KINEMATIC ANALYSIS OF A CHEBYSHEV SET OF APPROXIMATE CIRCLE-TRACING MECHANISMS

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Abstract

A set of four approximate circle-tracing mechanisms, discovered by Chebyshev, are analyzed and the possibility to receive some information on the accuracy of the coupler point trajectories of this set is examined. Furthermore, the paper provides kinematic algorithms to investigate angular accelerations and jerks of the followers and of the couplers of these four-bar linkages. The geometrical and kinematic results might be useful if the illustrated devices are used as robotic arms. Finally, the complete series of the coupler points trajectories and the kinematical characteristics are plotted and discussed.

Introduction

The theory of generating curves of points on the links of plane mechanisms has had notable practical application in the work of many scientists and engineers since the construction of machines requires that these points be driven along prescribed trajectories.

I. I. Artobolevsky, who examined many mechanical devices created by kinematicians during the second half of the last century and during this century, listed and classified an extensive quantity of them in his well known treatise "Mechanisms in Modern Engineering Design" [1]. A great number of devices are characterized by the specific aim of generating approximate curves, but the eminent Author gave the geometrical data without analyzing link displacements and kinematic properties. Artobolevsky's Handbook, thus, provides designers simply with the proportions of size of mechanism links.

It should be noted that in modern literature on the kinematic analysis of extant plane mechanisms, utilized to generate curves, an excellent article, based on Tesar's doctoral dissertation and published by D. Tesar and J. P. Vidosic [2] provides a detailed study of some four-bar approximate straight-line mechanisms. In this paper

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the Authors analyzed some diagrams and gave the length of the approximate straight-line output thus enabling designers to determine the parameters defining linkage with the suitable output. Furthermore, a previous kinematic analysis on a P. L. Chebyshev four-bar approximate straight-line mechanism indicated that it is feasible to draw a quasi-circular arc too, [3]. On this basis, it has seemed appropriate to investigate a set of four-bar approximate circle-tracing mechanisms conceived by Chebyshev, whose coupler curves may be suitably applied to robotic operations. In fact, for obvious reasons of construction simplicity, a coupler belonging to a closed-chain mechanism with a point capable of describing an approximate circular arc might serve advantageously as a special robot arm, [4]. As there are no pertinent considerations on the relative deviation of this set of Chebyshev four-bar linkages from the exact circle-tracing mechanisms, this report provides information on the accuracy of coupler trajectory. Moreover, it examines the possibility of approximating these quasi-circular arcs to those of the respective coupler osculating circles. Additionally, the couplers and the output link angular accelerations, the angular jerks, the accelerations of the points describing the quasi-circular arcs are plotted with respect to the input angle so as to allow the designer an ample choice among the Chebyshev set of four-bar linkages.

Kinematic algorithms

Fig. 1 shows a generic four-bar linkage in which a is the frame, b and d, respectively, the input link and the output link, c the connecting rod.

If $0xy$ is a system of fixed axes, α the input angle (positive counter-clockwise),

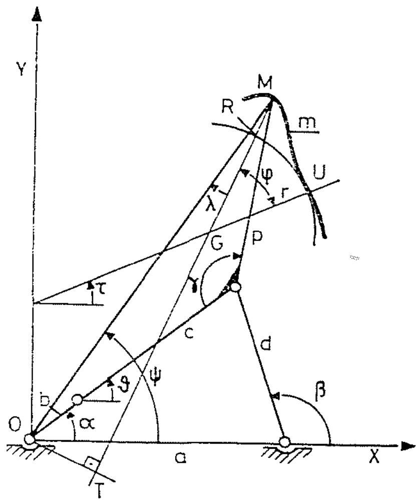

Fig. 1. Generical four-bar linkage in which a point M of the connecting rod c describes a quasi-circular arc m

β the output angle, ϑ the angle between the generical position of c and the x axis, γ the angle between c and p, the coordinates of a coupler point M are

$$\begin{aligned} x &= b\cos\alpha + c\cos\vartheta - p\cos(\gamma-\vartheta), \\ y &= b\sin\alpha + c\sin\vartheta + p\sin(\gamma-\vartheta). \end{aligned} \tag{1}$$

Assumpting that the configuration of the linkage lies in the first quadrant, the expressions of β and ϑ as functions of α are derived as follows:

$$\begin{aligned} \beta &= 2\tan^{-1}\left[(\sin\alpha - (\sin^2\alpha + B^2 + C^2)^{1/2})/(B+C)\right], \\ \vartheta &= 2\tan^{-1}\left[(\sin\alpha + (\sin^2\alpha + B^2 + D^2)^{1/2})/(B+D)\right], \end{aligned} \tag{2}$$

where

$$B = \cos\alpha - a/b,$$

$$C = (a^2+b^2-c^2-d^2)/(2bd) - (a/d)\cos\alpha,$$

$$D = (a/c)\cos\alpha - (a^2+b^2+c^2-d^2)/(2bc).$$

Eqs. (2) are expressed by means of the trigonometrical tangent since they present more accuracy in the neighborhood of $\alpha=0^0$ and there is no vagueness in the algebraic sign of β and ϑ, [5].

Referring to Fig. 1, if M describes segments approximating circular arcs and the point U belongs to one of these arcs, the accuracy of the quasi-circular arc with respect to that of a circumference, may be investigated by means of geometric ratio

$$\varepsilon = \overline{MR}/r, \tag{3}$$

where $r=\overline{GR}$ is the radius of the circle having G as center and

$$\overline{MR} = \pm\overline{OM}\cos\lambda \pm \overline{GT} - r.$$

In this expression $\overline{GT}$ is calculated as

$$\overline{GT} = (\overline{OG}^2 - \overline{OM}^2\sin^2\lambda)^{1/2}$$

and λ, the acute angle between the directions of $\overline{GM}$ and $\overline{OM}$, is a function of angle φ that is defined negative for the points of the coupler curve which are on the left of direction $\overline{GU}$. Furthermore, if τ is the angle that the line GU forms with the horizontal, the sign of ($\overline{OM}\cos\lambda$) is positive if $\tau\leqq\pi$ and negative if $\tau>\pi$; the sign of $\overline{GT}$ is positive if the angle $OGM<\pi/2$ and negative if $\geqq\pi/2$.

In regard to kinematical properties, the equations of the angular velocities and the angular accelerations of the follower and of the coupler are, respectively,

$$\begin{aligned} \dot\beta &= (\dot\alpha b/d)\sin(\alpha-\vartheta)/\sin(\beta-\vartheta), \\ \dot\vartheta &= (\dot\alpha b/c)\sin(\alpha-\beta)/\sin(\beta-\vartheta), \end{aligned} \tag{4}$$

and

$$\ddot{\beta} = [b\ddot{\alpha}\sin(\alpha-\vartheta)+b\dot{\alpha}^2\cos(\alpha-\vartheta)+c\dot{\vartheta}^2-d\dot{\beta}^2\cos(\beta-\vartheta)]/d\sin(\beta-\vartheta),$$
$$\ddot{\vartheta} = [b\ddot{\alpha}\sin(\alpha-\beta)+b\dot{\alpha}^2\cos(\alpha-\beta)-d\dot{\beta}^2+c\dot{\vartheta}^2\cos(\beta-\vartheta)]/c\sin(\beta-\vartheta). \tag{5}$$

Eqs. (4) and (5) are deduced differentiating the displacement equations of the extreme point of the follower d.

Consequently, the corresponding angular jerks are given by

$$\dddot{\beta} = [(E+F+H)\cos\vartheta-(J+K+L)\sin\vartheta]/d\sin(\beta-\vartheta),$$
$$\dddot{\vartheta} = [(E+F+H)\cos\beta-(J+K+L)\sin\beta]/c\sin(\beta-\vartheta), \tag{6}$$

in which

$$E = b(\dddot{\alpha}\sin\alpha+3\dot{\alpha}\ddot{\alpha}\cos\alpha-\dot{\alpha}^3\sin\alpha),$$

$$F = c(3\dot{\vartheta}\ddot{\vartheta}\cos\vartheta-\dot{\vartheta}^3\sin\vartheta),$$

$$H = d(-3\dot{\beta}\ddot{\beta}\cos\beta+\dot{\beta}^3\sin\beta),$$

$$J = -b(-\dddot{\alpha}\cos\alpha+3\dot{\alpha}\ddot{\alpha}\sin\alpha+\dot{\alpha}^3\cos\alpha),$$

$$K = -c(3\dot{\vartheta}\ddot{\vartheta}\sin\vartheta+\dot{\vartheta}^3\cos\vartheta).$$

$$L = d(3\dot{\beta}\ddot{\beta}\sin\beta+\dot{\beta}^3\cos\beta)$$

The kinematical characteristics of velocity V and of acceleration A of the coupler point M take the form

$$V = (\dot{x}^2+\dot{y}^2)^{1/2} \tag{7}$$

and

$$A = (\ddot{x}^2+\ddot{y}^2)^{1/2} \tag{8}$$

where the components $\dot{x}$, $\dot{y}$, $\ddot{x}$ and $\ddot{y}$ are expressed as

$$\dot{x} = -b\dot{\alpha}\left[\sin\alpha+\left(\sin\vartheta+p/c\sin(\gamma-\vartheta)\right)\sin(\alpha-\beta)/\sin(\beta-\vartheta)\right],$$

$$\dot{y} = +b\dot{\alpha}\left[\cos\alpha+\left(\cos\vartheta-p/c\cos(\gamma-\vartheta)\right)\sin(\alpha-\beta)/\sin(\beta-\vartheta)\right],$$

$$\ddot{x} = -b(\ddot{\alpha}\sin\alpha+\dot{\alpha}^2\cos\alpha)-c(\ddot{\vartheta}\sin\vartheta+\dot{\vartheta}^2\cos\vartheta)-p\left(\ddot{\vartheta}\sin(\gamma-\vartheta)-\dot{\vartheta}^2\cos(\gamma-\vartheta)\right),$$

$$\ddot{y} = b(\ddot{\alpha}\cos\alpha-\dot{\alpha}^2\sin\alpha)+c(\ddot{\vartheta}\cos\vartheta-\dot{\vartheta}^2\sin\vartheta)-p\left(\ddot{\vartheta}\cos(\gamma-\vartheta)+\dot{\vartheta}^2\sin(\gamma-\vartheta)\right).$$

The above geometrical and kinematic algorithms are used to analyse a set of Chebyshev four-bar approximate circle-tracing linkages.

Geometrical characteristics

Table 1 gives the geometrical characteristics of a four-bar linkages group that Chebyshev called approximate circle-tracing mechanisms, [1].

In order to verify the accuracy of the arcs described by mechanisms 1, 2, 3, 4, the following procedure is adopted:

— The inflection circles of the mechanism couplers are traced when the input cranks fold over the frame as ($\alpha=0°$) or are extended in a straight line with the frame as ($\alpha=180°$). In these cases the inflection circles result in a tangent to the geometrical axes of the followers of the four-bar linkages.

— The center G of the arc curvature on which point U lies, is determined.

— The osculation circles with a radius $r=\overline{GU}$ and three infinitely close points in common with the quasi-circular arcs at U, are consequently drawn.

Furthermore, taking into account the geometrical characteristics of the approximated arcs shown in Table 2, where the columns give the crank angle for which

Table 1

Geometrical characteristics of the set of the Chebyshev fourbar approximate circle-tracing mechanisms and type

Linkage	Artobolevsky classification	a	b	$c=d=p$	γ (degrees)	Type
1	679	2.94	1	3.12	240	crank-rocker
2	683	2.5	1	3	180	crank-rocker
3	681	1.36	1	1.55	110	crank-rocker
4	682	0.5	1	1.27	123	duble-crank

Table 2

Geometrical characteristics of the circles whose arcs are approximated to coupler curves segments

Linkage	α Angle (degrees)	$X(G)$ (b unit)	$Y(G)$ (b unit)	r (b unit)	τ (degrees)	Range α OF Investigation (degrees)
1	0	4.018	1.867	2.010	60	320—40
2	0	2.500	0.830	4.979	90	330—30
	180	2.500	21.709	16.835	270	70—290
3	0	1.507	−0.210	2.984	125	300—60
	180	2.582	−1.745	5.131	125	120—240
4	0	0.463	0.068	2.505	298 30′	0—360
	180	0.562	−0.115	2.648	118 30′	0—360

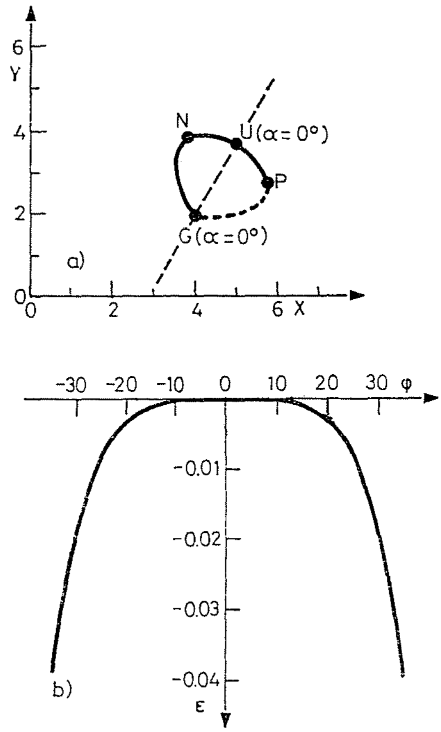

Fig. 2a), b). Linkage n. 1: a) the coupler curve described by M shows the quasi-circular arc NP and its respective center G; U is the osculating point of contact; b) geometric ratio ε versus the anomaly angle φ

the osculation circles are traced, coordinates X and Y of G, radius r, angle τ between r and the horizontal and/or the range of investigation of α, some observations can be derived as follows:

I — Four-bar linkage n. 1 (Artobolevsky classification 679): Fig. 2a shows the coupler curve described by point M and plotted by means of Eqs. (1). Because of the adopted geometrical configuration of the mechanism, point U lies in the middle of the approximate arc NP which is internally tangent to the osculation circle at $U(\alpha=0°)$; $\varepsilon \geqq -0.025$ (Fig. 2b) in this event, for $-30° \leqq \varphi \leqq 30°$ results.

II — Four-bar linkage n. 2 (Artobolevsky classification 683): point M traces the curve depicted in Fig. 3a; the coupler approximate arc NP is internally tangent to the osculation circle at $U\,(\alpha=0°)$ and for $-15° \leqq \varphi \leqq 15°$ it is visible that $\varepsilon \geqq -$ -0.0012, (Fig. 3b). The coupler approximate arc QS is externally tangent to the osculation circle at $U\,(\alpha=180°)$ and $\varepsilon \leqq 0.0006$ for $-4° \leqq \varphi \leqq 4°$ (Fig. 3c). In adopting this procedure, point U is the middle point of the approximate arcs NP and QS, respectively.

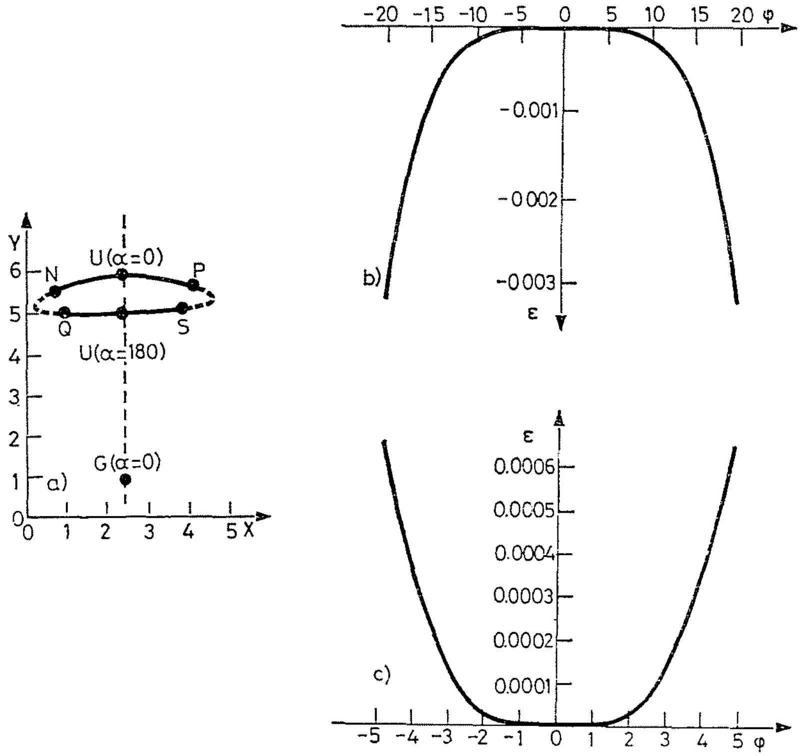

Fig. 3a), b), c). Linkage n. 2: a) the coupler curve described by M shows the quasi-circular arcs NP and QS and the NP arc center $G(\alpha=0)$; $U(\alpha=0)$ and $U(\alpha=180)$ are the osculating points of contact; b) geometric ratio ε versus the anomaly angle φ for the NP arc; c) geometric ratio ε versus the anomaly angle φ for the QS arc

III — Four-bar linkage n. 3 (Artobolevsky classification 681): in Fig. 4a, point M describes the approximate arcs NP and QS. The arc NP is externally tangent to the osculation circle at U ($\alpha=180°$) and $\varepsilon \leqq 0.01$ for $-40° \leqq \varphi \leqq 40°$, (Fig. 4b). The arc QS is externally tangent to the osculation circle at U ($\alpha=0°$) and $\varepsilon \geqq -0.008$ for $-15° \leqq \varphi \leqq 15°$ (Fig. 4c). In each of the above cases point U is the middle point of the approximate arcs.

IV — Four-bar linkage n. 4 (Artobolevsky classification 682): in this particular case (Fig. 5a), point M is considered to trace an approximate circle that is externally tangent to the osculation circle at U ($\alpha=0°$) and $\varepsilon < 0.005$ for $-90° \leqq \varphi \leqq 90°$, (Fig. 5b) in addition the maximum value of ε is -0.025. The approximate circle may also be regarded as internally tangent to the osculation circle at U ($\alpha=180°$) and $\varepsilon \geqq -0.14$ for $-180° \leqq \varphi \leqq 180°$ (Fig. 5c).

Therefore, coupler curves, whose approximate arcs are symmetrical about an axis, may be analyzed by means of the geometrical ratio. The introduction of ε as

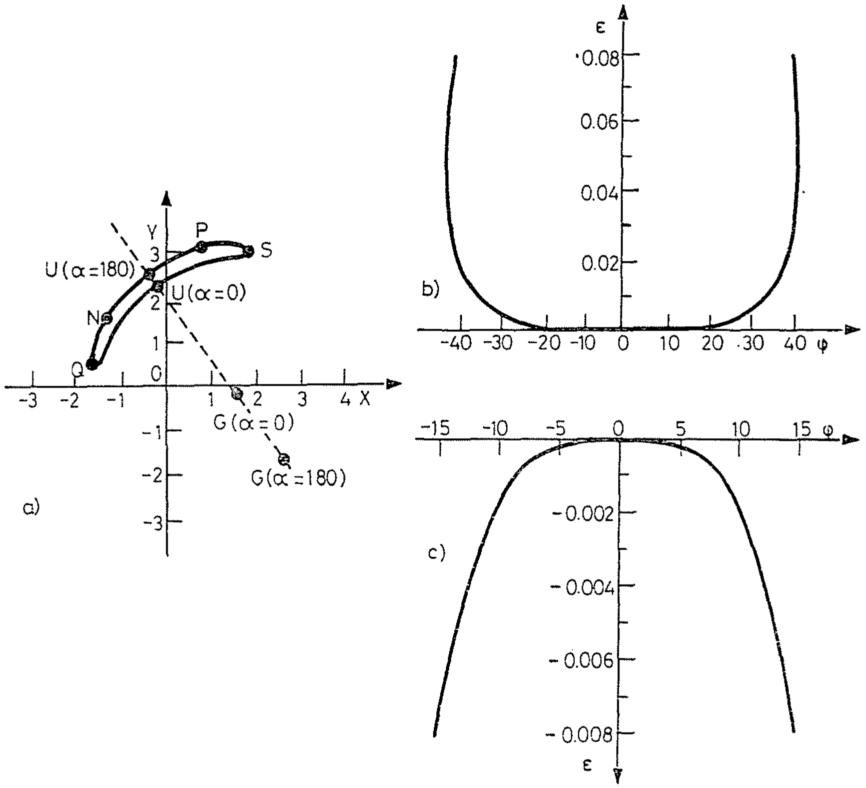

Fig. 4a), b), c). Linkage n. 3: a) the coupler curve described by M shows the quasi-circular arcs NP and QS and their respective centers $G(\alpha=0)$ and $G(\alpha=180)$; $U(\alpha=0)$ and $U(\alpha=180)$ are the osculating points of contact; b) geometric ratio ε versus the anomaly angle φ for the NP arc; c) geometric ratio ε versus the anomaly angle φ for the QS arc

a function of φ may be an appropriate investigation concerning the symmetrical properties of arcs with line GU as the axis of symmetry and the four above-mentioned examples prove the soundness of the method.

Computational results

The kinematic investigation of Chebyshev's four-bar approximate circle-tracing mechanisms is a suitable reference when the dynamical effects of the links of these four mechanisms have to be considered.

Therefore, by means of the first equation of (5) and the first equation of (6), the angular acceleration $\ddot{\beta}$ and the jerk $\dddot{\beta}$ of the follower of the four-bar linkage n. 1 are plotted with respect to (Fig. 6a). For a crank angle $\alpha=320°$, point M starts

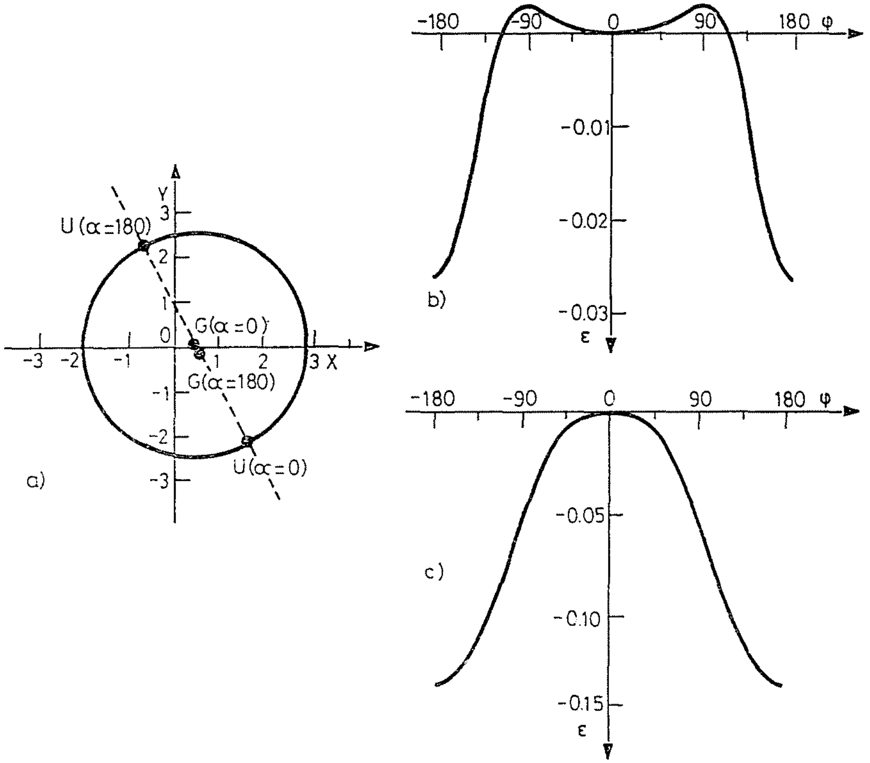

Fig. 5a), b), c). Linkage n. 4: a) the coupler curve described by M shows the quasi-exact circumference and the quasi-coincident centers $G(\alpha=0)$ and $G(\alpha=180)$ of the two arcs which give the entire curve; b) geometric ratio ε versus the anomaly angle φ for the arc with center $G(\alpha=0)$; c) geometric ratio ε versus the anomaly angle φ for the arc with center $G(\alpha=180)$

describing the quasi-circular arc from point P to point N ($\alpha=40°$) (Fig. 2a). In such a case values $\dddot{\beta}$ are higher than those corresponding to the other two arcs. The same consideration applies to the angular jerk $\dddot{\vartheta}$ of the coupler, is obtained from the second equation of (6), (Fig. 6b). In addition, (Fig. 6c) shows velocity and acceleration diagrams of point M, obtained, from Eqs. (7) and (8) resp. Because of jerk variations $\dddot{\beta}$ and $\dddot{\vartheta}$ this mechanism seems to be indoneus for fast motions.

The four-bar linkage n. 2 has the following characteristics: for a crank angle $\alpha=330°$, point M starts describing the quasi-circular arc from point P to point N ($\alpha=40°$) (Fig. 3a). Along arc PN, the values of $\dddot{\beta}$ (Fig. 7a), and of $\dddot{\vartheta}$, (Fig. 7b) are considerable with respect to those that relate to arc QS. (Fig. 7c) shows the quasi constant velocity and acceleration of M when it describes arc QS. This mechanism can be considered to have similar dynamic characteristics as the previous n. 1.

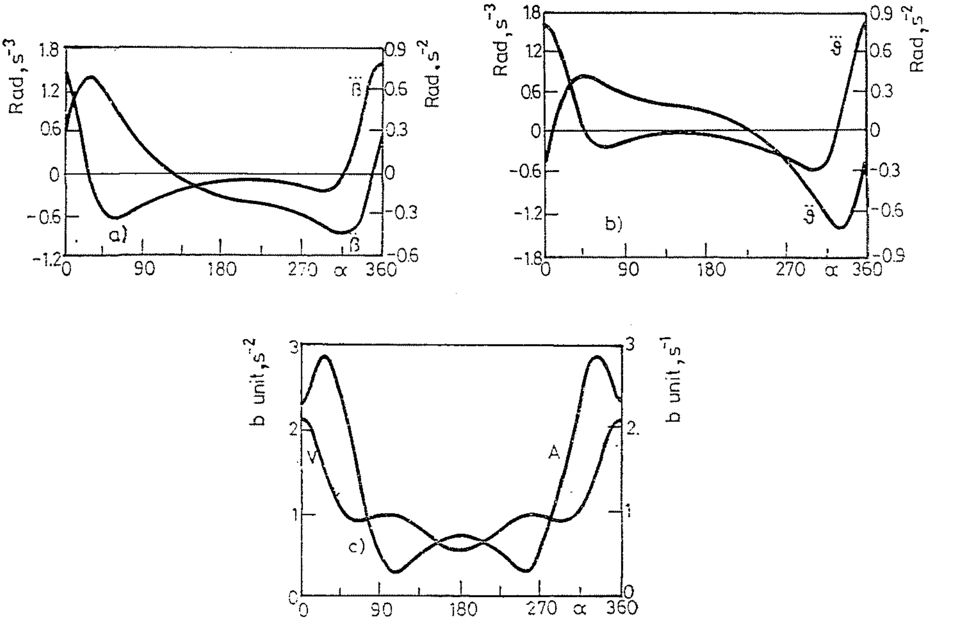

Fig. 6a), b), c). Linkage n. 1: a) angular acceleration $\ddot{\beta}$ and jerk $\dddot{\beta}$ of the follower as a function of the input angle α; b) angular acceleration $\ddot{\vartheta}$ and jerk $\dddot{\vartheta}$ of the coupler as a function of the input angle α; c) velocity V and acceleration A of point M as a function of the input angle α

In the four-bar linkage n. 3 the point M traces the two quasi-circular arcs PN and QS (Fig. 4a). For a crank angle $\alpha=300°$, M begins to describe the quasi-circular arc from P and it reaches N when $\alpha=60°$. The quasi-circular segment between Q and S is traced when the crank angle α takes, values of 120° and 240° resp. Values of $\dddot{\beta}$, $\dddot{\vartheta}$ and of the acceleration A, (Figs. 8a, 8b, 8c) are suitable to obtain constant kinematic properties during a long period of the crank revolution; nevertheless, the sudden increment of these kinematic characteristics, for some values of α, suggests the use of the mechanism with a very low number of crank revolutions.

The four-bar linkage n. 4 has a point M that describes an approximate circle, which may be thought to trace two arcs whose centers G are selected for $\alpha=0°$ and $\alpha=180°$ (Fig. 5a). This double-crank mechanism gives a good kinematic answer, (Figs. 9a), 9b), 9c)) and seems suitable to trace long, quasi-circular arcs without limiting the number of input crank revolutions.

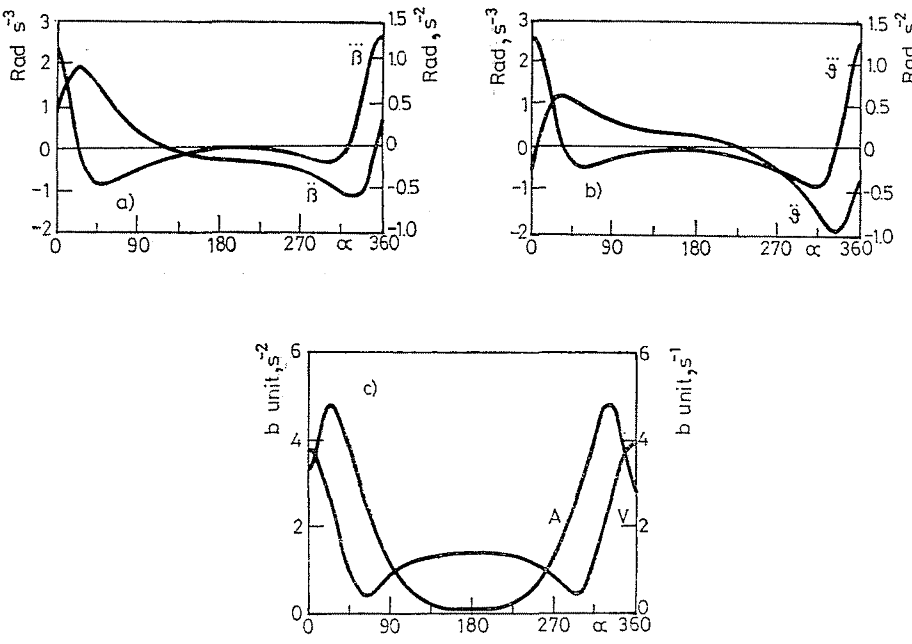

Fig. 7a), b), c). Linkage n. 2: a) angular acceleration $\ddot{\beta}$ and jerk $\dddot{\beta}$ of the follower as a function of the input angle α; b) angular acceleration $\ddot{\vartheta}$ and jerk $\dddot{\vartheta}$ of the coupler as a function of the input angle α; c) velocity V and acceleration A of point M as a function of the input angle α

Conclusion

The analysis of the illustrated set mechanisms focuses on the following:

a) the entire trajectories of the coupler point that had not been examined completely by Artobolevsky,

b) velocities and accelerations of point M describing the mentioned curves,

c) angular accelerations and jerks of the couplers and of the output links.

The investigation appears to be simplified if crank angle positions $\alpha = 0°$ and $\alpha = 180°$ are considered. In this case, in fact, the curvature centers of the approximate arcs can be obtained in a suitable manner; and probably the same methodology was used by Chebyshev in order to determine the coupler point path of the examined set of mechanisms.

Finally, this paper shows that each of the two coupler curves of four-bar linkages n. 683 and n. 681 describes two approximate arcs and not one, [1], and that the coupler curve of the four-bar linkage n. 682 gives an approximate circular trajectory.

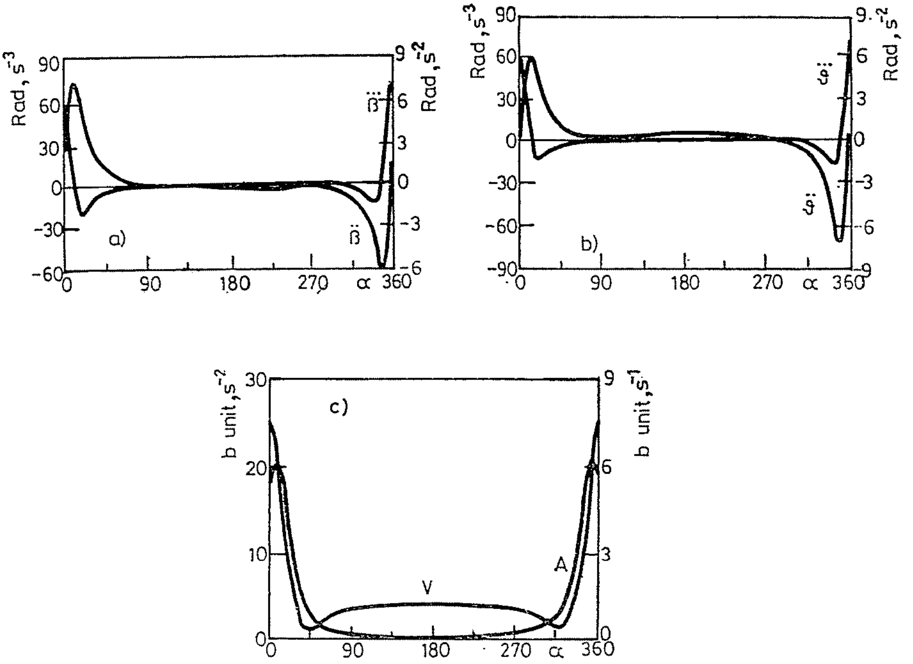

Fig. 8a), b), c). Linkage n. 3: a) angular acceleration $\ddot{\beta}$ and jerk $\dddot{\beta}$ of the follower as a function of the input angle α; b) angular acceleration $\ddot{\vartheta}$ and jerk $\dddot{\vartheta}$ of the coupler as a function of the input angle α; c) velocity V and acceleration A of point M as a function of the input angle α

On this subject, it may be observed that several devices discovered by Chebyshev have not been studied in detail or, perhaps, they have been forgotten; therefore, it would be desiderable that the development of modern robotics will urge kinematicians also to emphasize mechanisms of the past century.

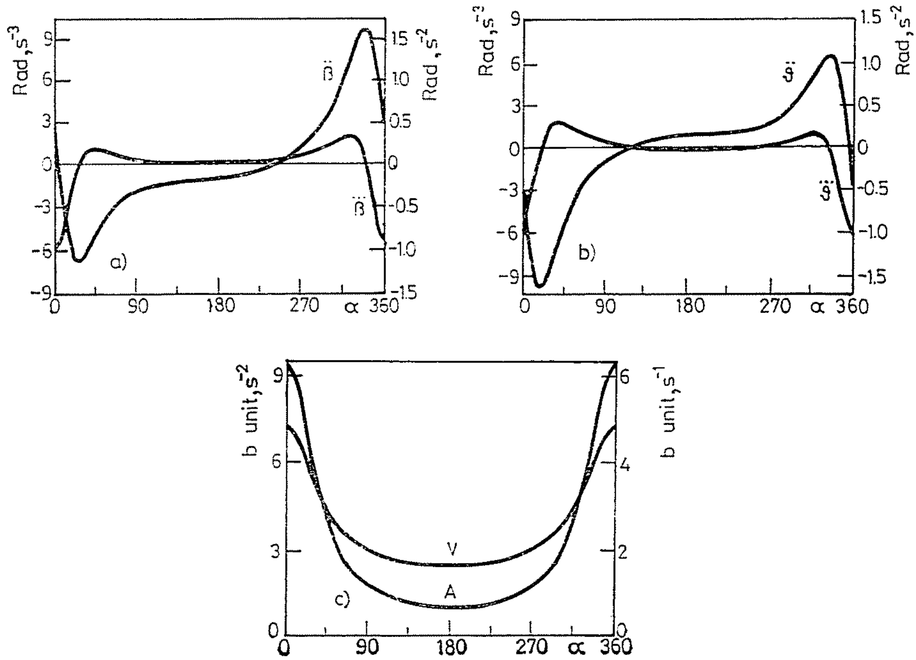

Fig. 9a), b), c). Linkage n. 4: a) angular acceleration $\ddot{\beta}$ and jerk $\dddot{\beta}$ of the output crank as a function of the input angle α; b) angular acceleration $\ddot{\vartheta}$ and jerk $\dddot{\vartheta}$ of the coupler as a function of the input angle α; c) velocity V and acceleration A of point M as a function of the input angle α

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