta_\infty$:

$$D_0(\lambda) = \lambda^2 + \frac{1+\mu_2}{\tau}\lambda + \frac{\mu_1}{\tau}\,, \tag{7a}$$

$$D_1(\lambda) = \lambda^3 + \frac{2}{\tau}\lambda^2 + \frac{1+\mu_2}{\tau^2}\lambda + \frac{\mu_1}{\tau^2}\,, \tag{7b}$$

$$D_2(\lambda) = \lambda^4 + \frac{6}{\tau}\lambda^3 + \frac{12}{\tau^2}\lambda^2 + \frac{8(1+\mu_2)}{\tau^3}\lambda + \frac{8\mu_1}{\tau^3}\,, \tag{7c}$$

$$D_\infty(\lambda) = \lambda + \mu_2\lambda e^{-\lambda\tau} + \mu_1 e^{-\lambda\tau}\,, \tag{7d}$$

respectively. In case of D_j, $j=0, 1, 2$, the Routh-Hurwitz criterion provides the necessary and sufficient conditions for the parameters $\mu_1\tau$ and μ_2 when the characteristic roots have negative real parts. For D_∞, the stability conditions can directly be deduced from the results in [1, 4] or from a recent paper of BOESE [5]. The results are presented in the following statement:

All the zeros of the characteristic function D_j in (7) have negative real parts if and only if

$$j = 0: \qquad \mu_1 > 0, \qquad \mu_2 > -1\,; \tag{8a}$$

$$j=1: \qquad 0<\mu_1\tau<2(1+\mu_2)\,; \tag{8b}$$

$$j=2: \qquad 0<\mu_1\tau<2(1+\mu_2)-\frac{2}{9}(1+\mu_2)^2\,; \tag{8c}$$

$$j=\infty: \qquad 0<\mu_1\tau<\sqrt{1-\mu_2^2}\,\arccos(-\mu_2)\,, \tag{8d}$$

respectively.

The stability charts defined by the conditions of this theorem are shown in *Fig. 2.* The better approximation of the discrete delay is used in the NFDE (2), the closer the stability limits are to the totally shaded central region of *Fig. 2.*

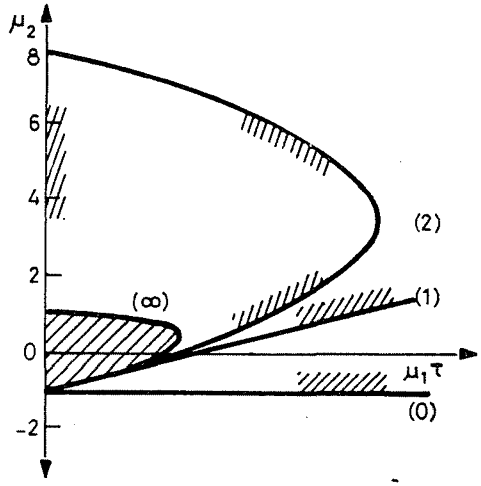

Fig. 2. Stability charts

Hopf Bifurcation

Let us consider $\eta=\eta_1$ according to (4) in the NFDE (2). If we introduce the new variables x_1, x_2 and x_3 by the formulae

$$x_1(t)=u(t)\,,$$

$$x_2(t)=\int\limits_{-\infty}^{0}\Big(\mu_1 u(t+\theta)+\mu_2\dot u(t+\theta)\Big)\Big(-\frac{\theta}{\tau^2}\mathrm{e}^{\theta/\tau}\Big)\mathrm{d}\theta\,,$$

$$x_3(t) = \int_{-\infty}^{0} \Big(\mu_1 u(t+\theta) + \mu_2 \dot{u}(t+\theta)\Big) \frac{1}{\tau} e^{\theta/\tau} d\theta ,$$

then the NFDE (2) can be transformed into a 3-dimensional system of ordinary differential equations (see [2, 6]):

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -\frac{1}{\tau} & \frac{1}{\tau} \\ \frac{\mu_1}{\tau} & -\frac{\mu_2}{\tau} & -\frac{1}{\tau} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \epsilon \begin{pmatrix} x_2^3 \\ 0 \\ \frac{\mu_2}{\tau} x_2^3 \end{pmatrix} . \tag{9}$$

The characteristic function of the linear part of (9) is just D_1 as shown in (7b). The inequalities (8b) give the stability limit

$$\mu_1 \tau = 2(1+\mu_2) , \qquad \mu_1 > 0 , \tag{10}$$

where D_1 has two pure imaginary zeros

$$\lambda_{1,2} = \pm i\omega , \qquad \omega = \sqrt{\frac{\mu_1}{2\tau}} \tag{11}$$

and a negative real one $\lambda_3 = -2/\tau$. We can easily choose a bifurcation parameter (e. g. μ_1 or μ_2 or τ) in a way that the characteristic roots $\lambda_{1,2}$ cross the imaginary axis with a non-zero velocity. Thus, there is a Hopf bifurcation at the critical parameters given by (10).

If we introduce the new variables y_1, y_2 and y_3 by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \tau \\ 0 & -\omega & 2 \\ \omega^2\tau & -\omega & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} ,$$

then we get the Poincaré normal form of *Eq.* (9) where the parameters are fixed at the critical values (10):

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & -\frac{2}{\tau} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$+ \frac{\epsilon}{\omega^2\tau^2+4} \begin{pmatrix} -\omega^3(\omega^2\tau^2+3)y_2^3 + \dots \\ -\frac{2}{\tau}\omega^2 y_2^3 + \dots \\ \dots \end{pmatrix} . \tag{12}$$

Since there are no terms of second degree of the variables, we can easily separate the first two equations with $y_3 = 0$ which describe the flow on the two dimensional centre manifold. Applying the ready-made formulae of [6, 7] to these equations in the centre manifold, we can directly prove that the

Hopf bifurcation is supercritical if $\epsilon > 0$. This means that orbitally asymptotically stable limit cycle exists around the unstable equilibrium when the parameters are close enough to the critical values. As the definition (3) of the parameters show, $\epsilon > 0$ yields $s_3 > 0$, that is the spring in the model of *Fig. 1* has hardening characteristics.

We get the same qualitative result if η_0 or η_2 is substituted into the NFDE (2). The investigation of the case of η_∞ is more difficult since the NFDE (2) remains infinite dimensional. However, if $\mu_1 > 0$, $\mu_2 = 0$, then *Eq.* (2) is a special retarded functional differential equation (RFDE)

$$\dot{u}(t) = -\mu_1 u(t-\tau) + \epsilon \mu_1^3 u^3(t-\tau)\,, \tag{13}$$

it is still infinite dimensional, though. Its linear part is well-known in the literature [4, 7]: at the critical parameter $\mu_1\tau = \pi/2$ (see (8d) when $\mu_2 = 0$), there are two pure imaginary characteristic roots

$$\lambda_{1,2} = \pm i\omega\,, \qquad \omega = \frac{\pi}{2\tau} \tag{14}$$

and all the infinitely many other characteristic roots have negative real parts. In order to reduce the infinite dimensional RFDE (13) at the critical parameters to the two dimensional centre manifold, we can use the operator differential equation form of (13) and the linear transformation in the same form as it has appeared in [7]. The short calculation results in the two dimensional system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \epsilon \frac{2\pi}{\pi^2+4}\frac{\omega^2}{\tau}\begin{pmatrix} -2y_2^3 + \dots \\ -\pi y_2^3 + \dots \end{pmatrix}, \tag{15}$$

where ω is given in (14). It has the same structure as the first two equations of (12), and we get the same qualitative result: if $\epsilon > 0$ then there is a supercritical Hopf bifurcation in the RFDE (13).

In spite of these results, the original NFDE (2) with η_∞ may have much more complicated attractors. This will be shown in the following section.

Chaos

Let us consider the NFDE (2) with η_∞ from (4), and let $\mu_1 = 0$, $\mu_2 \neq 0$. Then it has the special form

$$\dot{u}(t) = -\mu_2 \dot{u}(t-\tau) + \epsilon \mu_2^3 \dot{u}^3(t-\tau)\,. \tag{16}$$

It is easy to see that we have got a simple nonlinear iteration with respect to some discrete values of $\dot{u}$. If $|\mu_2| > 1$, then $\dot{u} = 0$ is unstable. If, in addition, $\epsilon > 0$ also holds, then there are two further trivial solutions for $\dot{u}$ which are also unstable if $|\mu_2|$ is large enough. As it is well-known from the literature [8], chaotic iteration may appear for certain parameter domains. As a matter of fact, the zero solution of the NFDE (2) with η_∞ has infinitely many characteristic roots with positive real parts if $|\mu_2| > 1$, as shown in [4, 5]. These refer to very complicated bifurcation phenomena at $\mu_1 = 0$, $\mu_2 = \pm 1$.

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