THE GENERALIZED D'ALEMBERT—LAGRANGE EQUATION

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Abstract

A generalised form of the D'Alembert—Lagrange equation is presented, which enables us to derive all kinds of equations of motion on basis of the same principles.

The D'Alembert—Lagrange equation is usually considered to be the most general equation of mechanics [1].

For a system of n particles this equation has the form

$$\sum_{i=1}^{n} (\mathbf{F}_i - m_i a_i) \cdot \delta r_i = 0 \tag{1}$$

where $\mathbf{F}_i$ is the sum of active forces applied to the i-th particle, m_i is its mass, a_i is its acceleration and δr_i is its virtual displacement.

The D'Alembert—Lagrange equation can be derived from the first and second Newton axioms, from the principle of superposition assuming that the constrains can be given as a result of some constrain force system. For an unknown constrain force system we have to assume that the sum of its virtual work is zero, that is, we have ideal constrains. The last assumption restricts the applicability of equation (1), because for a nonideal constrain system the virtual work of which does not satisfy the condition above, there is not any method to take into consideration in equation (1).

Let us assume that the constrains are nonideal constrains that is the sum of virtual works of the reaction forces is not zero.

Let us use Newtons first and second axioms, the principle of superposition and assume that the constrain can be considered as the action of some reaction forces.

The forces applied to the i-th particle can be divided into two groups, the group of known forces and the one of unknown forces. The sum of both groups of forces is denoted by $\mathbf{F}_i$ and $\mathbf{K}_i$, respectively. As in (1) we can obtain

$$\sum_{i=1}^{n} C_i \otimes (\mathbf{F}_i + \mathbf{K}_i - m_i a_i) = 0 \tag{2}$$

generalised D'Alembert—Lagrange equation, where C_i is some generability operator, $\otimes$ is some kind of multiplication. Equation (2) must be satisfied for any C_i, thus the properties of the unknown forces $\mathbf{K}_i$ can be taken into consideration in finding the appropriate C_i and [2].

For example if

$$\sum_{i=1}^{n} K_i \cdot \delta r_i = 0,$$

then, using C_i as δr_i virtual displacement and $\otimes$ as scalar multiplication $(\cdot)$, (2) implies (1). Another example can be given if all $\mathbf{K}_i$ are internal forces satisfying Newton's third axiom, that is, $\sum_{i=1}^{n} \mathbf{K}_i=0$, than with $C_i \doteq 1$, using $\otimes$ as multiplication

$$\sum_{i=1}^{n} \mathbf{F}_i - \sum_{i=1}^{n} m_i a_i = 0$$

is obtained, which is the well-known theorem of moment.

Let us further assume that $\mathbf{B}_{ij}$ and $\mathbf{B}_{ji}$ internal forces have the same line of action.

Let us denote by $\mathbf{r}_i$ the vector from some fixed point P to the i-th particle m_i. The assumption above means that

$$(\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{B}_{ij} = 0$$

but

$$\mathbf{K}_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \mathbf{B}_{ij}$$

thus

$$\sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{K}_i = \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{B}_{ij} = 0.$$

Now if in (2) $\mathbf{C}_i$ is substituted as $\mathbf{r}_i$, and the multiplication $\otimes$ is the cross product of vectors

$$\sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i - \sum_{i=1}^{n} \mathbf{r}_i \times m_i a_i = 0,$$

that is, the theory of kinetical moment to point P is obtained while $\sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i$ is the sum of the moments of forces on point P and $\sum_{i=1}^{n} \mathbf{r}_i \times m_i a_i$ is the kinetical moment of the system of material points.

Similarly, knowing some further properties of forces $\mathbf{K}_i$ other equations of mechanics can be obtained. These are well-known equations, as for example the equation of second order Lagrange or the equation of Appell [3]. In case of the second one $\mathbf{K}_i$ satisfies $\sum_{i=1}^{n} \mathbf{K}_i \cdot \delta a_i = 0$, where δa_i is the virtual acceleration of the i-th particle. Thus if $C_i \doteqdot \delta a_i$ is substituted into (2), and $\otimes$ is considered as a scalar product, we obtain Appell's equation. The Appell equation cannot be derived from (1). Usually it is introduced as a conclusion of Gauss's principle of least constrain.

From (2), the second order Lagrange equation and also Appell's equation can be derived without introduction of any other principle.

References

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