INTRINSIC VARIABLES OF CONSTITUTIVE EQUATION

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Abstract

Necessarily, investigation of the possibility of constitutive equations assumed to be of differential equation shape leads to the conclusion that, in addition to the assumed functions, the constitutive equations contain also so-called intrinsic variables.

In conventional material testing, the use of intrinsic variables permits a constitutive equation taking into consideration also the dynamic load to be written.

1. Necessity of intrinsic variables on the basis of mechanical test

In given material, the possibility of existence of the acceleration wave can be ensured if the material compatibility condition is satisfied [1].

With x_p, $p=1, 2, 3$ being the position co-ordinate, x_4 the time, σ_ϑ, ε_ψ, $\sigma_{\beta\hat{p}}$, $\varepsilon_{\gamma\hat{q}}$ co-ordinates of stress and strain tensor, respectively, and ($\vartheta, \psi, \ldots, \gamma=1, \ldots, 6$ and $\hat{p}, \hat{q}=1, \ldots, 4$) their derivative with respect to $x_{\hat{p}}$, the material compatibility conditions can be written, as follows:

Let $F_\alpha(\sigma_{\beta\hat{p}}, \varepsilon_{\gamma\hat{q}}, \sigma_\vartheta, \varepsilon_\psi, x_{\hat{s}})=0$ ($\alpha=1, 2, 3, \ldots, 6$) be the mass equation and $\varphi(x_{\hat{i}})=0$ ($\hat{i}=1, \ldots, 4$) the acceleration wave front, $\mathring{F}_\alpha$ and F_α the substitution value of constitutive equation before and after the wave front, respectively.

Accordingly, the material compatibility condition will be

$$f_\alpha \equiv F_\alpha - \mathring{F}_\alpha = 0.$$

$f_\alpha(\varphi_{\hat{i}}, x_{\hat{p}})=0$ contains derivatives $\dfrac{\partial\varphi}{\partial x_{\hat{i}}} \equiv \varphi_{\hat{i}}$ and thus it is a nonlinear partial differential equation system of the first order with respect to φ. Investigation of the integrability conditions of this equation leads to the conclusion that introduction of intrinsic variables is necessary.

The integrability conditions can be derived from the algebraic equation relating to $d\varphi_{\hat{k}}$.
Introduce notation

$$f_{\alpha|\hat{k}} \equiv \frac{\partial f_\alpha}{\partial \varphi_{\hat{k}}}, \quad f_{\alpha\hat{p}} \equiv \frac{\partial f_\alpha}{\partial x_{\hat{p}}}.$$

The algebraic equation:

$$f_{\alpha|\hat{k}}\, d\varphi_{\hat{k}} + f_{\alpha\hat{p}}\, dx_{\hat{p}} = 0.$$

Let $A_{\alpha\hat{k}} \equiv f_{\alpha|\hat{k}}$ and $dB_\alpha \equiv -f_{\alpha\hat{p}}\, dx_{\hat{p}}$. Then,

$$(C_{\alpha\hat{q}}) = \begin{bmatrix} A_{11} \dots A_{14} & dB_1 \\ \dots \dots & \\ \dots \dots & \\ \dots \dots & \\ \dots \dots & \\ A_{61} \quad A_{61} & dB_6 \end{bmatrix}$$

where $\check{q} = 1, \dots, 5$.

$C_{\alpha\check{q}}$ and $A_{\alpha\hat{k}}$ are of equal rank that is inhomogeneous algebraic equation

$$A_{\alpha\hat{k}}\, d\varphi_{\hat{k}} = dB_\alpha$$

is compatible.

Let rank $(A_{\alpha\hat{k}}) = 4$ that is $\det (A_{\hat{l}\hat{k}}) \neq 0$. Denote $\det (A_{\hat{l}\hat{k}}) \equiv A$ and let adj $(A_{\hat{p}\hat{l}}) \equiv K_{\hat{l}\hat{p}}$. Thus $A_{\hat{q}\hat{l}} K_{\hat{l}\hat{p}} = A \delta_{\hat{q}\hat{p}}$ and accordingly,

$$A\, d\varphi_{\hat{k}} = K_{\hat{l}\hat{k}}\, dB_{\hat{l}}.$$

But

$$d\varphi_{\hat{k}} = \frac{\partial \varphi_{\hat{k}}}{\partial x_{\hat{p}}}\, dx_{\hat{p}}$$

while

$$dB_{\hat{l}} = -\frac{\partial f_{\hat{l}}}{\partial x_{\hat{p}}}\, dx_{\hat{p}},$$

that is

$$A \frac{\partial \varphi_{\hat{k}}}{\partial x_{\hat{p}}} = -K_{\hat{l}\hat{k}} \frac{\partial f_{\hat{l}}}{\partial x_{\hat{p}}}$$

and thus

$$A \varphi_{\hat{k}\hat{p}} = -K_{\hat{l}\hat{k}} f_{\hat{l}\hat{p}}.$$

The condition of integrability is that $\varphi_{\hat{k}\hat{p}}$ be symmetric. This means that

$$K_{\hat{l}\hat{k}} f_{\hat{l}\hat{p}} = K_{\hat{l}\hat{p}} f_{\hat{l}\hat{k}}. \qquad (1.1)$$

Let (1.1) be multiplied with $A_{\hat{i}\hat{k}}$ and then with $A_{\hat{j}\hat{p}}$:

$$A_{\hat{j}\hat{p}} f_{\hat{i}\hat{p}} - A_{\hat{i}\hat{k}} f_{\hat{j}\hat{k}} = 0,$$

$$f_{\hat{j}|\hat{p}} f_{\hat{i}\hat{p}} - f_{\hat{i}|\hat{k}} f_{\hat{j}\hat{k}} = 0,$$

that is

$$(f_j, f_i) = 0, \tag{1.2}$$

Poisson bracket zero being the integrability condition.

Hence, there exist two relations between df_α in compliance with what has been said, such as e.g.

$$H_1(f_1, ..., f_6) = 0 \quad \text{and} \quad H_2(f_1, ..., f_6) = 0,$$

that is

$$H_A(f_\alpha) = 0, \quad (A = 1, 2).$$

Namely, in this case, $\frac{\partial H_A}{\partial f_\alpha} df_\alpha = 0$ will be the two relations to be found if the rank of $\frac{\partial H_A}{\partial f_\alpha}$ is 2. Thus 8 equations shall be satisfied instead of 6 equations to be found originally. Assume for this purpose that the number of unknown functions can be increased by addig h_1 and h_2. Thus in f_α, also h_1 and h_2 will appear in addition to the variables that have been existing so far and $H_A(f_\alpha, h_B) = 0$ where $(A, B = 1, 2)$.

Functions $f_{\bar{k}}$ with variables h_B added satisfy the Poisson bracket zero equation invariably. That is

$$(f_{\bar{i}}, f_{\bar{k}})^* = 0.$$

In detail:

$$\left(f_{\bar{i}|\bar{p}} + \frac{\partial f_{\bar{i}}}{\partial h_A} h_{A|\bar{p}}\right)\left(f_{\bar{k}\bar{p}} + \frac{\partial f_{\bar{k}}}{\partial h_B} h_{B\bar{p}}\right) - \left(f_{\bar{k}|\bar{p}} + \frac{\partial f_{\bar{k}}}{\partial h_C} h_{C|\bar{p}}\right)\left(f_{\bar{i}\bar{p}} + \frac{\partial f_{\bar{i}}}{\partial h_D} h_{D\bar{p}}\right) = 0,$$

that is

$$(f_{\bar{i}}, f_{\bar{k}}) + \frac{\partial f_{\bar{i}}}{\partial h_A}(h_A, f_{\bar{k}}) + \frac{\partial f_{\bar{k}}}{\partial h_B}(f_{\bar{i}}, h_B) + \frac{\partial f_{\bar{i}}}{\partial h_A}\frac{\partial f_{\bar{k}}}{\partial h_B}(h_A, h_B) \equiv 0, \tag{1.3}$$

here $(f_{\bar{i}}, f_{\bar{k}}) = 0$.

Since $f_{\bar{k}}$ and h_A can be selected optionally, equations (1.3) will be satisfied if $(h_A, f_{\bar{k}}) = 0$ and $(h_A, h_B) = 0$. The number of equations involved is $2.4 + 1 = 9$ with also the number of unknowns in them being 9.

h_A is expressed from H_C, and substituted into the equations, by means of f_α. Thus a homogeneous algebraic equation system with nonzero determinant is obtained for unknowns $(f_{\bar{k}}, f_{4+A})$ and (f_{4+A}, f_{4+B}) and therefore the unknowns are zero.

If the Poisson bracket zero condition is satisfied by f of number n from among the six f and the number of functions h is N, then the number of equations will be

$$nN + \binom{N}{2} \equiv N\left(n + \frac{N-1}{2}\right) \quad \text{while} \tag{1.4}$$

the number of unknowns

$$15-\binom{n}{2} = 15\frac{n(n-1)}{2}. \qquad (1.5)$$

The value of n can be 4, 3, 2. Let us see how many equations and unknowns are associated with these values.

n	N	Number of equations	Number of unknowns
	1	4	9
4	2	9	9
	3	15	9
	2	7	12
3	3	12	12
	4	18	12
	3	9	14
2	4	14	14
	5	20	14

h of number N for which the Poisson bracket zero condition is satisfied by total function system f_α can be found for any possible n, these h being called intrinsic variables of the constitutive equation. Thus the number of intrinsic variables will be 2, 3, 4 if the Poisson bracket zero condition is satisfied by 4, 3, 2 constitutive equations, respectively.

These intrinsic variables permit different mechanical or other physical interactions to be taken into consideration. For example the effect of thermodynamical processes upon motion can be taken into consideration, whereas thermodynamical processes are affected by motion.

2. Intrinsic variable in conventional material testing

The conventional method of material testing which has found wide use so far, is based on plotting of the $\sigma \sim \varepsilon$ diagram or the tensile curve. The curve of $\sigma=\sigma(\varepsilon)$ curve so obtained serves to determine the different material characteristics. In the test, stress σ and specific elongation ε can be determined as a function of one single variable that is as a function of time only. Restricting ourselves again to mechanical interaction, the $\sigma \sim \varepsilon$ diagram appears to change with the rate of tension in case of more materials. The idea of using $\sigma=\sigma(\varepsilon, \dot{\varepsilon})$ as the constitutive equation obtained in the test has, therefore, arisen. Hence, stress is a function of also rate of strain $\dot{\varepsilon}$ in addition to specific elongation ε. It can be shown that this function $\sigma=\sigma(\varepsilon, \dot{\varepsilon})$ can not be a constitutive equation [2].

However, the change of the $\sigma \sim \varepsilon$ diagram with the rate of tension is a fact proved experimentally. Let this fact be taken into consideration with an intrinsic variable η. Thus

$$\sigma = \sigma(\varepsilon, \eta), \tag{2.1}$$

or

$$\eta = W(\sigma, \varepsilon). \tag{2.2}$$

Assumptions are required to determine variable η. Assume on the basis of [3] that the material equation of dynamic tension takes in general the following shape:

$$F(\sigma_t, \varepsilon_t, \sigma, \varepsilon) = 0. \tag{2.3}$$

However, according to [3], equation

$$B\frac{\partial F}{\partial \sigma_t}+\frac{\partial F}{\partial \varepsilon_t} = 0 \tag{2.4}$$

will be associated with (2.3) if $B>0$ ($\varepsilon_t \equiv \dot{\varepsilon}$ and $\sigma_t \equiv \dot{\sigma}$ at present).

Taking into consideration (2.4), the differential of (2.3):

$$\frac{\partial F}{\partial \sigma_t}(\mathrm{d}\sigma_t - B\mathrm{d}\varepsilon_t)+\frac{\partial F}{\partial \sigma}\mathrm{d}\sigma+\frac{\partial F}{\partial \varepsilon}\mathrm{d}\varepsilon = 0. \tag{2.5}$$

With this compared with differential

$$\frac{\partial \sigma}{\partial \eta}\mathrm{d}\eta+\frac{\partial \sigma}{\partial \varepsilon}\mathrm{d}\varepsilon-\mathrm{d}\sigma = 0,$$

$$\mathrm{d}\eta = \mathrm{d}\sigma_t - B\mathrm{d}\varepsilon_t \tag{2.6}$$

provided also

$$\frac{\partial F}{\partial \sigma_t} \equiv \frac{\partial \sigma}{\partial \eta} \tag{2.7}$$

is satisfied.

A short calculation satisfies us on the basis of (2.6) that identity (2.7) is correct.

Considering now the differential of (2.2):

$$\mathrm{d}\eta = \frac{\partial W}{\partial \sigma}\mathrm{d}\sigma+\frac{\partial W}{\partial \varepsilon}\mathrm{d}\varepsilon, \tag{2.8}$$

which is zero along lines η=const. Along these lines,

$$\frac{\partial W}{\partial \sigma}\mathrm{d}\sigma_t+\frac{\partial W}{\partial \varepsilon}\mathrm{d}\varepsilon_t = 0. \tag{2.9}$$

However, also the left side of (2.6) is zero along these lines. Comparing equations

(2.6) and (2.9),

$$B = -\frac{\dfrac{\partial W}{\partial \varepsilon}}{\dfrac{\partial W}{\partial \sigma}}. \tag{2.10}$$

Thus the constitutive equation to be found:

$$\sigma = \sigma \left(\varepsilon, \sigma_t + \int \frac{\dfrac{\partial W}{\partial \varepsilon}}{\dfrac{\partial W}{\partial \sigma}} \mathrm{d}\varepsilon_t \right). \tag{2.11}$$

Using the experimental results, (2.11) can be used in case of given material.

References

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