FORMULATION OF COUPLED PROBLEMS OF THERMOELASTICITY BY FINITE ELEMENTS

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Abstract

When designing mechanical equipment, in numerous cases thermal effects have to be taken into consideration besides mechanical ones, especially if they are transient effects. In engineering practice these thermomechanical problems can be described by the different theories of thermoelasticity with adequate accuracy. This paper, after a brief survey of the basic equations of thermoelasticity, demonstrates its formulation by the finite element method.

Symbols

σ_{ij} — linear stress tensor
ε_{ij} — linear strain tensor
u_i — displacement vector
ϱ — mass density
F_i — external force per unit mass
R — strength of the internal heat source per unit mass
q_i — heat flux per unit area
s — entropy per unit mass
T — absolute temperature
T_0 — initial temperature
θ — temperature difference
C_{ijkl} — elasticity tensor
β_{ij} — thermoelasticity tensor
k_{ij} — thermal conductivity tensor
c — specific heat per unit mass at constant strain

λ, μ — Lamé elastic constants
β — thermal modulus
α — coefficient of linear thermal expansion
k — thermal conductivity
E — Young's modulus
ν — Poisson's ratio
τ — relaxation time
t — time
$\mathbf{r}$ — three-dimensional space vector
n_j — outward normal unit vector of the surface
q_n — heat flux in normal direction
a — coefficient of convective heat transfer
θ_w — surface temperature difference
θ_∞ — reference temperature difference
$(\dot{})$ — partial differentation with respect to time
$(\,.\,)_{,i}$ — partial differentation with respect to descartes coordinate x_i

1. Introduction

Mechanical equipments during operation are affected by various interactions, the most significant being the mechanical and thermal effects. Mechanical and thermal loads usually occur simultaneously and as a result, the displacement and temperature fields are created in close connection with each other. The two fields have to be defined simultaneously taking the relationship between them into account which in practice proves to be rather difficult.

Thermomechanical processes are described by the basic equations of continuum-mechanics and thermodynamics. In the solution of a variety of problems the application of thermoelasticity proves to be efficient. The foundations of thermoelasticity were laid by Duhamel and Neuman in the first half of the last century, widespread interest in this field, however, has not developed till the middle of the 20th century. The reason for the sudden growth of interest is that at that time the need for designing equipment that can operate at very high temperatures arose almost simultaneously in several dynamically developing areas of industry. Such areas were among others: production of high-speed aeroplanes, design of space vehicles, rocket and jet engines, technology of large turbines and the design of nuclear reactors.

Biot [1], Boley and Weiner [2], Parkus [3], Nowinski [4] and numerous other scientists have dealt with the solution of the problems, and as a result of their work the theory of classical linear thermoelasticity was created based on the solid foundation of reversible thermodynamics. In recent years modified, generalized versions of the classical theory have been published (Lord and Shulman [5], Green and Lindsay [6], Szekeres [10]).

The first part of this paper contains a brief summary of the basic equations of classical and generalized linear thermoelasticity. The second part introduces a finite element method which is suitable for solving two-dimensional problems.

2. Basic equations of linear thermoelasticity

2.1. Classical linear thermoelasticity

Classical linear thermoelasticity is based on the following fundamental equations [4]:

— kinematic relation:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i}), \tag{2.1}$$

— equations of motion:

$$\sigma_{ij,j}+\varrho F_i = \varrho \ddot{u}_i, \tag{2.2}$$

$$\sigma_{ij} = \sigma_{ji}, \tag{2.3}$$

— energy-scale equation:

$$q_{i,i}+\varrho(T_0\dot{s}-R) = 0, \tag{2.4}$$

— constitutive equations:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}+\beta_{ij}\theta, \tag{2.5}$$

$$q_i = -k_{ij}\theta_{,j}, \tag{2.6}$$

$$\varrho s = \frac{\varrho c}{T_0}\theta-\beta_{ij}\varepsilon_{ij}. \tag{2.7}$$

From the equation of motion (2.2) and the energy-scale equation (2.4) using the linear constitutive equations (2.5)—(2.7) we get the most general basic equations of linear thermoelasticity:

$$(C_{ijkl}\varepsilon_{kl})_{,j}+(\beta_{ij}\theta)_{,j}+\varrho F_i-\varrho\ddot{u}_i = 0, \tag{2.8}$$

$$(k_{ij}\theta_{,j})_{,i}-\varrho c\dot{\theta}+\varrho R+T_0\beta_{ij}\dot{\varepsilon}_{ij} = 0. \tag{2.9}$$

In the case of homogeneous, isotropic material with respect to (2.1), (2.8) and (2.9) can be transcribed as:

$$\mu u_{i,jj}+(\lambda+\mu)u_{k,ki}+\beta\theta_{,i}+\varrho F_i-\varrho\ddot{u}_i = 0, \tag{2.10}$$

$$k\theta_{,ii}-\varrho c\dot{\theta}+\varrho R+\beta T_0\dot{u}_{i,i} = 0. \tag{2.11}$$

These equations are the coupled field equations referring to u_i and θ field variables, for homogeneous and isotropic materials of the classical theory of linear thermoelasticity. The relationship between the displacement and temperature fields is shown by the third term of the equation of motion and the fourth term of the heat-conduction equation. In the absence of these the equations simplify to the Lamé equation of classical elasticity and the Fourier's heat-conduction equation of classical thermodynamics.

2.2. *Generalized linear thermoelasticity theory*

From the field equations of the classical theory the equation of motion (2.10) is hyperbolic type in case of known temperature field, so it provides finite speed for the spread of elastic disturbances. The heat-conduction equation (2.11) based on the classical Fourier's heat-conduction law is, however, parabolic, so the classical theory bears on itself the paradox of infinite speed of thermal signals. To resolve this, in the last two decades so-called generalized thermoelasticity theories were created (e.g. Lord and Shulman [5], Green and Lindsay [6], Szekeres [10]) which are based on hyperbolic type heat-conduction equations and these result in finite speed of thermal signals. In these the different authors modified the heat-conduction equation of thermoelasticity on the basis of different principles.

In the following we are going to deal with the generalized theory based on the modification of Fourier's classical heat-conduction law. Cattaneo's and Vernotte's modified version of the heat-conduction law is the following for anisotropic material:

$$\left(1+\tau\frac{\partial}{\partial t}\right) q_i = -k_{ij}\theta_{,j}. \tag{2.12}$$

If instead of equation (2.6) we regard the above equation as the constitutive equation describing heat flux, then the following equation can be derived for isotropic material:

$$k\theta_{,ii}+\left(1+\tau\frac{\partial}{\partial t}\right)(-\varrho c\dot{\theta}+\varrho R+\beta T_0\dot{u}_{i,i}) = 0. \tag{2.13}$$

This equation and the equation of motion (2.10) constitute the whole system of field equations of generalized thermoelasticity.

The relaxation time appearing in the equation is a new material property, the value of which according to literature is between 10^{-14} and 10^{-10} sec. The generalized heat-conduction equation (2.13) in $\tau=0$ case is naturally simplified to the classical heat-conduction equation (2.11).

2.3. Initial and boundary conditions

When formulating a specific practical problem, initial and boundary conditions have to be attached to the field equations of the classical or generalized theory. Initial conditions:

$$u_i(\mathbf{r}, t=0) = u_{i0}(\mathbf{r}), \quad \dot{u}_i(\mathbf{r}, t=0) = \dot{u}_{i0}(\mathbf{r}),$$
$$\theta(\mathbf{r}, t=0) = \theta_0(\mathbf{r}), \quad \dot{\theta}(\mathbf{r}, t=0) = \dot{\theta}_0(\mathbf{r}). \tag{2.14}$$

The inital condition referring to $\dot{\theta}$ has to be given only when the generalized theory is applied.

Boundary conditions:

Let us mark the examined range with 'V', its boundary with 'A'. The boundary conditions can be divided into two large groups: mechanical and thermal boundary conditions:

1. mechanical boundary conditions:
   a) kinematic boundary condition (prescribed displacement):

$$u_i|_{A_u} = \hat{u}_i, \tag{2.15}$$

   b) dynamic boundary condition:

$$\sigma_{ij} n_j|_{A_p} = \hat{p}_i. \tag{2.16}$$

2. thermal boundary conditions:
   a) prescribed temperature condition:

$$\theta|_{A_1} = \hat{\theta}, \tag{2.17}$$

   b) heat flow and convection boundary condition:

$$q_i n_i|_{A_2} = \hat{q}_n + a(\theta_w - \theta_\infty). \tag{2.18}$$

3. Finite element analysis of two dimensional thermoelastic problems

The field equations (2.10) and (2.13) of the generalized theory of linear thermoelasticity make a coupled partial differential equation system consisting of four equations. The closed form solution of this system belonging to given initial and boundary conditions with the exception of a few trivial cases cannot be produced for most practical problems. Modern computers and numerical methods have not been at the disposal of design-engineers' until very recently. For this reason to alleviate the problems various simplified forms of the basic equations were created by approximate assumptions based on practical experience. When designing heavy-duty equipment often the results gained from simplified equations are not satisfactory. In these cases the calculations have to be made on the basis of all the field equations of thermoelasticity. This problem can usually be solved with the application of numerical methods only. Among the different numerical methods the finite element is probably the only one, that can most effectively be used for solving the problems of thermoelasticity. In the following the main steps of deduction of a finite element method suitable for solving two-dimensional problems will be outlined.

When deducing finite element schemes suitable for solving mechanical problems the starting point is usually a variational principle. A variational principle exists for the problems of thermoelasticity as well, which was elaborated by Keramidas and Ting [8] through the utilization of Biot's [7] results. This principle contains the so-called heat displacement instead of temperature which makes taking heat flow and convection boundary conditions into account more difficult when solving practical problems. We would like to deduce the finite element scheme referring to the field variables of displacement and temperature, but there is no variational principle for these variables, so the scheme will be produced through a special weighted residual method, the Galerkin-method.

3.1. Discretization in space

In the case of two-dimensional problems only three scalar field functions, the 'x' directional 'u' and 'y' directional 'v' components of the displacement vector and the temperature difference have to be defined.

For finite element formulation of the problem the so-called finite element scheme referring to one element is needed. To deduce this the approximation of the field functions within one element will be given — in accordance with the basic principles of the method — with the local unknowns (u_i, v_i, θ_i) and the suitably chosen shape functions (N_i^d, N_i^θ):

$$u^E(x, y, t) = \sum_{i=1}^{n} u_i(t)\, N_i^d(x, y),$$

$$v^E(x, y, t) = \sum_{i=1}^{n} v_i(t)\, N_i^d(x, y), \tag{3.1}$$

$$\theta^E(x, y, t) = \sum_{i=1}^{n} \theta_i(t)\, N_i^\theta(x, y).$$

Let us now assume that the nodal values are the continuous functions of time, thus the discretization in time and space is separated. The N_i^d and N_i^θ symbols signify that various shape functions can be applied for the interpolation of displacement and temperature fields. The n index that appears in the equations is the number of nodal points in the applied finite element. Let us arrange the nodal unknowns of the element into one vector:

$$\delta^E = \begin{bmatrix} \mathbf{d}^E \\ \theta^E \end{bmatrix}, \quad \text{where} \quad \mathbf{d}^E = \begin{bmatrix} u_1 \\ v_1 \\ \vdots \\ u_n \\ v_n \end{bmatrix}, \quad \theta^E = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_n \end{bmatrix}, \tag{3.2}$$

and the shape functions into one matrix:

$$N = \begin{bmatrix} N^d & 0 \\ 0 & N^\theta \end{bmatrix}, \quad \text{where} \quad N^d = \begin{bmatrix} N_1^d & 0 & N_2^d & 0 \ldots N_n^d & 0 \\ 0 & N_1^d & 0 & N_2^d \ldots 0 & N_n^d \end{bmatrix}, \quad N^\theta = [N_1^\theta \ldots N_n^\theta]. \tag{3.3}$$

The approximation of the field functions within one element with the above symbols:

$$\begin{bmatrix} u^E \\ v^E \\ \theta^E \end{bmatrix} = N\delta^E. \tag{3.4}$$

Let us first do the discretization of the equation of motion. In order to take the dynamical boundary condition simply into account we start out from the following form of the equation:

$$\sigma_{ij,j} + \varrho F_i - \varrho \ddot{u}_i = 0. \tag{3 5}$$

If the approximation of the field functions (3.1) is inserted into equation (3.5), then we get a so-called residual that usually differs from zero (from here on the invar able symbols will be used):

$$\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = (\sigma^E \cdot \nabla + \varrho \mathbf{F} - \varrho \ddot{\mathbf{u}}^E) \neq \mathbf{0}. \tag{3.6}$$

The weighted residual method prescribes that the weighted integral value of this residual should be zero:

$$\int_{\Omega^E} S \cdot \mathbf{m} \, d\Omega = \mathbf{0}. \tag{3.7}$$

(Ω^E is the domain of the finite element). If we use the shape functions as S weighted functions, then we receive Galerkin's method:

$$\int_{\Omega^E} N^{d^T} \cdot (\sigma^E \cdot \nabla + \varrho \mathbf{F} - \varrho \ddot{\mathbf{u}}^E) \, d\Omega = \mathbf{0}. \tag{3.8}$$

We transform the first member that appears in the integral:

$$N^{d^T} \cdot (\sigma^E \cdot \nabla) = (N^{d^T} \cdot \sigma^E) \cdot \nabla - (\overset{\downarrow}{N}{}^{d^T} \cdot \sigma^E) \cdot \nabla = (N^{d^T} \cdot \sigma^E) \cdot \nabla - (N^{d^T} \circ \nabla) : \sigma^E. \tag{3.9}$$

We transform the area integral of the first term of the equation (3.9) into a line integral with the help of the Gauss—Ostrogradsky theorem which enables us to take the dynamical boundary condition (2.16) into consideration:

$$\int_{\Omega^E} (N^{d^T} \cdot \sigma^E) \cdot \nabla \, d\Omega = \int_{S^E} N^{d^T} \cdot \sigma^E \cdot \mathbf{n} \, dS = \int_{S_p^E} N^{d^T} \cdot \hat{\mathbf{p}} \, dS_p . \tag{3.10}$$

By integrating term by term in equation (3.8) and using equations (3.1) and (3.10) we get:

$$\int_{\Omega^E} (N^{d^T} \circ \nabla) : \sigma^E d\Omega - \int_{S_p^E} N^{d^T} \cdot \hat{\mathbf{p}} \, dS_p - \int_{\Omega^E} N^{d^T} \varrho \mathbf{F} d\Omega + \int_{\Omega^E} N^{d^T} \varrho \ddot{\mathbf{u}}^E \, d\Omega = \mathbf{0}. \tag{3.11}$$

The constitutive equation referring to the stress tensor and the approximation of the field functions (3.1) together with equation (3.11) give the following matrix-differential equation:

$$M^{dd} \ddot{\mathbf{d}}^E + K^{dd} \mathbf{d}^E + K^{d\theta} \boldsymbol{\theta}^E = \boldsymbol{\Phi}. \tag{3.12}$$

This matrix equation is the form of equation of motion discretized in space, which refers to one element.

The quantities appearing in the equation (3.12) are as follows:

$$M_{ij}^{dd} = \int_{\Omega^E} \varrho \begin{bmatrix} N_i^d N_j^d & 0 \\ 0 & N_i^d N_j^d \end{bmatrix} d\Omega, \tag{3.13/a}$$

$$K_{ij}^{dd} = \int_{\Omega^E} \left[\begin{array}{c:c} D_1 \frac{\partial N_i^d}{\partial x} \frac{\partial N_j^d}{\partial x} + D_3 \frac{\partial N_i^d}{\partial y} \frac{\partial N_j^d}{\partial y} & D_2 \frac{\partial N_i^d}{\partial x} \frac{\partial N_j^d}{\partial y} + D_3 \frac{\partial N_i^d}{\partial y} \frac{\partial N_j^d}{\partial x} \\ \hdashline D_2 \frac{\partial N_i^d}{\partial y} \frac{\partial N_j^d}{\partial x} + D_3 \frac{\partial N_i^d}{\partial x} \frac{\partial N_j^d}{\partial y} & D_1 \frac{\partial N_i^d}{\partial y} \frac{\partial N_j^d}{\partial y} + D_3 \frac{\partial N_i^d}{\partial x} \frac{\partial N_j^d}{\partial x} \end{array} \right] d\Omega, \tag{3.13/b}$$

$$K_{ij}^{d\theta} = \int_{\Omega^E} \beta \left[\begin{array}{c} \frac{\partial N_i^d}{\partial x} N_j^\theta \\ \hdashline \frac{\partial N_i^d}{\partial y} N_j^\theta \end{array} \right] d\Omega, \qquad i, j = 1, \ldots, n \tag{3.13/c}$$

$$\Phi_i^d = \int_{S_p^E} N_i^d \begin{bmatrix} \hat{p}_x \\ \hat{p}_y \end{bmatrix} dS_p + \int_{\Omega^E} \varrho N_i^d \begin{bmatrix} F_x \\ F_y \end{bmatrix} d\Omega, \tag{3.13/d}$$

where D_1, D_2, D_3 material properties in the case of plane strain are:

$$D_1 = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}, \quad D_2 = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad D_3 = \frac{E}{2(1+\nu)}, \tag{3.14/a}$$

in case of plane stress they are:

$$D_1 = \frac{E}{1-\nu^2}, \quad D_2 = \frac{E\nu}{1-\nu^2}, \quad D_3 = \frac{E}{2(1+\nu)}. \tag{3.14/b}$$

Let us now consider the following form of the heat-conduction equation:

$$q_{i,i} + \varrho c\dot{\theta} - \varrho R - \beta T_0 \dot{u}_{i,i} = 0. \tag{3.15}$$

To discretize equation (3.15) similarly to the discretization of the equation of motion, Galerkin's method is used:

$$\int_{\Omega^E} \mathbf{N}^{\theta T} \cdot (\mathbf{q}^E \cdot \nabla + \varrho c\dot{\theta}^E - \varrho R - \beta T_0 \dot{\mathbf{u}}^E \cdot \nabla)\, d\Omega = \mathbf{0}. \tag{3.16}$$

The first term can be transformed here as well:

$$\mathbf{N}^{\theta T} \cdot (\mathbf{q}^E \cdot \nabla) = (\mathbf{N}^{\theta T} \circ \mathbf{q}^E) \cdot \nabla - (\mathbf{N}^{\theta T} \circ \nabla) \cdot \mathbf{q}^E. \tag{3.17}$$

The heat flow and convection boundary conditions are taken into account similarly to the dynamical boundary condition:

$$\int_{\Omega^E} (\mathbf{N}^{\theta T} \circ \mathbf{q}^F) \cdot \nabla\, d\Omega = \int_{S_2^E} \mathbf{N}^{\theta T} (\mathbf{q}^E \cdot \mathbf{n})\, dS_2 =$$

$$= \int_{S_2^E} (\mathbf{N}^{\theta T} \hat{q}_n\, dS_2 + \int_{S_2^E} \mathbf{N}^{\theta T} a(\theta^E - \theta_\infty)\, dS_2. \tag{3.18}$$

By doing the integration in equation (3.16) term by term and taking equations (3.17) and (3.18), and modified Fourier's law into account we can derive the following equation:

$$\int_{\Omega^E} \mathbf{N}^{\theta T} \circ \nabla) k \nabla \theta^E\, d\Omega + \left(1 + \tau \frac{\partial}{\partial t}\right) \Big[\int_{\Omega^E} \mathbf{N}^{\theta T} \varrho c \dot{\theta}^E\, d\Omega - \int_{\Omega^E} \mathbf{N}^{\theta T} \varrho R\, d\Omega -$$

$$- \int_{\Omega^E} \mathbf{N}^{\theta T} \beta T_0 \dot{\mathbf{u}}^E \cdot \nabla d\Omega + \int_{S_2^E} \mathbf{N}^{\theta T} \hat{q}_n\, dS_2 + \int_{S_2^E} \mathbf{N}^{\theta T} a(\theta^E - \theta_\infty) dS_2 \Big] = \mathbf{0}. \tag{3.19}$$

Using the approximation (3.1) from equation (3.19) we get the form of the modified heat-conduction equation that is discretized in space:

$$\boldsymbol{M}^{\theta d} \ddot{\mathbf{d}}^E + \boldsymbol{M}^{\theta\theta} \ddot{\boldsymbol{\theta}}^E + \boldsymbol{C}^{\theta d} \dot{\mathbf{d}}^E + \boldsymbol{C}^{\theta\theta} \dot{\boldsymbol{\theta}}^E + \boldsymbol{K}^{\theta\theta} \boldsymbol{\theta}^E = \boldsymbol{\Phi}^\theta, \quad \text{where} \tag{3.20}$$

$$M_{ij}^{\theta d} = \int_{\Omega^E} \tau \beta T_0 \left[N_i^\theta \frac{\partial N_j^d}{\partial x} \middle| N_i^\theta \frac{\partial N_j^\theta}{\partial y} \right] d\Omega, \tag{3.21/a}$$

$$M_{ij}^{\theta\theta} = \int_{\Omega^E} \tau \varrho c N_i^\theta N_j^\theta\, d\Omega, \tag{3.21/b}$$

$$C_{ij}^{\theta d} = \int_{\Omega^E} \beta T_0 \left[N_i^\theta \frac{\partial N_j^d}{\partial x} \middle| N_i^\theta \frac{\partial N_j^d}{\partial y} \right] d\Omega, \tag{3.21/c}$$

$$i, j = 1, \ldots, n$$

$$C_{ij}^{\theta\theta} = \int_{\Omega^E} \varrho c N_i^\theta N_j^\theta \, d\Omega + \int_{S_2^E} \tau a N_i^\theta N_j^\theta \, dS_2, \tag{3.21/d}$$

$$K_{ij}^{\theta\theta} = \int_{\Omega^E} k \left(\frac{\partial N_i^\theta}{\partial x} \frac{\partial N_j^\theta}{\partial x} + \frac{\partial N_i^\theta}{\partial y} \frac{\partial N_j^\theta}{\partial y} \right) d\Omega + \int_{S_2^E} a N_i^\theta N_j^\theta \, dS_2, \tag{3.21/e}$$

$$\Phi_i^\theta = \int_{S_2^E} a \theta_\infty N_i^\theta \, dS_2 + \int_{\Omega^E} \varrho (R + \tau \dot{R}) N_i^\theta \, d\Omega - \int_{S_2^E} (\hat{q}_n + \tau \dot{\hat{q}}_n) N_i^\theta \, dS_2. \tag{3.21/f}$$

On the basis of equations (3.12) and (3.20) the matrix form referring to one element of the coupled field equations discretized in space:

$$\begin{bmatrix} M^{dd} & 0 \\ M^{\theta d} & M^{\theta\theta} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{d}}^E \\ \ddot{\boldsymbol{\theta}}^E \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ C^{\theta d} & C^{\theta\theta} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{d}}^E \\ \dot{\boldsymbol{\theta}}^E \end{bmatrix} + \begin{bmatrix} K^{dd} & K^{d\theta} \\ 0 & K^{\theta\theta} \end{bmatrix} \begin{bmatrix} \mathbf{d}^E \\ \boldsymbol{\theta}^E \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}^d \\ \boldsymbol{\Phi}^\theta \end{bmatrix}. \tag{3.22}$$

From the equations referring to the elements the so-called global matrix differential equation-system of the whole structure can be constituted:

$$\boldsymbol{M}\ddot{\boldsymbol{\delta}} + \boldsymbol{C}\boldsymbol{\delta} + \boldsymbol{K}\dot{\boldsymbol{\delta}} = \boldsymbol{\Phi}. \tag{3.23}$$

$\boldsymbol{\delta}$ is the vector of nodal unknowns, $\boldsymbol{M}$ the generalized mass matrix, $\boldsymbol{C}$ the generalized capacitance matrix, $\boldsymbol{K}$ the generalized stiffness/conductivity matrix and $\boldsymbol{\Phi}$ the generalized nodal force vector.

4.2. Discretization in time

Discretization in time can be best done by three node quadratic time finite elements. The time dependence of the nodal unknowns of division according to space can be approximated with the help of the nodal unknowns ($\boldsymbol{\delta}_{k-1}$, $\boldsymbol{\delta}_k$, $\boldsymbol{\delta}_{k+1}$) of the $2\Delta t$ length time finite element and the quadratic time shape functions (M_{k-1}, M_k, M_{k+1}):

$$\boldsymbol{\delta}(t) = M_{k-1}(t)\boldsymbol{\delta}_{k-1} + M_k(t)\boldsymbol{\delta}_k + M_{k+1}(t)\boldsymbol{\delta}_{k+1}. \tag{3.24}$$

If this approximation is inserted into equation (3.23) and the weighted residual method is applied by choosing various weight functions, then we get various time-schemes for the discretization of equation (3.23). From these the so-called quadratic Galerkin-scheme received through the application of the M_{k+1} weighting function shows the most favourable stability and accuracy characteristics [9]:

$$\left[M+\frac{3}{2}C\Delta t+\frac{4}{5}K(\Delta t)^2\right]\boldsymbol{\delta}_{k+1}=\left[2M+2C\Delta t-\frac{2}{5}K(\Delta t)^2\right]\boldsymbol{\delta}_k+$$

$$+\left[-M-\frac{1}{2}C\Delta t+\frac{1}{5}K(\Delta t)^2\right]\boldsymbol{\delta}_{k-1}-\frac{1}{5}(\Delta t)^2\boldsymbol{\Phi}_{k-1}+\frac{2}{5}(\Delta t)^2\boldsymbol{\Phi}_k+\frac{4}{5}(\Delta t)^2\boldsymbol{\Phi}_{k+1}, \tag{3.25}$$

where $\boldsymbol{\Phi}_{k-1}$, $\boldsymbol{\Phi}_k$, and $\boldsymbol{\Phi}_{k+1}$ are the values of the load vector in the nodal points of the time element. Using scheme (3.25) the solution of the matrix differential equation-system (3.23) can be reduced to the solution of the following linear system of equations

$$K_{eff}\boldsymbol{\delta}_{k+1}=\boldsymbol{\Phi}_{eff}. \tag{3.26}$$

The kinematic and prescribed temperature conditions have to be taken into account when solving the so-called effective equation-system (3.26).

With the help of the quadratic Galerkin-scheme we can determine the $\boldsymbol{\delta}_{k+1}$ value in the $(k+1)$ 'th time step if the value of the nodal unknowns of the $(k-1)$ and k'th steps are known. When starting out, the scheme, in case $k=1$, to calculate $\boldsymbol{\delta}_2$, $\boldsymbol{\delta}_0$ and $\boldsymbol{\delta}_1$ are needed. $\boldsymbol{\delta}_1$ can be produced by the application of the various starting methods from the vectors $\boldsymbol{\delta}_0$ and $\dot{\boldsymbol{\delta}}_0$ which are defined by the initial conditions. One of these methods is the following, so-called Crank—Nicholson method:

$$\left[\frac{2}{\Delta t}M+C+\frac{\Delta t}{2}K\right]\boldsymbol{\delta}_1=\left[\frac{2}{\Delta t}M+C-\frac{\Delta t}{2}K\right]\boldsymbol{\delta}_0+2M\dot{\boldsymbol{\delta}}_0+\frac{\Delta t}{2}[\boldsymbol{\Phi}_0+\boldsymbol{\Phi}_1]. \tag{3.27}$$

From the second step on with the help of scheme (3.25) the approximate solution of the matrix differential equation-system can be produced step by step.

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