POSSIBLE CONSTITUVE EQUATIONS OF
A DYNAMICALLY LOADED CONTINUUM
TAKING INTO ACCOUNT SMALL DEFORMATIONS

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Abstract

Definition of the concept of the possible constitutive equation and determination of the possible constitutive equations of dynamic tension.

1. Introduction

To test a moving solid continuum, equations of motion

$$\varrho \dot{v}^p = \sigma^{pq}{}_{;q} + b^p \tag{1}$$

$$\sigma^{pq} = \sigma^{qp} \quad (p, q = 1, 2, 3) \tag{2}$$

shall be used above all. Unknown functions in these equations are mass density ϱ, acceleration field $\dot{v}^p$, and the stress field described by symmetric tensor σ^{pq}. The number of unknown functions is ten, and the number of equations to be written altogether is tree. The number of unknown functions is by six more than the number of equations even if density is considered constant for the solid continuum to a good approximation.

Should the state of deformation of the continuum be described by tensor ε^{pq}, then, for a small deformation, the rate of deformation $\dot{\varepsilon}_{pq}$, and the velocity gradient $v_{p;q}$ is interconnected by equations

$$2\dot{\varepsilon}_{pq} = v_{p;q} + v_{q;p} \tag{3}$$

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There are six additional unknowns in the equation written above so that there has been no progress in the mathematical formulation of the continuum movement, again new equations have to be written. If only mechanical effects are taken into consideration in the course of movement, then the six missing equations shall be written in such a way that the number of unknown functions will not increase as a result. These missing equations are called constitutive equations. Constitutive equations of this type are e.g. Hooke's law for elastic body, or often the Prandtl—Reuss equation with the Mises yield condition for plastic body.

In equations (1) and (3), the dot above v^p and ε_{pq} indicates the time derivative while the subscript after the semicolon in the subscript of σ^{pq} and v_p the covariant derivative, in the case of a quanfity or a product the identical subscript and superscript indicates a summation with respect to them.

In the right side of equation (1) b^p is the density of body force.

Constitutive equations are equations of fundamental importance for the investigation of equilibrium or motion of a continuum. In addition to the Hooke law and the Prandtl—Reuss equation, many constitutive equations are known today, and even theories to explain the derivation of constitutive equations are available. Below a theory devised to take into consideration the peculiarities of a moving continuum supposing small deformation is presented. The theory assumes that this deformation is brought about by dynamic load and that suitable experiments are required to write the constitutive equation for a continuum of given material.

2. The acceleration wave

If a continuum is loaded by a system of forces, the changes taking place at the point of load will not appear simultaneously at each point of the continuum. At least one surface $\varphi(x^p, t)=0$ can be identified in the continuum at every instant which separates domains V^+ and V^- that is the domains where the effect of the load has already appeared and those still being in the state before loading, respectively. Assume that a wave takes place in the continuum in the course of movement. $\varphi(x^p, t)=0$ is the wave front and

$$c = -\frac{\varphi_4}{\sqrt{g^{sr}\varphi_s\varphi_r}} \tag{4}$$

wave propagation velocity [1]. In (4), $\varphi_4=\dfrac{\partial\varphi}{\partial t}$ is the partial time derivative while $\varphi_s=\dfrac{\partial\varphi}{\partial x^s}$ the partial space derivative and g^{sr} the contravariant metric tensor. In the

following, the Cartesian system of coordinates will be used, $g^{pq}=g_{pq}=\delta_{pq}$ being the Kronecker delta.

Wave front φ may be observed experimentally, if e.g. acceleration function $\dot{v}^p$ has a jump along the wave front. Let this jump be denoted by $[\dot{v}^p]$. If $\dot{v}^{p-}$ is the acceleration before and $\dot{v}^{p+}$ the acceleration after the wave front, then the jump will be

$$[\dot{v}^p] = \dot{v}^{p+} - \dot{v}^{p-}. \tag{5}$$

We can speak of acceleration wave if along the wavefront, $[v^p]=0$, the velocity is continuous, and $[\dot{v}^p]\neq 0$.

In the case of the acceleration wave, the kinematic and dynamic compatibility conditions [2] are as follows:

$$2\varkappa_{pq}\varphi_4 = v_p\varphi_q + v_q\varphi_p \tag{6}$$

and

$$\varrho v^p\varphi_4 = \mu^{pq}\varphi_q \tag{7}$$

must be fulfilled with b^p being continuous, $[b^p]=0$.

$\varkappa_{pq}$, v_p and μ^{pq} are the general amplitudes of deformation rate, acceleration wave and stress wave, respectively. Equations (6) and (7) result from equations (3) and (1), respectively, taking into consideration that, if a function G is continuous over φ but their derivatives have jump along φ, then, with symbols $\dfrac{\partial G}{\partial x^p}=G_p$ and $\dfrac{\partial G}{\partial t}=G_4$

$$[G_p] = g\varphi_p \quad \text{and} \quad [G_4] = g\varphi_4, \tag{8}$$

respectively.

3. Possible constitutive equations

Back to constitutive equations, the usual conditions are disregarded that is the continuum is not assumed to be elastic or plastic or viscous nor to be of some more general property that is isotropic, homogeneous, or even scleronomous. Instead, it is assumed that [2]

a) an acceleration wave can be induced in the continuum,
b) the acceleration wave propagates at finite velocity, and
c) there exist independent wave front families with at least one number identical with the number of independent variables.

Let the constitutive equation be

$$F_\alpha(\varepsilon^{pq}, \sigma^{rs}, \varepsilon^{pq}_{;s}, \sigma^{rs}_{;w}, \varepsilon^{pq}_{;4}, \sigma^{rs}_{;4}, x^p, t) = 0$$

$$(\alpha = 1, 2, 3, 4, 5, 6) \tag{9}$$

Taking F_α on both sides of $\varphi = 0$ front and subtracting them from each other, we obtain with the use of (8)

$$f_\alpha = F_\alpha^+ - F_\alpha^- = F_\alpha(\varepsilon^{pq}, \sigma^{rs}, \varepsilon^{pq-} + \varkappa^{pq}\varphi_{\hat{s}}, \sigma^{rs-} +$$

$$+\mu^{rs}\varphi_{\hat{w}}, x^p, t) - F_\alpha(\varepsilon^{pq}, \sigma^{rs}, \varepsilon^{pq-}_{;\hat{s}}, \sigma^{rs-}_{;\hat{w}}, x^p, t) = 0$$

$$(\hat{s}, \hat{w} = 1, 2, 3, 4) \tag{10}$$

this equation is called material compatibility condition [2].

Condition a) means that there exists a solution φ for equation system (6), (7), (10). According to condition b), this solution φ results a finite velocity c), defined on the basis of (4). Finally, according to c), the solution of equations (6), (7), and (10) includes at least four independent families of function $\varphi(x^p, t)$. These requirements set limits to constitutive equations (9). Such equations $F_\alpha = 0$ are called possible constitutive equations.

4. Dynamic tension of ribbon-type body

For the sake of a better understanding let us investigate a ribbon-type body clamped at point $x = 0$, the axis of which being the x-axis, and which is loaded by dynamic tension to end $x = l$ in such a way that an acceleration wave will take place [3]. Assume a uniaxial stress state in the ribbon.

Matrix of stress tensor σ^{pq} in Cartesian coordinate system x, y, z is:

$$\begin{bmatrix} \sigma(x, t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Equation of motion:

$$\varrho v_t = \sigma_x + b, \tag{11}$$

equation (3):

$$\varepsilon_t = v_x, \tag{12}$$

and the assumed constitutive equation:

$$F(\sigma, \varepsilon, \sigma_x, \sigma_t, \varepsilon_x, \varepsilon_t, x, t) = 0 \tag{13}$$

In the above equations, subscripts x and t indicate partial time and space derivation, respectively.

Now compatibility equations (7), (6), and (10) can be written, as follows:

$$\varrho v \varphi_t = \mu \varphi_x, \tag{14}$$

$$\varkappa \varphi_t = v \varphi_x, \tag{15}$$

and

$$f = F(\sigma, \varepsilon, \sigma_x^- + \mu\varphi_x, \sigma_t^- + \mu\varphi_t, \varepsilon_x^- + \varkappa\varphi_x, \varepsilon_t^- +$$
$$+ \varkappa\varphi_t, x, t) - F(\sigma, \varepsilon, \sigma_x^-, \sigma_t^-, \varepsilon_x^-, \varepsilon_t^-, x, t) = 0 \qquad (16)$$

For φ, (16) is a first-order nonlinear partial differential equation the characteristic equation of which being

$$\frac{dx}{f\varphi_x} = \frac{dt}{f\varphi_t} = \frac{d\varphi}{\varphi_x f\varphi_x + \varphi_t f\varphi_t}.$$

Considering the first equation of the characteristic equation system, we have

$$(F_{\sigma_t}\mu + F_{\varepsilon_t}\varkappa)\frac{dx}{dt} = F_{\sigma_x}\mu + F_{\varepsilon_x}\varkappa. \qquad (17)$$

On the basis of (14) and (15), equation

$$\varrho\varkappa\left(\frac{dx}{dt}\right)^2 = \mu$$

is valid between $\varkappa$ and μ. Using this result, $\varkappa$ can be replaced with μ in (17) and after suitable reduction,

$$\mu(\varrho F_{\sigma_t}c^3 - \varrho F_{\sigma_x}c^2 + F_{\varepsilon_t}c - F_{\varepsilon_x}) = 0 \qquad (18)$$

is obtained [4].

In the course of transformation, (4) has been taken into consideration according to which now

$$\frac{dx}{dt} \equiv c = -\frac{\varphi_t}{\varphi_x}. \qquad (19)$$

In (18) μ can not be zero and thus

$$\varrho F_{\sigma_t}c^3 - \varrho F_{\sigma_x}c^2 + F_{\varepsilon_t}c - F_{\varepsilon_x} = 0. \qquad (20)$$

The coefficients of the third-degree equation of c in (20) are real according to condition a). Hence, there exists at least one real c. In the following, the quality of the roots of the third-degree equation of real coefficient in (20) shall be investigated with a view to satisfy conditions b) and c). In this investigation, the possible variables of functions F are identified.

In the present case, condition c) requires that at least *2* independent families of wave front be available. This means that there are two different real roots of (20). Hence, in the general case, three families of wave front may occur.

It can be shown by quite a simple calculation that the constitutive equation of shape

$$F(\sigma, \varepsilon, \varepsilon_t) = 0$$

that has been recommended earlier in more cases in the literature is not possible [4].

The conditions described for the possible constitutive equations are fulfilled by the Hooke law and the Prandtl—Reuss equations.

From among the possible constitutive equations, that for a continuum of given material can be determined by experimental investigation of the acceleration wave. For copper, the author's experimental investigations seem to prove a constitutive equation of the following shape [3].

$$F(\sigma_t, \varepsilon_t, \varepsilon_x) = 0.$$

5. Comments

— Mathematical and experimental investigation of the existence and properties of the acceleration wave shall reasonably be taken as a basis for investigation of the constitutive equations of continua [3].

— Mathematical investigation of the acceleration wave may show what functions and mechanical characteristics shall be observed in the course of experimental investigation [5].

— A general investigation of equations (6), (7), and (10) under conditions a), b), and c) puts the general properties of continua like isotropy and homogeneity in a new light [5].

— The correctness of the assumption of the uniaxial stress state can be confirmed experimentally, and the results obtained in investigation of the uniaxial stress state by means of equation (20) can be extended also to the general case [2], [5].

References

1. SMIRNOV, W. I.: Lehrgang der höheren Mathematik. VEB Deutscher Verlag der Wissenschaften. Berlin (1958).
2. BÉDA, Gy.: Investigation of the mechanical basic equations of solid bodies by means of acceleration wave. Periodica Polytechnica Mechanical Engineering. Vol. *28*. Nos 2—4. (1984).
3. BÉDA, Gy.: Method to investigate plastic waves (Candidate's Thesis) Miskolc (1959).
4. BÉDA, Gy.: Method zur Bestimmung der Materialgleichung. NME Közleményei *XXII.* kötet (271—286) (1962).
5. BÉDA, Gy.: Possible constitutive equations of the moving plastic body (Doctoral Dissertation) Budapest (1982).

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