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ON THE EFFECT OF THE NORMAL FORCE ON THE TORSIONAL VIBRATION OF SYSTEMS

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Abstract

This paper considers the effect of a normal force on the torsional vibration of systems. In the solution of this problem, a continuum-mechanical examination of the strain exceeding the linear theory has an important role. In the first part of the paper this examination of the strain will be given, and in the second part the effect of the normal forces to the vibration of a torsional system will be considered based on the results of the first part [5].

Keywords: torsional vibration coupled with normal force, nonlinear vibration, continuum-mechanical examination.

Examination of the Strain

Small strains are supposed where the linear theory is generally used. Then we may use the strain and stress tensors to describe the state referring to the undeformed state of the body. Thus, in the following, the Lagrangian strain tensor and the II. Piola–Kirchoff stress tensor can be used, supposing – similarly to the linear theory of elasticity — that the connection between these tensors can be given by the Hooke's law [1].

Keeping some nonlinear elements of strain, more exact results can be produced than by the usual linear theory of elasticity.

It is not necessary to keep all the nonlinearities. In the following, two quantities will be considered to be equal if they differ from each other in elements which include the third or higher powers of small quantities of strain as multipliers. These terms are negligible.

The continuum-mechanical examinations suppose that the beam has neither body forces, nor surface forces but there are normal forces and torsional torque acting on the ends of the beam. Moreover, it is assumed that the beam is long enough to have a middle part where it is approximately true that during the deformation the cross-sections remain planes, normal to the axes and in their planes they deform equally.

Co-ordinates and Strain Tensor

Fig. 1 shows the coordinates R, Φ, Z and the basis $\mathbf{e}_R$, $\mathbf{e}_\Phi$, $\mathbf{e}_Z$ belonging to the point P of the beam being in an undeformed state. In this state the position vector of point P is $\mathbf{R} = R\mathbf{e}_R + Z\mathbf{e}_Z$. During the deformation, the point P gets to position P' where the co-ordinates of P' are r, ϕ, z, and the coordinates of the displacement are ρ, $Z\Psi$, ζ. The position vector of P' is $\mathbf{r}$.

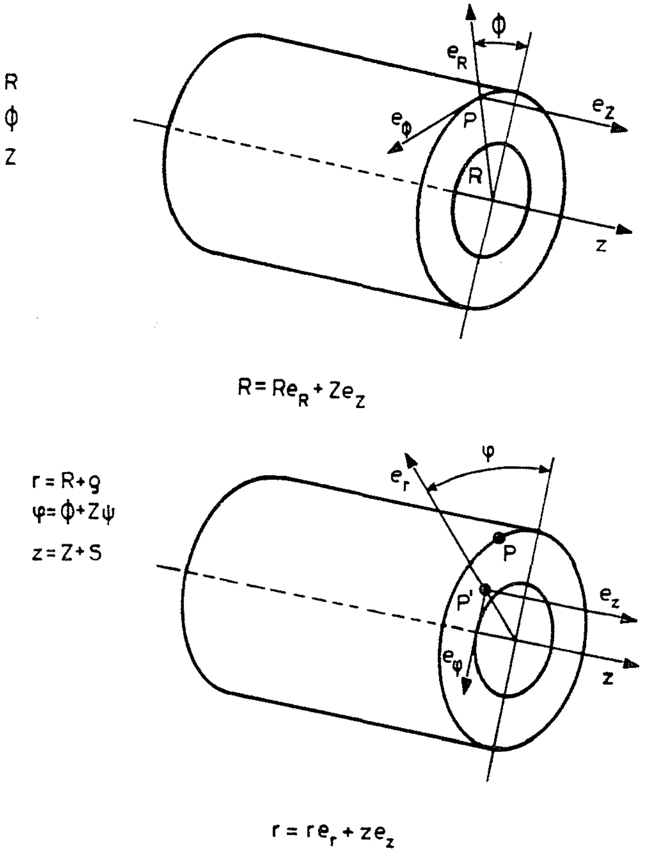

Fig. 1. Co-ordinates and basis vectors for undeformed and for deformed state of a tube

Hence,

$$r = R + \rho\,,$$

$$\phi = \Phi + Z\Psi\,,$$

$$z = Z + \zeta .$$

Due to the symmetry and the earlier assumptions referring to the deformation of the cylindrical tube, the displacements are:

$$\rho = \rho(R) ,$$

$$\Psi = \text{ const.}$$

$$\zeta = \zeta(Z) = kZ , \qquad k = \text{ const.}$$

For the derivatives of the coordinates of the displacements, let us use the following notations:

$$\frac{\partial \rho}{\partial R} = \rho_R ,$$

$$\frac{\partial \zeta}{\partial Z} = \zeta_z .$$

During the deformation, the elementary lengths S_i of the coordinate lines passing through the point P change to s_i, and their originally rectangular angles become deformed [3].

a. With the help of the arc lengths S_i, the basis $\mathbf{G}_i$ belonging to the undeformed state and the basis $\mathbf{g}_i$ belonging to the deformed state of the beam can be represented as vectors tangent to the line elements.

$$\mathbf{G}_i = \frac{\partial \mathbf{R}}{\partial S_i} , \qquad \mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial S_i} = (1 + \varepsilon_i) \cdot \mathbf{e}_i = (1 + \varepsilon_i) \cdot \frac{\partial r}{\partial s_i} ; \qquad \mathbf{e}_i \cdot \mathbf{e}_i = 1 .$$

b. With the help of $\mathbf{G}_i$ and $\mathbf{g}_i$ we can define the Lagrangianstrain tensor required to the examinations:

$$\mathbf{H} = \frac{1}{2} \cdot (\mathbf{g} - \mathbf{G}) ,$$

where $\mathbf{g}$ and $\mathbf{G}$ are the metric tensors belonging to the undeformed and deformed state of the body, respectively.

In the following, the strain tensor will be used with some modifications.

$$H_{ik} = \frac{1}{2} \cdot (g_{ik} - G_{ik}) \qquad \text{if} \quad i \neq k\,,$$

$$H_{ik} \cong \frac{1}{2} \cdot (\varepsilon_i + \varepsilon_k) \qquad \text{if} \quad i = k\,,$$

or

$$H_{ii} = \varepsilon_i\,.$$

The modified strain tensor will be denoted by $\mathbf{L}$. The matrix of the strain tensor referring to the concrete problem is:

$$\underset{(R\Phi Z)}{\mathbf{L}} = \begin{bmatrix} \rho_R & 0 & 0 \\ 0 & \rho/R & (R+2\rho)\Psi/2 \\ 0 & (R+2\rho)\Psi/2 & R^2\Psi^2/2 + \zeta_z \end{bmatrix}\,,$$

where

$$\rho_R = \frac{\partial \rho}{\partial R}\,,$$

$$\zeta_Z = \frac{\partial \zeta}{\partial Z}\,.$$

To get the unknown functions $\rho(R)$ and $\zeta(Z)$ we have to use the equations of the equilibrium, and the boundary conditions. For that reason, we need the matrix of stress tensor, too. The elements of the stress tensor are:

$$\frac{\sigma}{2G} = \frac{1-\nu}{1-2\nu} \cdot \rho_R + \frac{\nu}{1-2\nu} \cdot \left[\frac{\rho}{R} + \left(\frac{R^2\Psi^2}{2} + \zeta_z\right)\right]\,,$$

$$\frac{\sigma}{2G} = \frac{1-\nu}{1-2\nu} \cdot \frac{\rho}{R} + \frac{\nu}{1-2\nu} \cdot \left[\rho_R + \left(\frac{R^2\Psi^2}{2} + \zeta_z\right)\right]\,,$$

$$\frac{\sigma}{2G} = \frac{1-\nu}{1-2\nu} \cdot \left(\frac{R^2\Psi^2}{2} + \zeta_z\right) + \frac{\nu}{1-2\nu} \cdot \left[\rho_R + \frac{\rho}{R}\right]\,,$$

$$\frac{\tau}{2G} = \frac{1}{2} \cdot (R + 2\rho) \cdot \Psi\,,$$

$$\tau_{12} = \tau_{21} = \tau_{13} = \tau_{31} = 0\,.$$

From the first equation of the equilibrium we get an ordinary linear Euler-type differential equation for the function $\rho(R)$:

$$R \cdot \rho_{RR} + \rho_R - \frac{\rho}{R} = -\frac{\nu}{1-\nu} \cdot R^2\Psi^2 .$$

The general solution of this equation is:

$$\rho = -\frac{1}{8} \cdot a_0 \cdot R^3\Psi^2 + C_1 R + C_2 R ,$$

where $a_0 = \nu/(1-\nu)$, C_1 and C_2 are constant values which can be computed from the boundary conditions.

If the values of the normal force and the torsional torque are given, the unknown values of Ψ and ζ_z can be obtained from the stress distribution functions $\tau_{23} = \tau_{23}(R)$ and $\sigma_3 = \sigma_3(R)$, reducing them to the center of the cross-section.

Deformation of a Thin-Walled Cylindrical Tube

Let us consider an elementary part of a tube being in the equilibrium (*Fig. 2*). The length of it will be $\mathrm{d}Z$ and the central angle $2\mathrm{d}\Phi$. Supposing the quantities depending on the thickness of the tube and also the stress functions are to be approximated with its Taylor-series as a function of the distance from the middle surface of the tube. The approximation stops at the linear terms of the series:

$$\rho = -\nu\eta R ,$$

$$\Psi = \text{ const.}$$

$$\zeta = \Big(-\frac{R_0^2\Psi^2}{2} + \eta\Big) \cdot Z ,$$

where R_0 is the middle radius of the tube and η is the specific value of stretching.

Deformation of a Solid Cylinder

In this case, the solution we obtain is formally just the same as before:

$$\zeta = \Big(-\frac{R_0^2\Psi^2}{2} + \eta\Big) \cdot Z ,$$

but in this expression we get

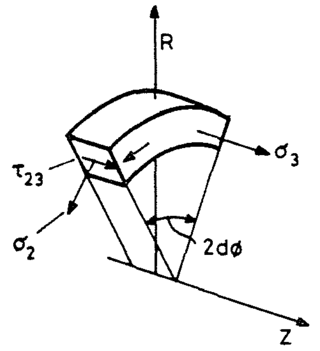

Fig. 2. An elementary part of a tube in the state of equilibrium

$$R_0 = \frac{R_2}{\sqrt{2}},$$

where R_2 is the radius of the cylinder.

Function $\rho(R)$ is:

$$\rho = \frac{1}{8} \cdot a_0 \left(R_2^2 - R^2 \right) \cdot R\Psi - \frac{R\nu\sigma_{03}}{E},$$

where σ_{03} is the average value of the tension on the surface.

On the Properties of Torsional Systems

Cylindrical Thin-Walled Tube as a Torsional Spring

First we consider a system that consists of an elastic tube with a disc on its end (*Fig. 3*). Let us denote the length of the tube by l, the mass moment of inertia of the disc by J and its mass by m. Let a normal force F and a torsional torque M be acting on the disc.

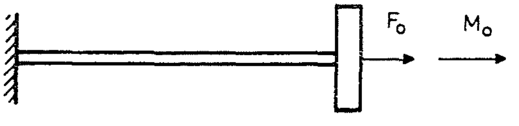

Fig. 3. Torsional vibration system

The motions of the disc are supposed to be described by a displacement w in the direction of axis z and by a rotation around axis z. The equations of the motion of the system can be given with the help of the Lagrangian equations [2]. The kinetic energy of the system will be as follows:

$$2T = m\dot{w}^2 + J\dot{\alpha}^2 .$$

Displacement w includes the stretching y caused by the normal force and the change of the length of the beam ξ produced by torsion

$$w = y + \xi ,$$

where $\xi = R^2\Psi^2 \cdot l/2 = R^2\alpha^2/2l$, R is the middle radius of the tube.

For the left side of the Lagrangian equations — keeping only the linear terms — we get the following expression:

$$\frac{\mathrm{d}}{\mathrm{d}t} \cdot \left(\frac{\partial T}{\partial \dot{\alpha}}\right) - \frac{\partial T}{\partial \alpha} = J\ddot{\alpha} ,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \cdot \left(\frac{\partial T}{\partial \dot{y}}\right) - \frac{\partial T}{\partial y} = m\ddot{y} .$$

We can get the right side of the equations with the help of the expressions of the power of internal and external forces:

$$P_I = -\frac{1}{2}\int_{(V)} \underline{\underline{\sigma}}..\mathbf{L}\mathrm{d}V = -\frac{I_pG}{l} \cdot \alpha\dot{\alpha} + \frac{EA}{l} \cdot y\dot{y} ,$$

$$P_E = M\dot{\alpha} + F \cdot (\dot{y} + \dot{\xi}) = \left(M + \frac{FR^2\alpha}{l}\right) \cdot \dot{\alpha} + F\dot{y} ,$$

where I_p denotes the polar moment of the cross-section to the center, A is the area of the cross-section, E is the Young modulus and G is the modulus of rigidity. The general forces are as follows:

$$Q_\alpha = M - \frac{1}{l} \cdot (I_pG + FR^2) \cdot \alpha ,$$

$$Q_y = F - \frac{EA}{l} \cdot y .$$

The equations of the motion are:

$$J\ddot{\alpha} + \frac{1}{l} \cdot (I_pG + FR^2) \cdot \alpha = M ,$$

$$m\ddot{y} + \frac{EA}{l} \cdot y = F .$$

It can be seen that the first equation includes the effect of the normal force on the torsional vibrations, but the equation describing the longitudinal motion of the system does not change due to the torsion.

Let us use the following notation in the first equation

$$c = \frac{I_p G}{l} + \frac{F R^2}{l} = \frac{I_p G}{l} \cdot \left(1 + \frac{\sigma}{G}\right) = c' + c'' .$$

For the materials commonly used, the value of σ/G cannot exceed 5%. But in the case of some kind of plastics, this value can be as high as 30% [4]. The equation of the torsional vibration can be written in the usual form:

$$J\ddot{\alpha} + c\alpha = M ,$$

where c includes the effect of the normal forces.

Torsional Vibration of a Shaft Fixed at Both Ends

Fig. 4 shows an elastic shaft with only one disc on it. The part of the shaft denoted '0–1' has the value of stiffness c_1, the other part '1–2' has the value of stiffness c_2 and the mass moment of inertia of the disc is J. There are no external forces or moments acting on the disc. The motion of the disc is characterized by the rotation α around the axis z.

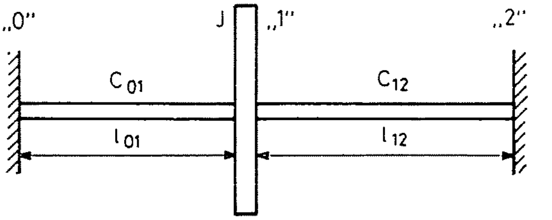

Fig. 4. Torsional vibration system fixed at both ends

In that case, the specific change of the '0–1' part of the shaft in the direction z would be ξ_{z1}, due to the torsion and the specific change of the part '1–2' would be ξ_{z2}, if the end '2' could move freely. The whole change of the shaft caused by the torsion would be as follows:

$$y_{2z} = \xi_{z1} l_1 + \xi_{z2} l_2 .$$

With the help of the specific value of the rotation, this expression can be written in the following form:

$$y_{2t} = -\frac{1}{2} \cdot R^2 \Psi_1^2 l_1 - \frac{1}{2} \cdot R^2 \Psi_2^2 l_2 \,,$$

where $\Psi_1 = (\alpha_1 - \alpha_0)/l_1 = \alpha_1/l_1$,
$\Psi_2 = (\alpha_2 - \alpha_1)/l_2 = \alpha_1/l_2$.

Let us make the end '2' free, but let a normal force F and a torsional torque M be acting on this end. The following conditions at the '2' cross-section have to be satisfied:

$$y_2 = 0 \,,$$

$$\alpha_2 = 0 \,.$$

The equation of the motion of the disc is:

$$J\ddot{\alpha} + (c_1 + c_2) \cdot \alpha = M \,,$$

where

$$c_1 = c_1' + c_1'' = \frac{J_p G}{l_1} + \frac{J_p}{l_1} \,, \qquad \sigma = \frac{F}{A} \,,$$

$$c_2 = c_2' + c_2'' = \frac{J_p G}{l_2} + \frac{J_p}{l_2} \,,$$

$$c = J_p G \cdot \left(1 + \frac{\sigma}{G}\right) \cdot \left(\frac{1}{l_1} + \frac{1}{l_2}\right) = J_p G \cdot \left(1 + \frac{\sigma}{G}\right) \cdot \frac{1}{l_0}$$

supposing that the cross-sections of the parts at the shaft are equal.

To satisfy the given conditions, the value of the stretching caused by the normal force acting on the end '2' should be equal to the change of the length caused by the torsion

$$y_{2t} = -y_{2s} \,,$$

$$y_{2s} = \frac{F}{EA} \cdot \left(l_1 + l_2\right) = \frac{\sigma}{E}\left(l_1 + l_2\right) .$$

From this we can obtain the following expression:

$$\sigma = \frac{1}{2} \cdot E R^2 a_1^2 \cdot \frac{1}{l_0 \cdot (l_1 + l_2)} \,.$$

Substituting this value into the equation of the motion, we get:

$$J\ddot{\alpha}_1 + \frac{J_p G}{l_0} \cdot \alpha_1 + \frac{J_p E R^2}{2 l_0 \cdot (l_1 + l_2)} \cdot \alpha_1^3 = 0 \,.$$

We can see that taking into account the change of the length due to the torsion the equation describing the motion of the system is a nonlinear differential equation (Duffing-type).

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