SOME ASPECTS IN THE DYNAMICAL TASKS OF THERMOELASTICITY: HEAT CONDUCTION PROBLEMS, NUMERICAL EXPERIMENTS WITH LONG BARS

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Abstract

The formulation of the dynamical task of thermoelasticity and the search for the possible solutions is — because of the character of the problem — rather complicated. In the course of our investigations we followed a complex method. The theoretical investigations referring to the basic equations were supplemented by physical and numerical experiments. As a result we got new considerations referring to the modification of the law of heat conduction, experimental arrangements that serve to control the former, and numerical results referring to the thermal shock of long bars that can be used in technical practice.

Introduction

Problems of thermoelasticity can be examined by means of the basic equations of continuummechanics and thermodynamics. These equations are usually grouped [9]: the natural laws, the equation of motion and the first and second law of thermodynamics falling within the first, the geometrical and thermokinematical equations within the second, the constitutive equations within the third group. The constitutive equations are the so-called constitutive equation, the state equation and the law of heat conduction. The fourth group consists of the initial and boundary conditions.

As the equations of the first group are based on numerous experiences, and those of the second are definitional relations, recent investigations referring to the basic equations of mechanics, are first of all aimed at the equations of the third group.

The investigations can be theoretical or experimental and within these categories of physical or numerical nature.

Our goal is to introduce the method and some results of our project made in the field of the dynamical tasks of thermoelasticity in accordance with this division. The considerations related to the modification of the law of heat conduction will be introduced and a physical experimental possibility will be demonstrated in connection with this. Then the results of the experiments executed with the numerical model of the thermal shock of long bar will be outlined.

Modification of the law of heat conduction

The idea of modifying Fourier's law of heat conduction was first raised by Maxwell in 1867. Researchers have been dealing with this problem ever since. As any interdisciplinar problem, this also contains numerous difficulties. Here we would only like to discuss one of them.

Among our circumstances researchers of different training and field of interest, e.g. among physicists those dealing with thermodynamics and mechanics, among mechanicians those interested in continuummechanics and finally thermal energeticians deal with the question. Furthermore, those mathematicians who investigate the mathematical considerations of the problem can be added to the above list of researchers. Thus because of the distribution of those dealing with the problem according to profession, certain problems of authority emerge, which question the existence of the whole topic sometimes. The latter, however, is rather a question of „science politics" than of science itself.

The existence of the problem is indicated by the growing number of articles published in the field.

Chandrasekhariah's [1] survey published in 1986 examines thermoelasticity in great detail with the application of the „second sound" phenomenon. The number and publication date of the apx. 300 articles that he quotes shows the validity of the question (Fig. 1.). And this is only one, thermoelastical consideration of the phenomenon. The figure also shows that the running up took place in the second half of the 1960's, first half of the 1970's.

The author divides the articles he processed into three groups. The first contains the articles that discuss conventional tasks of thermoelasticity, the second contains the articles based on basic equations supplemented with a relaxation member while the articles based on the basic equations supplemented with a temperature rate type member belong to the third group. In our earlier works [2, 3, 4] the possibility of second sound in heat conduction and in thermoelasticity is investigated systematically. This way — in the case of a completely generalized investigation — the modi-

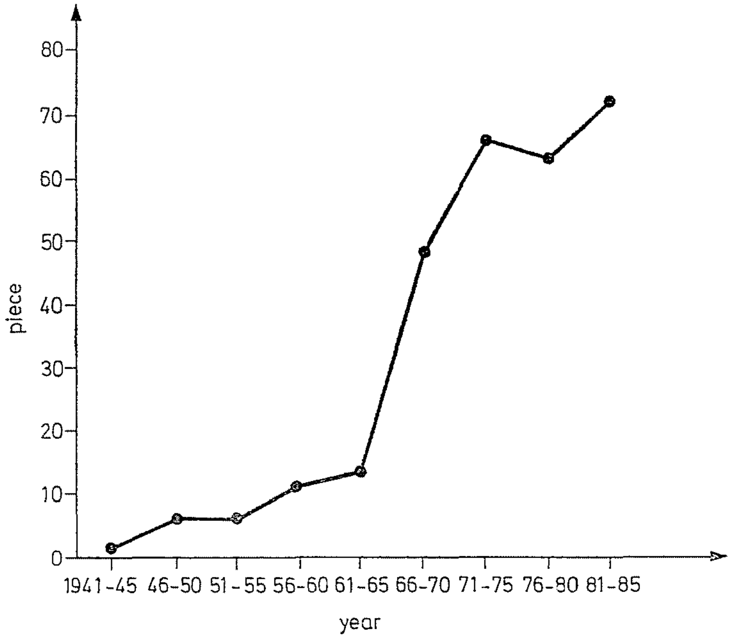

Fig. 1

fication of the law of heat conduction, of the equation of state or else does not have to be assumed in advance.

Coleman, Hrusa and Owen [5] in their paper published in 1986 investigate the mathematical considerations of the second sound phenomenon; furthermore the stability of the balance of the nonlinear, hyperbolic system describing the heat transfer in solid continuum on the basis of second sound.

The 1987 paper of Morro and Ruggeri [6] investigates the relationship between the internal energy and second sound of solid continua.

Day [7] in his 1987 article analysis the second law of thermodynamics on the basis of such a constitutive system which presumes the second sound phenomenon.

The paper of Donato and Fusco [8] published in 1987 analyses a secondary, quasi-linear equation which appears e.g. in the investigation of heat conduction on the basis of second sound.

In the following we are going to deal with some questions and ideas that emerge in connection with the modification of the law of heat conduction.

The problem written on the basis of the Fourier law results in a parabolic differential equation from which infinite value is obtained for the velocity of temperature disturbance propagation. It follows obviously that by modification of the law of heat conduction, a hyperbolic equation can be obtained e.g. in the following form:

$$aT_{xx} = T_t + \tau T_{tt}, \tag{1}$$

from where the heat propagation velocity is limited:

$$v_T = \sqrt{a/\tau} \tag{2}$$

However, the following questions arise in the this case:

a) What is the purpose of modification?
b) Why has the adequacy of the Fourier law never, for about 200 years, been questioned?
c) What is the scope of problems the solution of which is affected?
d) In relation with what has been said above, the need arises for the determination of an additional material property called relaxation time and denoted with τ on the basis of macrostructure or microstructure tests or the combination of both.
e) What considerations should be taken as a basis for modification?
f) What physical interpretation or experimental proof is required for the modified equations?

Let us answer the questions. The primary point in modification is a rather theoretical one as there can be no infinite velocity of propagation in nature whereas the secondary point that is the increase of the accuracy in certain problems is rather practical. Which are these problems?

When equation (1) is examined, it can be seen that there is no point about a modification in case of stationary or quasistatic problems since the value of $\frac{\partial^2 T}{\partial t^2}$ is zero in both cases. The need for modification arises in dynamic problems only, and even so only if very fast processes are involved.

Let us see an example. Let us assume that the temperature changes with time asymptotically with T_0 being the amplitude and $2t_0$ the oscillation time. Now on the basis of

$$T = T_0 \sin \frac{\pi}{t_0} t, \tag{3}$$

$$\dot{T}_{\max} = T_0 \frac{\pi}{t_0}, \quad \ddot{T}_{\max} = T_0 \left(\frac{\pi}{t_0}\right)^2. \tag{4}$$

As can be seen, the relation of the time derivatives of equation (1) as compared with each other is determined by the quotient

$$\frac{\tau \ddot{T}_{\max}}{\dot{T}_{\max}} = \frac{\tau \pi}{t_0} \tag{5}$$

that is, the error resulting from neglecting the second time derivative will be significant if τ and t_0 are commensurable. Hence, in respect of our investigations, the velocity depends partly on the process t_0 and on the other hand, on the material property τ. Now in answering question a) and partly question c), we arrived also at the answer to question b).

Ever since it existed, the Fourier law has been one of the most efficient models of physics. Although the need for its modification arose in principle as early as in 1867, its practical inadequacy, proved by numerical examples and possibly experimentally, has become known only in the recent decades, presumably because the number of so-called high-speed processes keeps increasing: changes take place very rapidly in the field of reactor engineering, space exploration, supersonic flight, modern weapons, magnetohydrodynamic generators, high-speed internal combustion engines, etc. However, as a matter of fact, part of the research in these fields is going on behind closed doors and neither the actual problems are known to, nor the results are available for, research workers of an average level of information.

Otherwise the fact that thermoelasticity ranges among the first fields where the adequacy of the Fourier law has been questioned cannot be considered to be accidental: the reason for it lies presumably in the significant difference between the elastic wave and the thermodynamic phenomenon accompanying it, that is the velocity of temperature disturbance propagation.

As a numerical example, let us assume an internal combustion engine with a speed of $n=3000$ r.p.m. In this case, the half period of oscillation will be $t_0=0{,}04$ s. Assume that a temperature change of $T_0=100$ K takes place in the cylinder wall per cycle. With a value of $\tau=10^{-3}$ s taken as the relaxation time, $\tau\pi/t_0=0{,}08$ will be obtained, that is, its neglection results in an error of 8% in this case.

The value of the relaxation time τ varies for each material. Maurer [10] calculated the relaxation times for metals on the basis of microstructure considerations but he obtained questionably low values. For copper, we obtained a value of $10^{-1} \div 10^{-3}$ in our measurements [3] where also the uncertainty in order of magnitude was permitted.

A conditio sine qua non for the introduction of the modified law of heat conduction is the knowledge of appropriate material characteristics that is, in the present case, of τ. The question arises why the value of τ is so uncertain. The reason for this uncertainty is a dual one. On the one hand, it is diffucult to produce the value of τ by measurement and consideration and, on the other hand, the need for it arises only late and seldom. The problem is still more confused by the fact that, instead of the model corresponding to the modification, solid state physics designed first of all to determine the value of τ in connection with heat conduction uses a temperature dependent heat conduction factor [11], which varies in the Fourier law. As has been mentioned, the determination of the value of τ to a sufficient accuracy is presumably possible by means of marcrostructural and microstructural investigations.

Considerations concerning the modification of the law of heat conduction have been discussed in detail in our earlier works [2, 4]. We arrived there at the result that e.g. in case of invariable state equation, the law of heat conduction could be written in the general case, for a one-dimensional problem, in the following form:

$$h = \varkappa T_x - \tau h_t + \omega_2 h_x + \omega_3 T + \omega_4 T_t, \tag{6}$$

where $h = \varkappa T_x$ is the Fourier law, ω_i ($i=2, 3, 4$) are further material properties, unknown for the time being.

In this case, the heat conduction equation will be

$$a T_{xx} = T_t + \tau T_{tt} - \left(\omega_2 + \frac{\omega_4}{\varrho c}\right) T_{tx} - \frac{\omega_3}{\varrho c} T_x \tag{7}$$

from which (1) will result if $\omega_i = 0$.

Question (f) still remains to be answered. Interesting thoughts are raised by the physical interpretation.

Namely, after the law of heat conduction in the form of $h = \varkappa T_x$ had been formulated by Fourier, it became obvious that heat is flowing from the warmer spot to the colder spot and the intensity of flow is proportional to the temperature gradient. However, doubt comes up that this physical aspect has been the result of the well-memorized, and thus accepted, relationship. The doubt is still more justified considering the fact that e.g. in case of anisotropic materials, this physical interpretation is by far not so simple. Of course, the problem is a rather philosophical one and it is related to the theory of cognition. Obviously, Fourier's law was formulated in compliance with Fourier's age: both the requirements and the experimental possibilities were different at that time. We might be right in saying that today Fourier would formulate the law of heat conduction in a different form, and we would not dream of accepting it on the basis of contemporary interpretation.

For that very reason, the second part of the question that is, experimental proof becomes still more important. Let us discuss it in detail.

Experimental possibilities with the modified law of heat conduction

It is difficult in general, and in the present case in particular, to find the possibility of producing an experimental proof. Hence, looking for the simplest way, we try to prove the law of heat conduction according to (6) for the stationary case, using the following method:

The model shown in Fig. 2. shall be prepared. Heat shall be introduced at a constant rate at one end of a long, thin metal bar heat-insulated along the mantle.

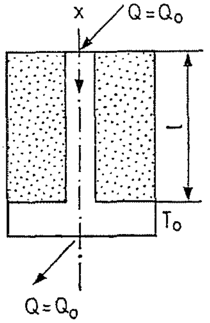

Fig. 2

The other end of the metal bar shall be connected to a heat reservoir of unlimited capacity. That means that here also, the heat dissipation is constant and thus the problem can be considered to be stationary.

Accordingly,

$$h = \varkappa T_x + \omega_2 h_x + \omega_3 T \tag{8}$$

applies in place of (6). Because of further properties of the experimental setup,

$$h = h_0 = \varkappa T_x + \omega_3 T. \tag{9}$$

This differential equation shall be solved for $\omega_3 = 0$ and $\omega_3 \neq 0$. Taking into consideration boundary condition $T(x=l) = T_0$; the following solutions are obtained:

$$T = T_0 + \frac{h_0}{\varkappa}(x - l), \quad \omega_3 = 0 \tag{10}$$

$$T = \left(T_0 - \frac{h_0}{\omega_3}\right), \, e^{-\frac{\omega_3}{\varkappa}(x-l)} + \frac{h_0}{\omega_3}, \quad \omega_3 \neq 0. \tag{11}$$

Now with function $T(x)$ determined by measurement compared with (10) and (11), the necessity and the way of modification, respectively, can be determined.

Numerical experiments with long bars

It is customary to reduce the basic equations of thermoelasticity [9] to the equation system that contains the generalized equations of motion and of heat conduction with the unknown functions of displacement and temperature. On condition of isotropy, inhomogeneity and one dimensionality and that the deformations and $\partial T/\partial s$ are small and the processes reversible, the equations are as follows:

$$\frac{E}{\varrho} u_{xx} - u_{tt} - \frac{E\alpha_t}{\varrho} \Theta_x = \frac{(E\alpha_t)_x}{\varrho} \Theta - \frac{E_x}{\varrho} u_x, \tag{12}$$

$$\Theta_{xx} - \left(\frac{c\varrho}{\varkappa} + \frac{\gamma}{\varkappa} u_x\right) \Theta_t - \frac{\gamma}{\varkappa} T u_{xt} = -\frac{\varkappa_x}{\varkappa} \Theta_x. \tag{13}$$

Mention has to be made of the fact that the third member on the left in equation (13) does not appear in the technical literature, this is a result of our earlier investigations.

Our numerical experimental equipment is based on these equations. Let us apply the equations to the long bar in Fig. 3. The initial and boundary conditions that appear in the figure are the following. At the beginning of the investigation the bar is in rest condition and its temperature is constant along its length. At the fixed end the displacement is zero and the temperature — in accordance with the infinite heat container — is constant. At the free end of the bar we transfer heat according to given rules, and this will determine the temperature and the strain of the end of the bar.

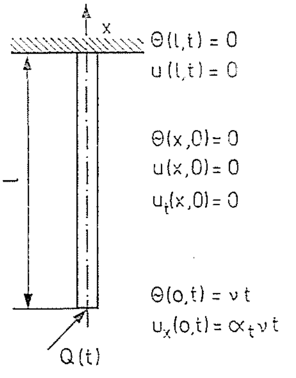

Fig. 3

Let us transcribe equations (12) and (13) into difference — equations and search for the solution with the help of finite differences. By introducing the $\Delta x = h$ and $\Delta t = k$ symbols we get

$$\frac{E}{\varrho h^2}[u(x+h,t)-2u(x,t)+u(x-h,t)]-\frac{1}{k^2}[u(x,t+k)-2u(x,t)+u(x,t-k)]-$$

$$-\frac{E\alpha_t}{\varrho h}[\Theta(x+h,t)-\Theta(x,t)]=\frac{1}{\varrho}\frac{d}{dx}(E\alpha_t)\Theta(x,t)-\frac{1}{\varrho}\frac{dE}{dx}\frac{1}{h}[u(x+h,t)-u(x,t)], \tag{14}$$

$$\frac{1}{h^2}[\Theta(x+h,t)-2\Theta(x,t)+\Theta(x-h,t)]-\left[\frac{c\varrho}{\varkappa}+\frac{\gamma}{\varkappa h}(u(x+h,t)-\right.$$

$$\left.-u(x,t))\right]\frac{1}{k}[\Theta(x,t+k)-\Theta(x,t)]-\frac{\gamma}{\varkappa}(T+\Theta(x,t))\frac{1}{hk}[u(x+h,t+k)-$$

$$-u(x,t+k)-u(x+h,t)+u(x,t)]=-\frac{1}{\varkappa k}\frac{d\varkappa}{dx}[\Theta(x+h,t-\Theta(x,t)]. \tag{15}$$

With equations (14) and (15) the displacement and temperature field can be produced. To make the method work, linear extrapolation at the boundaries is needed. According to this in case of $t=0$, the following equation will be valid for $u(x, t-k)$:

$$u(x, -k) = u(x, 0) - u_t(x, 0)k \tag{16}$$

and in case of $x=h$ for $u(x-h, t)$:

$$u(0, t) = u(h, t) - u_x(0, t) \cdot h = u(h, t) - \alpha_t vth. \tag{17}$$

With this model we get the numerical experimental results for steel bar that are going to be outlined here.

Thermal shock of long homogeneous, isotropic bar by cooling (Fig. 4). The curves show the displacement and the temperature as function of space with the

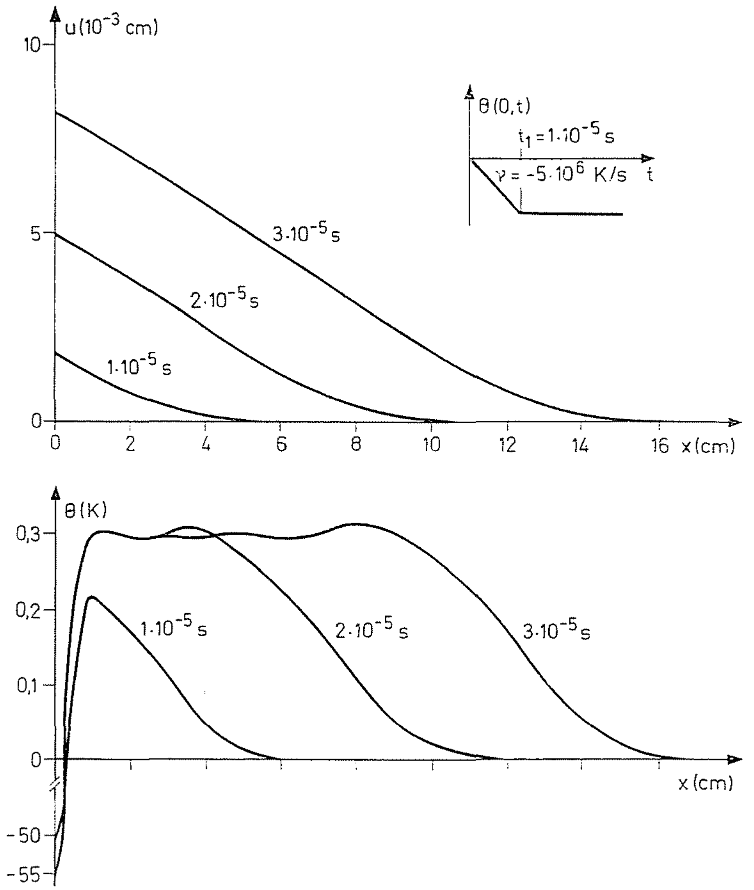

Fig. 4

parameter of time. The results are in good agreement with the earlier ones. The displacement is positive, and the temperature at the very end of the bar is negative, but after a while it turns to positive. In good agreement with Nowinski's [12] result in this range of energy there is very little difference between the warming up and cooling. Of course the results also depend on the parameters.

In connection with Fig. 4 it has to be mentioned that in some of the earlier result the profile of the temperature curve in the range of $x=0\div 1$ cm is not visible. In this figure one can see the more exact — schematically valuable — form. In spite of the manifold deformation of the measure the coincidence with the results of Lord & Shulman [13], Francis & Lindholm [14], Nickell & Sackman [15] and Chu & Dodge [16] is good.

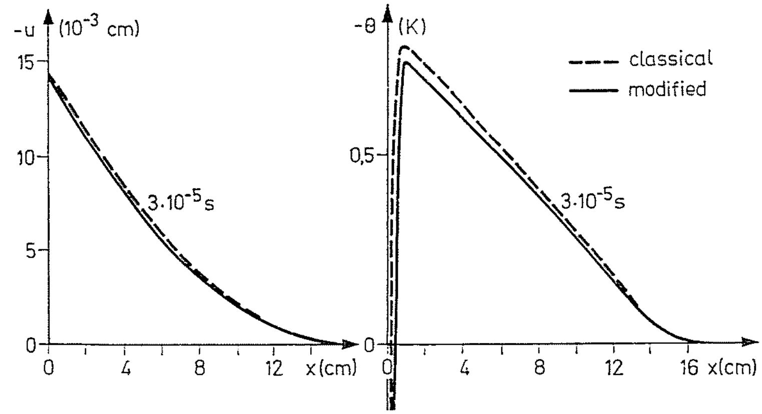

Fig. 5

Effect of the neglection of the new term $u_x \Theta_t$ (Fig. 5). The curves show the displacement and the temperature as function of place in a fixed time. One can see the effect of this term, which is not big, about 4—6%, but not negligible. Of course the effect of this term depends on the model and the parameter. As it contains the temperature rate it mainly depends on the rate of the process. It is shown by the numerical results (see the curves) that at the beginning where the changes are faster the differences between the classical and modified solutions are bigger.

As a result of the numerical calculation it can be stated that the practical effect of the term $u_x \Theta_t$ is visible.

Thermal shock of an isotropic, inhomogeneous bar (Fig. 6). The change of the inhomogeneity is linear along the bar and in the figure one can see four different cases. In the first one the bar is homogeneous. In the second case the extent of the inhomogeneity is 20%. This means that the physical constants change 20% along the bar from one end to the other. In our calculations it results in only some percent

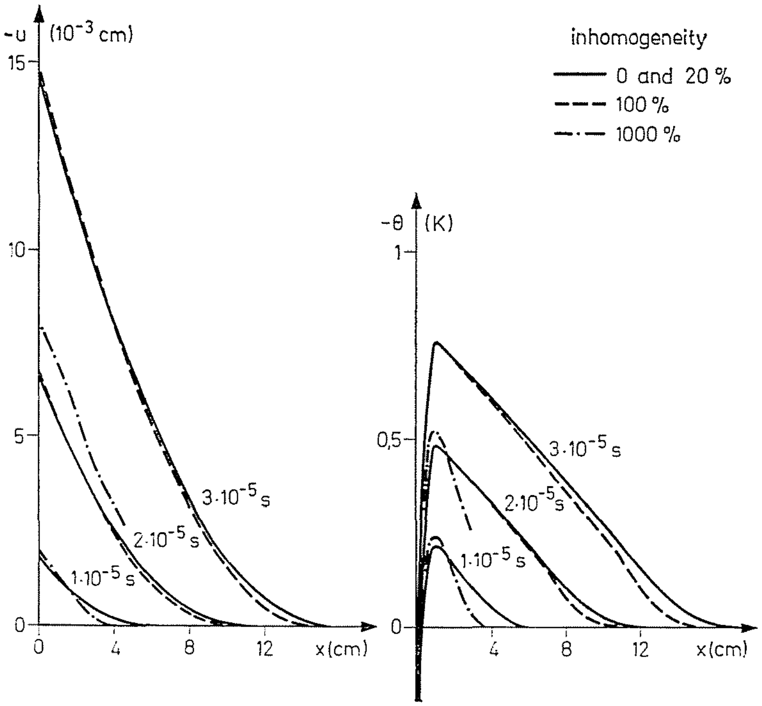

Fig. 6

changes. This is the case of the not perfect, but acceptable technology. The figures show that in the displacement and temperature field three is no obvious difference.

In the third case the inhomogeneity is 100%, it produces a 5—6% change in our calculations. This is the case of the bad and not acceptable technology.

In the fourth case the inhomogeneity is 1000%. There is no reality of this case, as it is shown by the calculation method also, which does not work in this case, there is no convergence.

The practical serviceability and the fact that no references were at hand about this case in scientific and technical literature make this result all the more significant.

Symbols

a = coefficient of temperature conduction
T = temperature
τ = relaxation time
x = locus

t = time

$(\ )_x = \frac{\partial()}{\partial_x}$

$(\ )_t = \frac{\partial()}{\partial_t}$

v_T = velocity of heat propagation

h = intensity of heat flux

$\varkappa$ = coefficient of heat conduction

ω_i = constants

ϱ = mass density

c = specific heat

l = length of bar

s = entropy density

u = displacement

E = Young's modulus

α_t = coefficient of linear expansion

Θ = temperature difference

v = rate of temperature change

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