ON DIFFERENTIAL FORMS OF THE CONSTITUTIVE EQUATIONS FOR ELASTO-PLASTIC SOLIDS

GY. BÉDA and L. SZABÓ

Department of Technical Mechanics,
Technical University, H-1521 Budapest
Received February 14, 1989

Abstract

Actually, several theories are in use for describing elasto-plastic deformation. A wide variety of forms exist for constitutive equations expressing theories of plasticity. Here, a general form of constitutive equations will be given for a significant group of theories of small elasto-plastic deformations. It is pointed out that this incremental constitutive equation arises as a special case of the theory developed by Béda [1] for the general description of plastic behaviour.

1. Introduction

Several theories of plasticity have been published to describe small elasto-plastic deformations. Equations of the suggested theories of plasticity are mostly of an incremental form. But these equations may, structurally, be of rather different forms and comprise several different scalar or tensor type functions as material characteristics.

An accurate description of the plastic behaviour of materials requires to disclose fields of applicability of the existing theories of plasticity. Part of these examinations is a theoretical and numerical comparative analysis of various theories of plasticity. Comparison of theories of plasticity is significantly eased by writing different constitutive equations in the same form.

First, such a recapitulative, incremental constitutive equation will be detailed, followed by the analysis of the relation between the constitutive equation of general form or system of criteria — as suggested by Béda [1, 2] to describe plastic behaviour, — and the recapitulative, incremental material equation.

2. Constitutive equation of some theories of plasticity

Incremental constitutive equations for theories suggested to describe small elasto-plastic deformations normally comprise terms:

$d\sigma_{ij}$ — stress tensor increment;
$d\varepsilon_{ij}$ — strain tensor increment;
σ_{ij} — stress tensor;
ε_{ij} — strain tensor;
$\left.\begin{matrix}\eta_{ij}\\ \xi\end{matrix}\right\}$ — plastic internal variables.

In case of the Prandtl—Reuss theory, these terms figure in the well-known equation

$$de_{ij} = \frac{1}{2G} ds_{ij} + d\lambda s_{ij} \tag{1a}$$

$$d\sigma_{ii} = K d\varepsilon_{ii} \tag{1b}$$

where de_{ij} is the increment of the strain deviator, s_{ij} is stress's deviator, $d\lambda$ is a plastic parameter to be determined with the aid of the loading — unloading criterion, G — is the shear modulus, K — is the bulk modulus.

Concerning the Prandtl—Reuss equation, an incremental equation:

$$d\sigma_{ij} = \left(D_{ijkl} - \frac{9G^2}{\bar{\sigma}^2(3G+H')} s_{ij} s_{kl}\right) d\varepsilon_{kl} \tag{2}$$

derived from (1) is known, where D_{ijkl} is the elasticity tensor conform to Hooke's law, $\bar{\sigma}$ — Mises's effective stress, H' — is the slope of the stress — plastic strain curve. Another theory of plasticity, also regarded as classic, Hencky—Nadai's theory of deformation [3], involves the constitutive equation of incremental form

$$d\sigma_{ij} = \frac{2G}{1+3G/H_s}\left[d\varepsilon_{ij} + \frac{\nu+(1+\nu)\dfrac{G}{H_s}}{1-2\nu}\delta_{ij} d\varepsilon_{kk} - \right.$$

$$\left. - \frac{9G^2\left(1-\dfrac{H'}{H_s}\right)}{\bar{\sigma}^2(3G+H')} s_{ij} s_{kl} d\varepsilon_{kl}\right] \tag{3}$$

including a single material characteristic parameter H_s in excess to (2), secant modulus of the uniaxial stress/plastic. Equations (2) and (3) may be given in a common, recapitulative form with distinct parameters H' and H_s causing the deviation. This is of the form:

$$d\sigma_{ij} = \left[\frac{2G}{1+b} T_{ijk} + KL_{ijk} - 2G\left(\frac{a}{1+a} - \frac{b}{1+b}\right) n_{ij} n_{kl}\right] d\varepsilon_{kl} \tag{4a}$$

or

$$d\sigma_{ij} = D^{ep}_{ijkl} d\varepsilon_{kl} \tag{4b}$$

where

$$T_{ijkl} = I_{ijkl} - \frac{1}{3} L_{ijkl}$$

$$I_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$L_{ijk} = \delta_{ij}\delta_{kl}$$

$$n_{ij} = s_{ij}/(s_{kl}s_{kl})^{1/2}$$

δ_{ij} — is the Kronecker delta.

Parameters a and b in Eq. (4) are:

for the Prandtl—Reuss theory: $a = 3G/H'$, $b = 0$

for the Hencky—Nadai theory: $a = 3G/H'$, $b = 3G/H_s$.

Besides summarizingly comprising constitutive equations for the Prandtl—Reuss and the Hencky—Nadai theories, Eq. (4) comprises several other theories of plasticity via parameters a and b. In the following, further such theories will be described. Several theories of plasticity include a condition of plasticity, and to this, a kind of yield surface. Now, in case of hardening materials, in plastic deformation, the yield surface is characterized by various changes, such as the rise of peaks, corners. Theories permitting to describe phenomena of this kind are termed corner theories. Subsequently, interpretation of parameters a and b will be given for three such parameters. The first is the Christoffersen—Hutchinson theory [4], in fact, a modification of the Hencky—Nadai theory, where parameters a and b are:

$$a = \frac{3G}{H'} F_1(\dot{\varepsilon}^p, \sigma, \Theta_c) \tag{5a}$$

$$b = \frac{3G}{H_s} F_2(\dot{\varepsilon}^p, \sigma, \Theta_c) \tag{5b}$$

where functions F_1 and F_2 depend on the vertex angle Θ_c of stress, on the rate of plastic deformation, and on the yield surface.

Modification of the Prandtl—Reuss theory brings about another corner equation suggested by Hughes—Shakib [5], defining a modulus of plasticity $h(\alpha)$ depending on the load direction (direction of the strain increment). Now, parameters a and b become:

$$a = \frac{3G}{h'(\alpha)} \quad \text{and} \quad b = 0. \tag{6}$$

The third corner theory is due to Gotoh [6] combining the theories of improvement

Prandtl—Reuss's and Hencky—Nadai's. In this case, parameters a and b are

$$a = \frac{3G}{H'} M \left(\frac{1-c_1}{\cos \alpha} + c_1\right) \tag{7a}$$

$$b = \frac{3G}{H_s} M \tag{7b}$$

where

$$M = c_1 + (1-c_1) \cos \beta$$

$$c_1 = \cos \beta_0/(1+\cos \beta_0)$$

where β_0 is the vertex angle of the yield surface varying in plastic deformation, and b is the angle included between vectors of stress increment, and the instantaneous stress deviator. Finally, parameters a and b will also be defined for Iliushin's geometrical theory [7], that involves no yield surface, neither division of the toral deformation to plastic and elastic parts. Irrespective of that, the constitutive equation for the theory can be brought to form (4). Expressions for a and b in Iljushin's theory are:

$$a = \frac{2G}{N} - 1, \quad b = \frac{2G}{P} - 1 \tag{8}$$

where P and N are the so-called Iljushin's functions.

Constitutive equation (4) refers to several other time-independent theories (Valanis' endochronic, Prager's Dafalias—Popov's), detailed in [8].

For most of time-independent theories (those by Perzyna, Bodner—Partom, and some creep theories) the constitutive equations can be given in a common, comprehensive equation

$$\mathrm{d}\sigma_{ij} = \boldsymbol{D}^{(ep)}_{ijkl}\, \mathrm{d}\varepsilon_{kl} + \mathrm{d}R_{ij} \tag{9}$$

where tensor $\boldsymbol{D}^{(ep)}_{ijkl}$ is formally identical to (4) but the included parameters a and b are also time-dependent. Tensor $\mathrm{d}\boldsymbol{R}$ involves dependence on the strain rate. Equations (4) and (9) facilitate the theoretical and numerical comparison of the different theories. For instance, angles included between magnitudes $\mathrm{d}\boldsymbol{\sigma}$ and $\mathrm{d}\boldsymbol{\varepsilon}$ with vector $\mathbf{n}$ interpreted in conformity with the instantaneous stress state are related by

$$\cos \alpha = \frac{(1+a) \cos \beta}{[(1+b)^2 + (a-b)(a+b+2) \cos^2 \beta]^{1/2}} \tag{10}$$

where

$$\cos \alpha = \frac{n_{ij}\, \mathrm{d}\varepsilon_{ij}}{(\mathrm{d}\varepsilon_{kl}\, \mathrm{d}\varepsilon_{kl})^{1/2}}$$

$$\cos \beta = \frac{n_{ij}\, \mathrm{d}\sigma_{ij}}{(\mathrm{d}\sigma_{ij}\, \mathrm{d}\sigma_{ij})^{1/2}}.$$

3. Derivation of the incremental constitutive equation based on [2]

Theories above arose from different considerations, Equations (4) and (9) point, however, to the possibility to recapitulate constitutive equations for these theories in an identical incremental form. Next it will be shown that the method developed for generally describing plastic behaviour [1] and the group of constitutive equations to be derived from it [9] comprise incremental equations (4) and (9). The possibility to derive incremental equations (4) and (9) from the theory developed for generally describing plastic behaviour and the related constitutive equations [9] permits to generalize them at different levels. In conformity with conditions in [10]:

a) an acceleration wave can be induced in the continuum,
b) the acceleration wave propagates at finite velocity, and
c) there exist independent wave from families of at least a number identical with the number of independent variables,

general form of the constitutive equation is:

$$F_\alpha(\sigma_{\vartheta\hat{p}}, \varepsilon_{r\hat{q}}, \sigma_\lambda, \varepsilon_\varkappa, x_i) = 0$$

where $\alpha, \ldots, \varkappa = 1, 2, \ldots, 6$; $p, q, \ldots, i = 1, 2, 3, 4$; σ_λ and $\varepsilon_\varkappa$ are coordinates of the stress and strain, x_1, x_2, x_3 space coordinate and $x_4 = t$ the time coordinate

$$\sigma_{\vartheta\hat{p}} = \frac{\partial \sigma_\vartheta}{\partial x_{\hat{p}}} \text{ etc.}$$

The differential of F_α:

$$dF_\alpha = \frac{\partial F_\alpha}{\partial \sigma_{\vartheta\hat{p}}} d\sigma_{\vartheta\hat{p}} + \frac{\partial F_\alpha}{\partial \varepsilon_{\gamma\hat{q}}} d\varepsilon_{\gamma\hat{q}} + \frac{\partial F_\alpha}{\partial \sigma_\lambda} d\sigma_\lambda + \frac{\partial F_\alpha}{\partial \varepsilon_\varkappa} d\varepsilon_\varkappa + \frac{\partial F_\alpha}{\partial x_i} dx_i = 0.$$

An acceleration wave may exist in the body thus F_α satisfies the condition

$$\frac{\partial F_\alpha}{\partial \sigma_{\vartheta\hat{p}}} - L_{\vartheta\gamma\hat{p}\hat{q}} \frac{\partial F_\alpha}{\partial \varepsilon_{\gamma\hat{q}}} = 0. \tag{11}$$

Thus dF_α

$$\frac{\partial F_\alpha}{\partial \varepsilon_{\gamma\hat{q}}} (L_{\vartheta\gamma\hat{p}\hat{q}} \, d\sigma_{\vartheta\hat{p}} + d\varepsilon_{\gamma\hat{q}}) + \frac{\partial F_\alpha}{\partial \sigma_\lambda} d\sigma_\lambda + \frac{\partial F_\alpha}{\partial \varepsilon_\varkappa} d\varepsilon_\varkappa + \frac{\partial F_\alpha}{\partial x_i} \partial x_i = 0.$$

The form of the first term shows that instead of $\sigma_{\vartheta\hat{p}}$ and $\varepsilon_{\gamma\hat{p}}$ only one $\eta_{\beta j}$ can be used as variable, thus the constitutive equation becomes:

$$F_\alpha(\eta_{\beta j}, \sigma_\lambda, \varepsilon_\varkappa, x_i) = 0. \tag{12}$$

The condition (1) in this case can be satisfied if

$$\frac{\partial \eta_{\beta j}}{\partial \sigma_{\vartheta\hat{p}}} - L_{\vartheta\gamma\hat{p}\hat{q}} \frac{\partial \eta_{\beta j}}{\partial \varepsilon_{\gamma\hat{q}}} = 0. \tag{13}$$

The differential of (12) is

$$\frac{\partial F_\alpha}{\partial \eta_{\beta j}}\left(\frac{\partial \eta_{\beta j}}{\partial \sigma_{\vartheta \hat{p}}}\mathrm{d}\sigma_{\sigma\hat{p}}+\frac{\partial \eta_{\beta j}}{\partial \varepsilon_{\gamma\hat{q}}}\mathrm{d}\varepsilon_{\gamma\hat{q}}\right)+\frac{\partial F_\alpha}{\partial \sigma_\lambda}\mathrm{d}\sigma_\lambda+\frac{\partial F_\alpha}{\partial \varepsilon_\varkappa}\mathrm{d}\varepsilon_\varkappa+\frac{\partial F_\alpha}{\partial x_i}\mathrm{d}x_i=0$$

that is

$$\mathrm{d}\eta_{\beta j}=\frac{\partial \eta_{\beta j}}{\partial \sigma_{\vartheta\hat{p}}}\mathrm{d}\sigma_{\vartheta\hat{p}}+\frac{\partial \eta_{\beta j}}{\partial \varepsilon_{\gamma\hat{q}}}\mathrm{d}\varepsilon_{\gamma\hat{q}}$$

which is with (13):

$$\mathrm{d}\eta_{\beta j}=\frac{\partial \eta_{\beta j}}{\partial \varepsilon_{\gamma\hat{q}}}(L_{\vartheta\gamma\hat{p}\hat{q}}\,\mathrm{d}\sigma_{\vartheta\hat{p}}+\mathrm{d}\varepsilon_{\gamma\hat{q}}). \tag{14}$$

Equation (14) can also be used in calculating.

Special cases:

Let

$$\eta_{\beta j}=L_{\vartheta\beta\hat{p}j}\,\sigma_{\vartheta\hat{p}}+\varepsilon_{\beta j},$$

then

$$L_{\vartheta\beta\hat{p}j}=L_{\vartheta\beta\hat{p}j}(\sigma_\lambda,\varepsilon_\varkappa,x_i)$$

after some further reduction

$$L_{\vartheta\beta\hat{p}j}=L_{\vartheta\beta}\,\delta_{\hat{p}j}$$

$\eta_{\beta j}$ multiplied by $\mathrm{d}x_j$ and after addition

$$\mathrm{d}\eta_\beta=L_{\vartheta\beta}\,\mathrm{d}\sigma_\vartheta+\mathrm{d}\varepsilon_\beta.$$

A) Let $\mathrm{d}\eta_\beta=0$, so the differential form of the possible constitutive equation is

$$L_{\vartheta\beta}\,\mathrm{d}\sigma_\vartheta+\mathrm{d}\varepsilon_\beta=0 \tag{I.}$$

or

$$\mathrm{d}\sigma_\vartheta=D^{(ep)}_{\vartheta\beta}\,\mathrm{d}\varepsilon_\beta \tag{15}$$

where

$$D^{(ep)}_{\vartheta\beta}=-L^{-1}_{\vartheta\beta} \tag{16}$$

B) Let $\mathrm{d}\eta_\beta=B_\beta\,\mathrm{d}t$, now the differential form is

$$L_{\vartheta\beta}\,\mathrm{d}\sigma_\vartheta+\mathrm{d}\varepsilon_\beta=B_\beta\,\mathrm{d}t \tag{II.}$$

or

$$\mathrm{d}\sigma_\vartheta=D^{ep}_{\vartheta B}\,\mathrm{d}\varepsilon_\beta+\mathrm{d}R_\vartheta \tag{17}$$

where

$$D^{ep}_{\vartheta\beta}=-L^{-1}_{\vartheta\beta} \tag{18}$$

$$\mathrm{d}R_\vartheta=\mathrm{d}t\,L^{-1}_{\vartheta\beta}\,B_\beta \tag{19}$$

Equations (15) and (17) permit to derive an incremental constitutive equation of a more general form to describe small elasto-plastic deformations.

4. Conclusion

Constitutive equations of small plastic deformations can be derived in incremental form from one equation. The recapitulated theories of plasticity are special cases of the theory based on the existence of a wave of acceleration.

Acknowledgement

This research was supported by the OTKA under Contract: OTKA 5—103. The authors gratefully acknowledges this support.

References

1. BÉDA, GY. (1982): A mozgó képlékeny test lehetséges anyagtörvényei, doktori értekezés, (in Hungarian).
2. BÉDA, GY. (1987): Possible Constitutive Equations of the Moving Plastic Body, Advances in Mechanics, Vol. *10*. No. 1, pp. 65—87.
3. YAMADA, Y.—HAUNG, Y.—NISHIGUCHI, I.: Deformation theory of Plasticity and its installation in the finite element analysis routin.
4. CHRISTOFFERSEN, J.—HUTCHINSON, J. W. (1979): A class of phenomenological corner theories of plasticity, J. Mech. Phys. Solids Vol. *27*. 465—487.
5. HUGHES, T. R. J.—SHAKIB, F. (1986): Pseudocorner theory: a simple enhaucement of J_2-flow theory for applications involving non-proportional loading, Eng. Comput. *3*, 116—120.
6. GOTOH, M. (1986): A class of plastic constitutive equations with vertex effect — I General theory, Int. J. Solids Struct. *21*, 1101—1116.
7. ILJUSIN, A. A. (1963): Plaszticsnoszty, M. Izd. vo AN SzSzSzR.
8. SZABÓ, L. (1988): A képlékenységtan elméletei és néhány általánosításuk (Candidate's Thesis), Budapest (in Hungarian).
9. BÉDA, GY. (1988): Differential forms of the possible constitutive equations, News Letter, Technical Univ. Budapest, Vol. *6*, No. 2. p. 3—5.
10. BÉDA, GY. (1988): Possible constitutive equations of a dynamically loaded continuum taking into acount small deformations, Periodica Polytechnica, Vol. *33*/1—2.

Dr. Gyula BÉDA }
László SZABÓ } H-1521, Budapest.