STRESSES IN SYMMETRICALLY LOADED STEPPED CIRCULAR PLATES USING FINITE ELEMENT METHOD

M. G. SHEBL*—J. UJ

Department of Technical Mechanics
Technical University, H-1521 Budapest

Received November 10, 1987
Presented by Prof. Dr. Gy. Béda

Abstract

This paper deals with theoretical and numerical investigations for the stresses induced in circular plates with variable thickness and exposed by static loads. The effects of different types of loads on stresses are considered. The radial and hoop stresses are obtained for stepped circular plates widely used in industry. Two different forms of such plates are considered, one has a raised central region and the other has an unraised one. The results are obtained by finite element method using a tapered annulus bending element. The graphs presented in this work show the effect of the different values of step radius, type of the applied load and the effect of the different boundary conditions on the stresses from the point of view of design. When the step radius vanishes or equals to the plate radius, the plate becomes of uniform thickness. Consequently, the results obtained can be compared with the values of exact solutions. The limits of this simple finite element model for stepped plates are investigated by a more advanced finite element model having three dimensional axisymmetric annular element.

Keywords

Finite element method, tapered annulus bending element, radial and hoop stresses, stepped circular plates, symmetric transverse loads, boundary conditions.

* Assistant professor, Dept. of Mechanical Desing, Faculty of Eng. Tech., Helwan Univ., Cairo, Egypt.

Introduction

A considerable work has been done for the calculation of stresses induced in circular plates due to different loading and boundary conditions. The earliest research work was done by Timosenko [1], who found the mathematical expressions required in solving such problems for plates with different forms. The stepped plates are one of these plates which are widely used in engineering applications. Therefore, a complete stress analysis for such plates with different boundary conditions is carried out using finite element method.

A computer program having a tapered annulus bending element [2] is used in obtaining the stress results for plates with different step radii. Complete design charts for the maximum stresses induced in each plate are presented in this work to provide the designer with the optimal selection for plates which have minimum induced stresses.

Governing equations

Figures 1 shows the cross section for two particular forms of such stepped circular plates. These plates can be described by two zones I and II. Zone I represents a plate with uniform thickness h_1, while zone II represents annular plate with uniform thickness h_2. The common radius r_1 between the two zones is called the step radius while the radius a represents the outer plate radius.

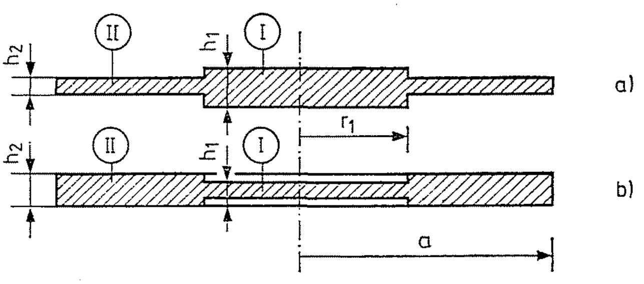

Fig. 1. Stepped Circular Plates
a) plate with central region raised; b) plate with central region unraised

Tapered Annulus Bending Element

The element used in computation is shown in Fig. 2. This element is qualified for computing the nodal displacements and stresses in circular plates of varying thickness under lateral loading. This loading can be a combination of uniform pressure and concentrated load.

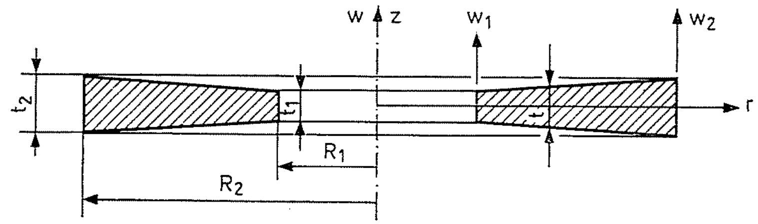

Fig. 2. Tapered Annulus Under Lateral Loading

The element is defined by two nodal circles, one internal and one external. It has two degrees of freedom per node, making a total of four degrees of freedom per element.

Since a linear variation in thickness of the element is assumed, the plate thickness at radius r can be determined by

$$t = mr + c, \tag{1}$$

where

$$m = \frac{t_2 - t_1}{R_2 - R_1} \quad \text{and} \quad c = \frac{t_1 R_2 - t_2 R_1}{R_2 - R_1}. \tag{2}$$

The assumed displacement function for lateral deflection w is given by

$$w = \alpha_1 + \alpha_2 r + \alpha_3 r^2 + \alpha_4 r^3 = [N(r)]^T[\alpha], \tag{3}$$

and the four nodal displacements per element are expressed in the form

$$[U^e] = [w_1 \theta_1 w_2 \theta_2]^T. \tag{4}$$

The relation between the unknown coefficients $\alpha_1, \alpha_2, \ldots, \alpha_4$ and the nodal displacement is given by

$$[U^e] = \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & R_1 & R_1^2 & R_1^3 \\ 0 & 1 & 2R_1 & 3R_1^2 \\ 1 & R_2 & R_2^2 & R_2^3 \\ 0 & 1 & 2R_2 & 3R^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = [C][\alpha], \tag{5}$$

then

$$[\alpha] = [C]^{-1}[U^e]. \tag{6}$$

Equation (3) can be written in the form

$$w(r) = [N(r)]^T[C^{-1}][U^e]. \tag{7}$$

The elastic strain energy in an element has the form [4]

$$U = \pi \int_{R_1}^{R_2} D\left(w''^2 + \frac{2\nu}{r} w' w'' + \frac{1}{r^2} w'^2\right) r\,dr, \tag{8}$$

where

$$D = Et^3/12(1-\nu^2).$$

By using the deflection expression given by Eq. 7. and its derivatives, the strain energy will be of form

$$U = \frac{1}{2}[U^e]^T[C^{-1}]^T[K^*][C^{-1}][U^e],$$

where

$$[K^*] = 2\pi \int_{R_1}^{R_2} D\left\{[N''][N'']^T + \frac{2\nu}{r}[N'][N'']^T + \frac{1}{r^2}[N'][N']^T\right\} r\,\mathrm{d}r, \tag{9}$$

and the

$$[K^e] = [C^{-1}]^T[K^*][C^{-1}]$$

is the element stiffness matrix.

If nodal displacements are known, the largest radial and hoop bending stresses can be computed by

$$\sigma_r = 6M_r/t^2, \quad \sigma_h = 6M_h/t^2 \tag{10}$$

where the bending moments are

$$\begin{bmatrix} M_r \\ M_h \end{bmatrix} = D\begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix}\begin{bmatrix} w'' \\ w'/r \end{bmatrix} = D\begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix}[B(r)][C]^{-1}[U^e] \tag{11}$$

where

$$[B(r)] = \begin{bmatrix} 0 & 0 & 2 & 6r \\ 0 & 1/r & 2 & 3r \end{bmatrix}.$$

Numerical results

Plates with single step as described in Fig. 1 are used to obtain the numerical results of stresses at different values of the step radius. The computations are performed at different types of boundary conditions and loading. The two thicknesses h_1 and h_2 for plates with central region raised are 4 and 2 mm, respectively. While in plates with central region unraised, h_1 and h_2 are 2 and 4 mm, respectively. The outer diameter for all plates is 300 mm, hence the computation processes are executed for plates with different step radius ratios r_1/a of 0, .2, .4, .6, .8 and 1. These plates are supposed to be made of steel with Young's modulus of $2 \cdot 10^{11}$ N/m^2 and Poisson's ratio of 0.3. Since the regions I and II shown in Fig. 1 are of uniform thickness, then t_1 and t_2 shown in Fig. 2 are equal for each element.

The applied transverse distributed load is taken 3500 N/m^2, while the concentrated load is chosen to be equal to the resultant load applied by the distributed load which is approximately 250 N. Two boundary conditions are considered, one for simple support and the other for clamped outer edge. The numerical results for plates are obtained by using 15 elements of uniform thickness.

In the case of distributed load, Figs 3 and 4 show the radial bending stresses induced in simply supported plates with raised and unraised central region, respectively. Figs 5 and 6 show the hoop stresses for these plates. Figures 7—10 show stress description as before, only for clamped supported stepped plates. In the case of concentrated load, Figs 11 and 12 show the radial bending stresses induced in simply supported stepped plates for the two forms of plates, while Figs 13 and 14 are concerned with the hoop stresses. Figures 15—18 represent the stress description for clamped plates.

Discussion and conclusions

Generally, the results shown in the figures are describing the stress distribution induced in stepped plates. The relation between these stresses and the radius ratio is divided into two behaviours. The first one shows the stresses in plate region I, while the other shows the stresses in plate region II. These behaviours are proportional to the corresponding uniform plates which have the same thicknesses. It is known that in case of concentrated load, the stress value at the plate centre is infinite and its numerical computation is impossible. Therefore, the stress at this region is computed at a radius of 10^{-6} m, which is very close to the plate centre.

Fortunately, when the plate radius ratio vanishes or equals to one, the plate becomes of uniform thickness.

The stress curves in this case are continous when these results can easily be checked with those of exact equations (1) in order to satisfy the validity of the solution procedure.

The stress curves are summarized here by very important graphs from the point of view of design. In the case of distributed loads, Figs 19 and 20 show the maximum radial and hoop bending stresses induced in the two forms of stepped plates for simply and clamped supported plates, respectively. Similarly, Figs 21 and 22 are the corresponding curves for concentrated load. From these curves we can see that there are critical points or critical step ratios which determine the optimal plate form having minimum stresses. Hence in the case of distributed load and for all boundary conditions, it is preferred to use plates with central region unraised until step radius ratio of 0.8 and vice versa after this value. However, in the case of concentrated load, the critical step radius for similar description is 0.1. Such results can be obtained for other circular plates with variable thicknesses which have more complicated forms.

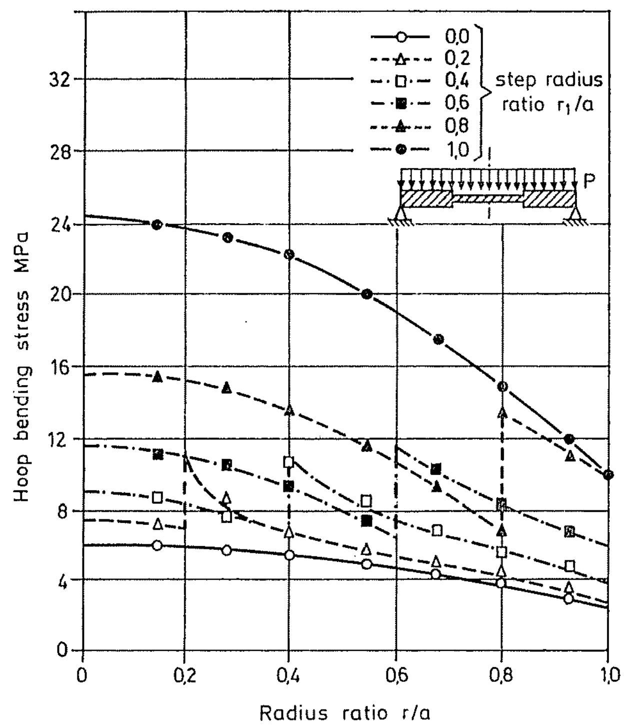

Fig. 3. Radial stresses in simply supported stepped circular plates with central region raised for distributed load

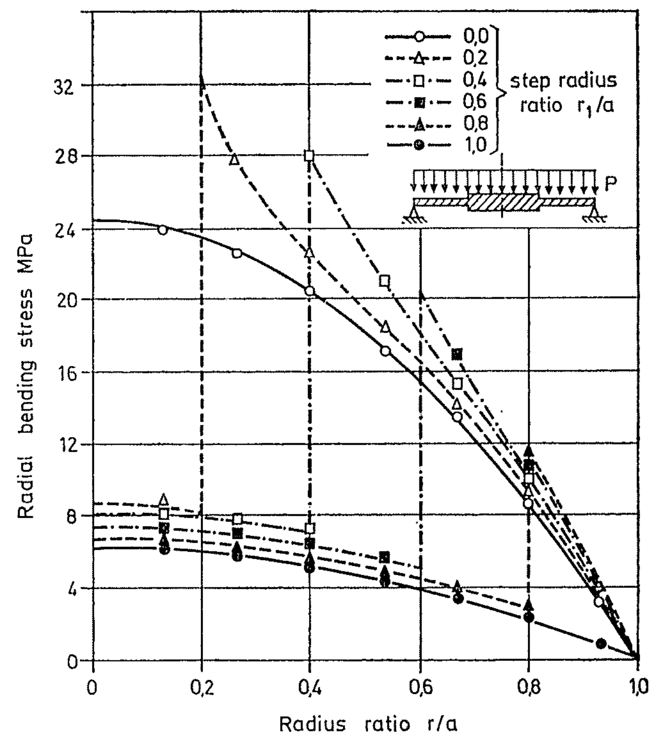

Fig. 4. Radial stresses in simply supported stepped circular plates with central region unraised for distributed load

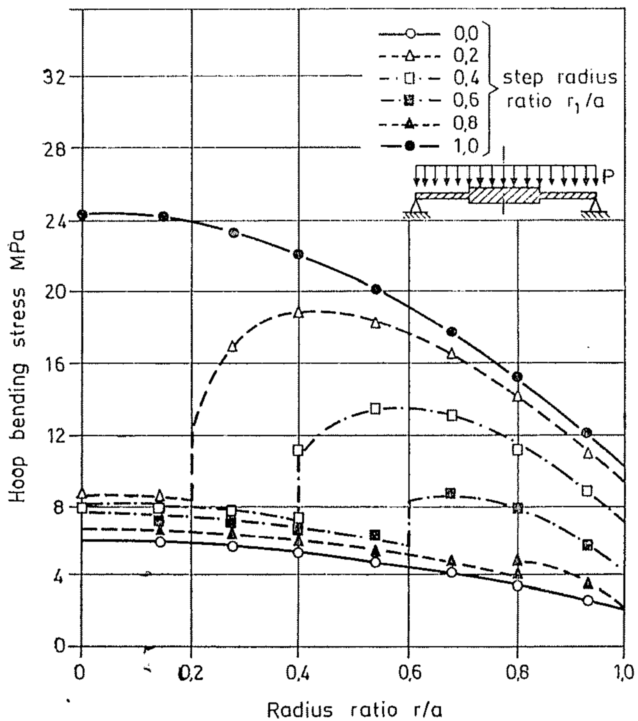

Fig. 5. Hoop stresses in simply supported stepped circular plates with central region raised for distributed load

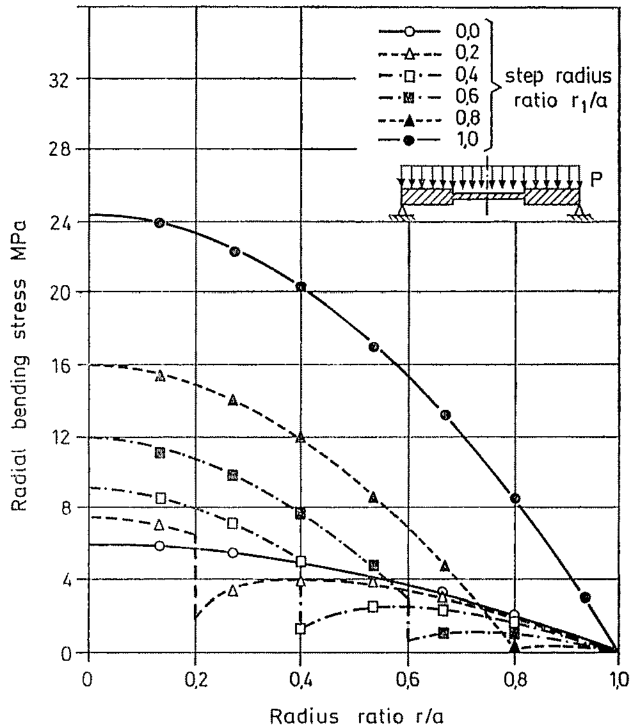

Fig. 6. Hoop stresses in simply supported stepped circular plates with central region unraised for distributed load

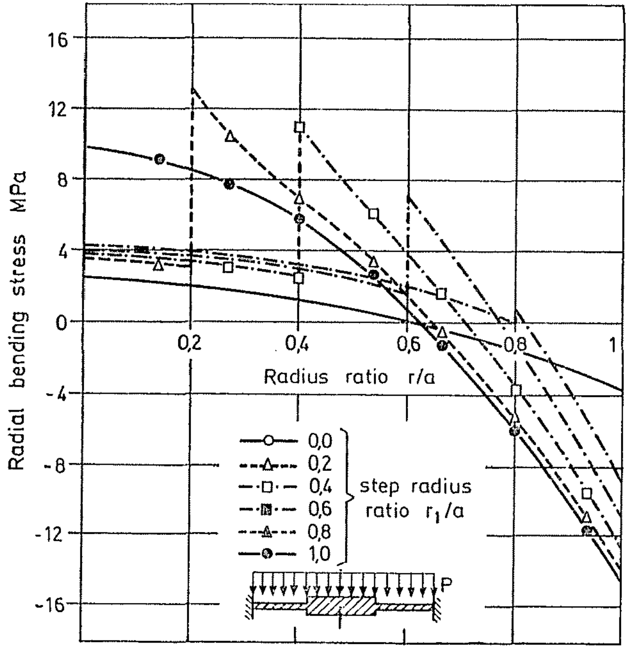

Fig. 7. Radial stresses in clamped (supported) stepped circular plates with central region raised for distributed load

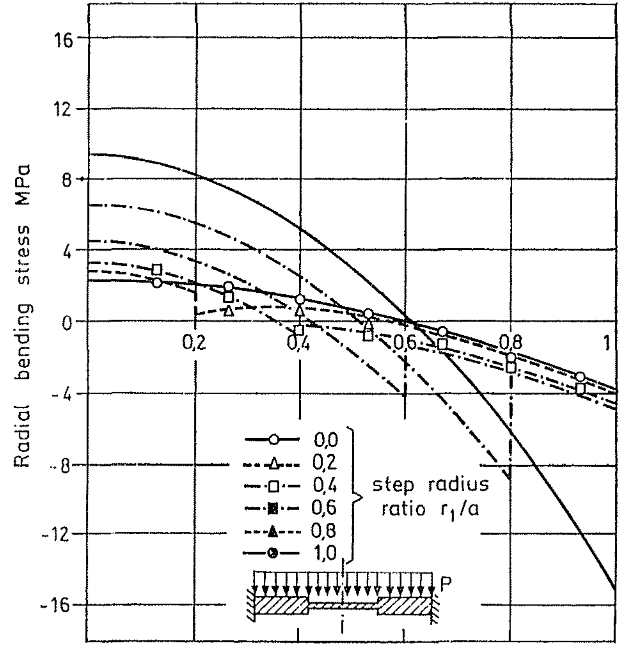

Fig. 8. Radial stresses in clamped (supported) stepped circular plates with central region unraised for distributed load

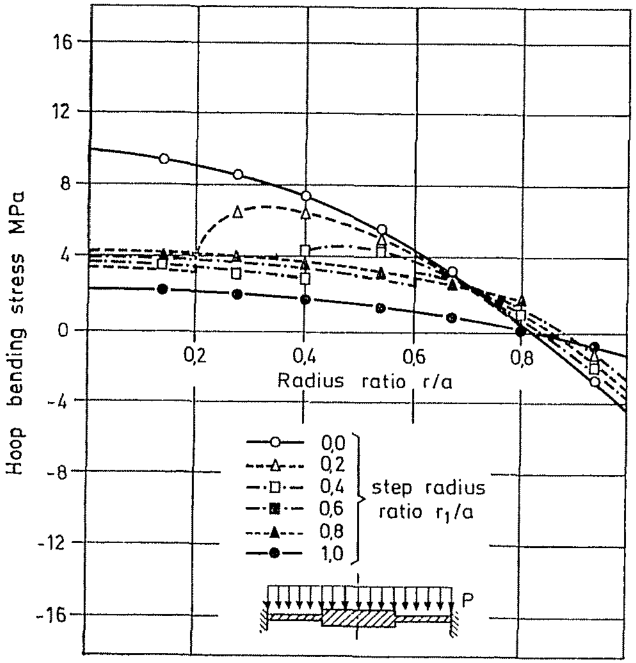

Fig. 9. Hoop stresses in clamped (supported) stepped circular plates with central region raised for distributed load

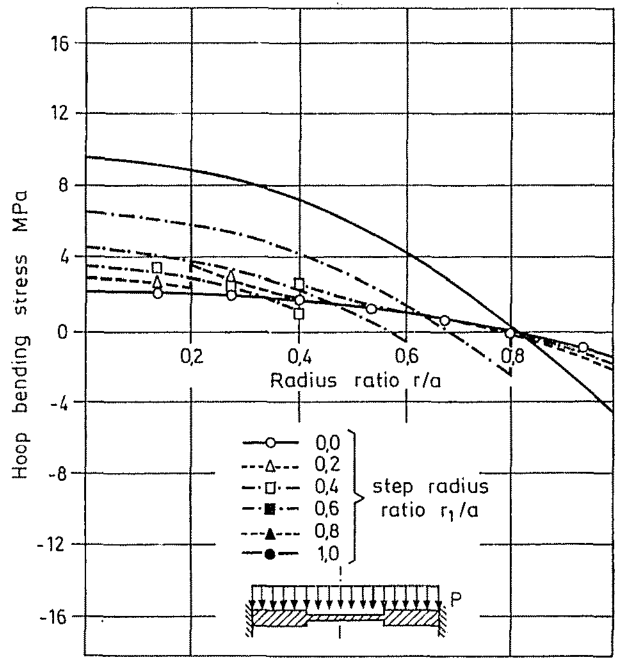

Fig. 10. Hoop stresses in clamped (supported) stepped circular plates with central region unraised for distributed load

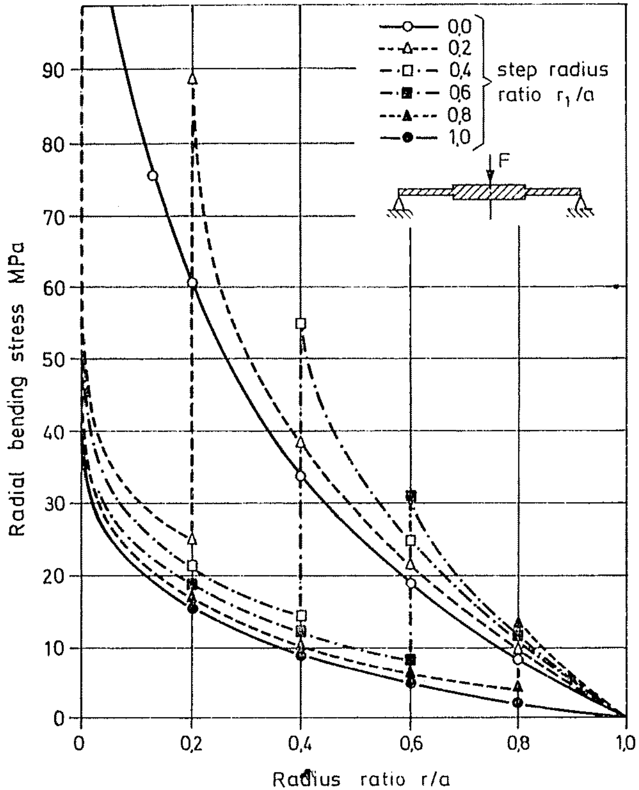

Fig. 11. Radial stresses in simply supported stepped circular plates with central region raised for concentrated load

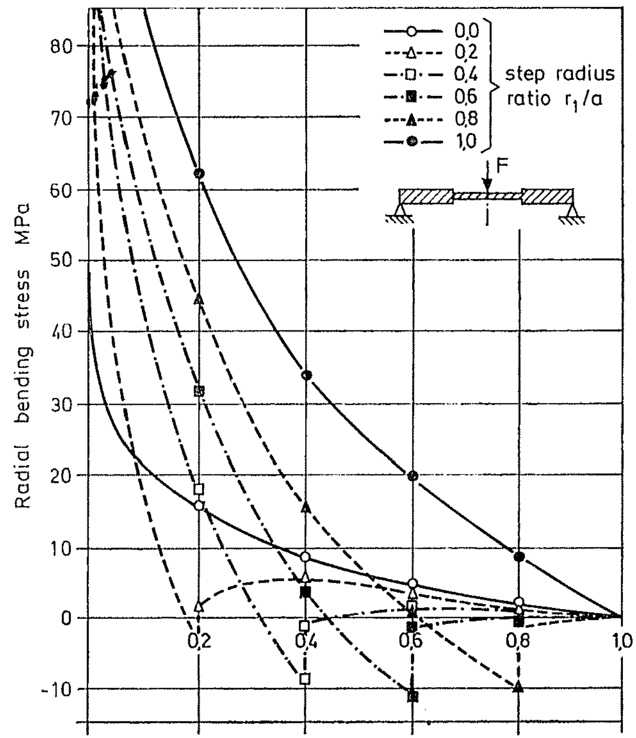

Fig. 12. Radial stresses in simply supported stepped circular plates with central region unraised for concentrated load

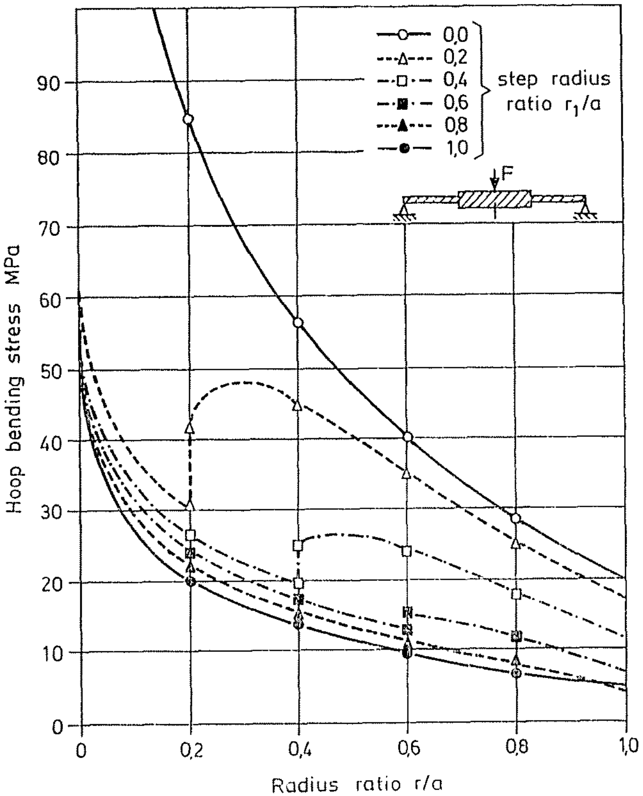

Fig. 13. Hoop stresses in simply supported stepped circular plates with central region raised for concentrated load

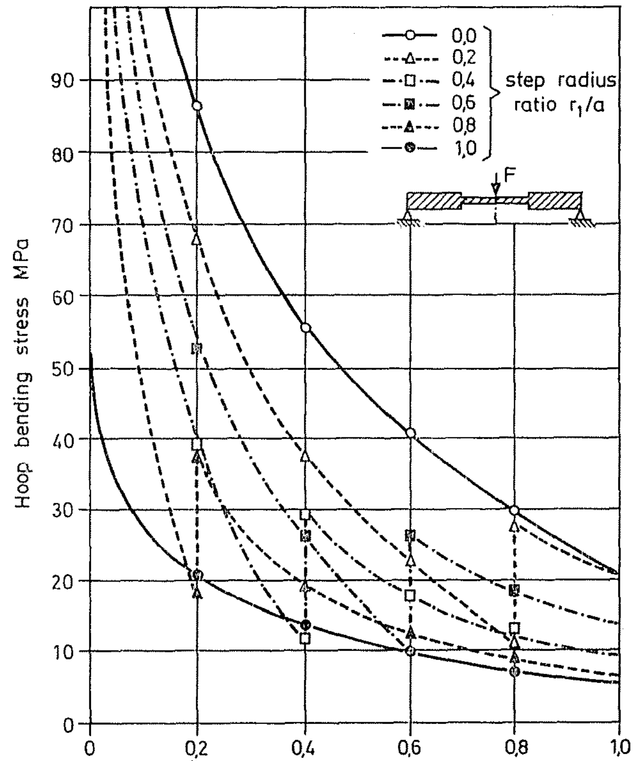

Fig. 14. Hoop stresses in simply supported stepped circular plates with central region unraised for contrentrated load

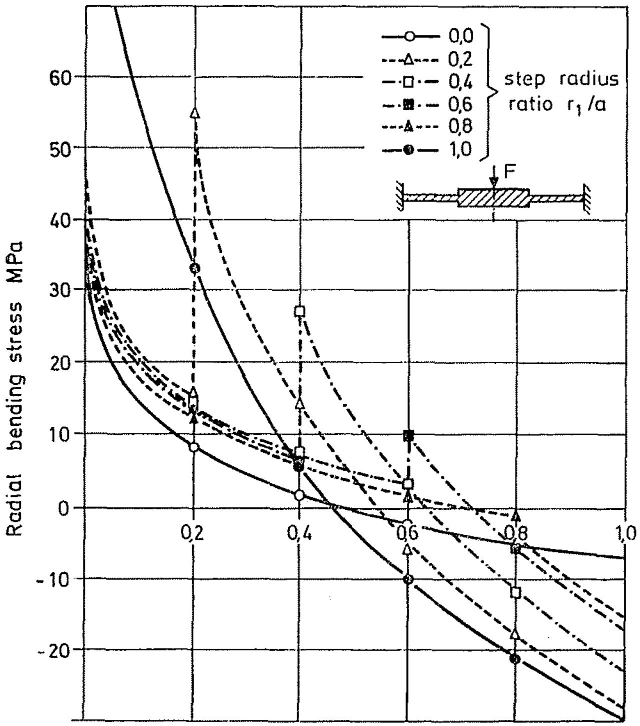

Fig. 15. Radial stresses in clamped stepped circular plates with central region raised for contrentrated load

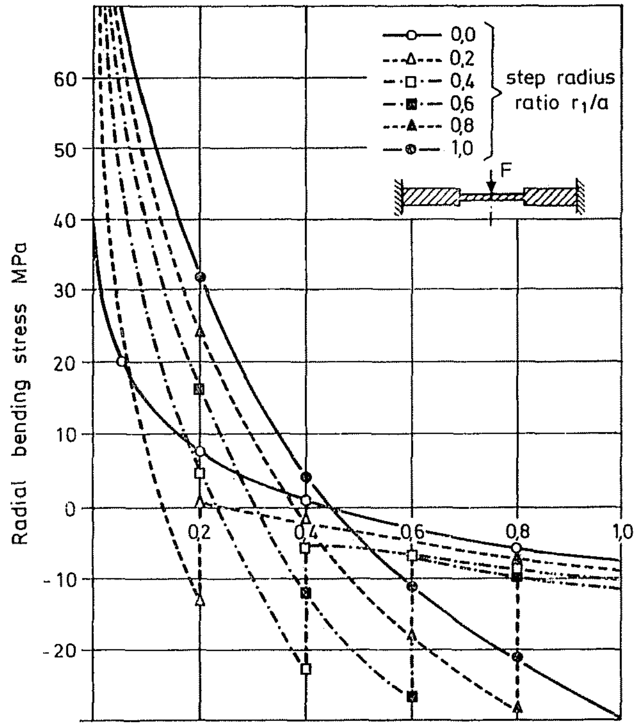

Fig. 16. Radial stresses in clamped stepped circular plates with central region unraised for contrentrated load

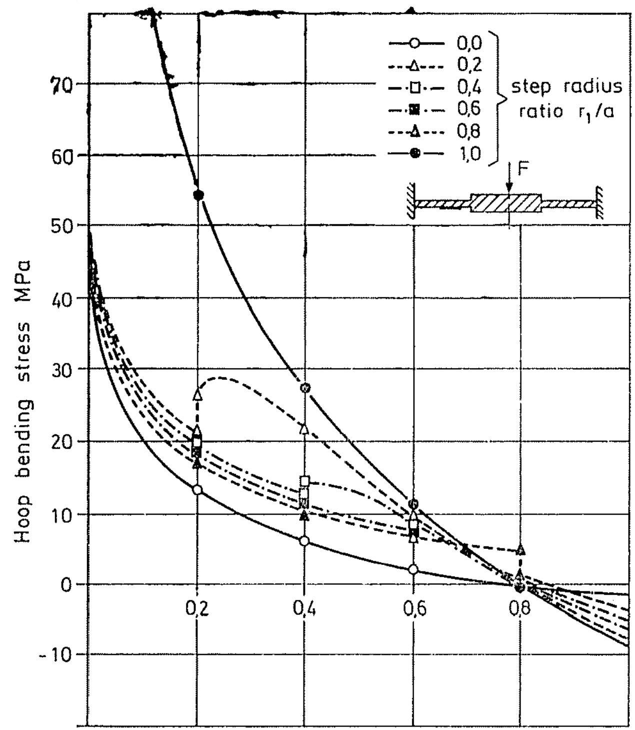

Fig. 17. Hoop stresses in clamped stepped circular plates with central region region raised for contrentrated load

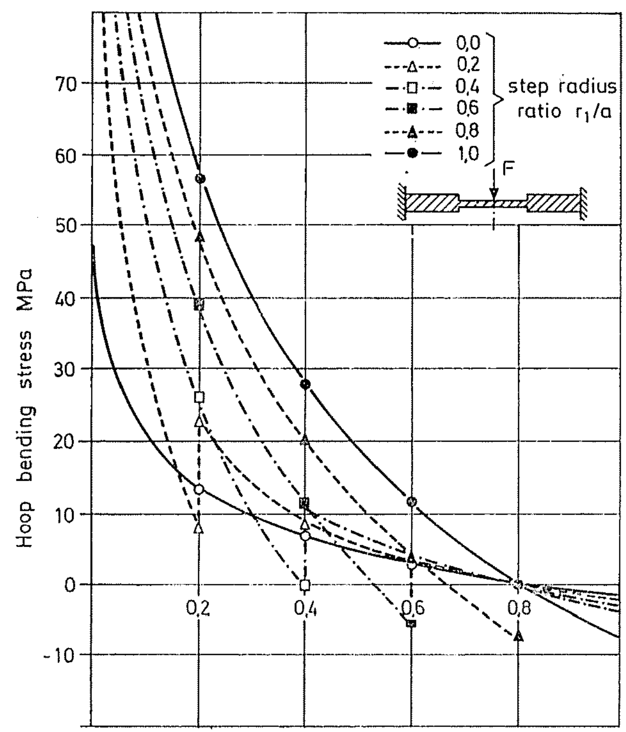

Fig. 18. Hoop stresses in clamped stepped circularl pates with central region unraised for contrentrated load

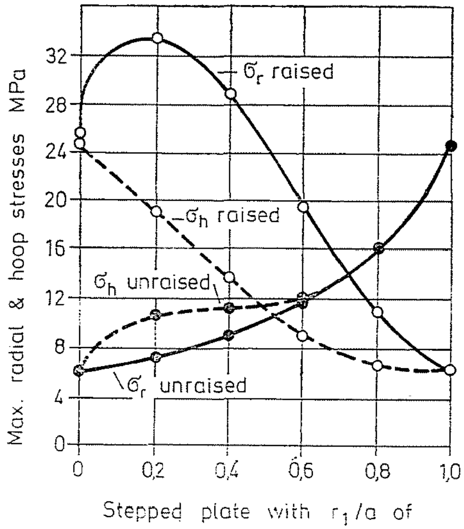

Fig. 19. Maximum radial and hoop bending stresses for distributed loads. Simply supported plates

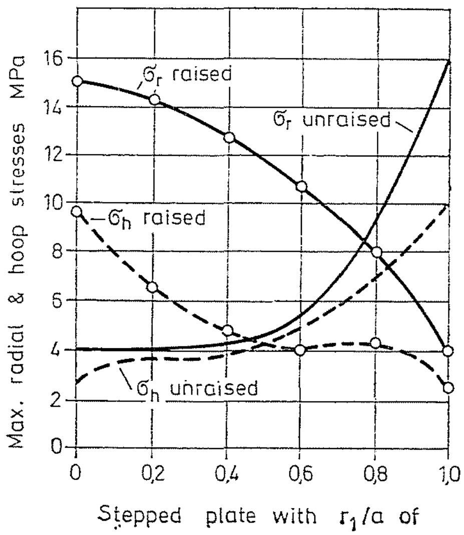

Fig. 20. Maximum radial and hoop bending stresses for distributed loads. Clamped plates

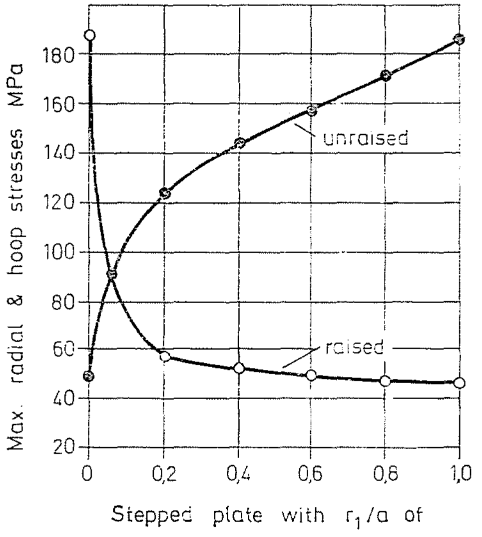

Fig. 21. Maximum radial and hoop stresses for contrentrated load. Simply supported plates

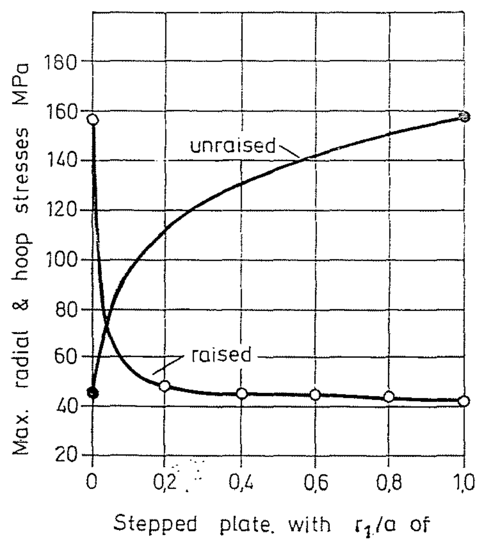

Fig 22. Maximum radial and hoop stresses for contrentrated load. Clamped plates

Nomenclature

E	modulus of elasticity,
h_1, h_2	thicknesses of zones I and II of the stepped plate,
M_h, M_r	hoop and radial bending stresses,
r	plate radius,
R_1, R_2	radii at internal and external nodal circles,
t	plate thickness at radius r,
t_1, t_2	thicknesses at internal and external nodal circles,
w	transverse plate displacement,
w_1, w_2	nodal transverse displacements at internal and external nodal circles,
Θ_1, Θ_2	angular displacements at internal and external nodal circles,
ν	Poisson's ratio,
σ_h, σ_r	hoop and radial bending stresses.

References

1. TIMOSENKO, S.: "Theory of Plates and Shells". McGraw-Hill, 2nd edition, 1959.
2. ROSS, C. T. F.: "Finite Element Programs for Axisymmetric Problems in Engineering". Ellis Horwood limited, 1984.
3. ROSS, C. T. F.: "Axisymmetric Finite Elements for Circular Plates and Cylinders". Trans. R. T. N. A. *116*, 1974.
4. JAEGER, L. G.: "Elementary Theory of Elastic Plates". Pergamon Press, Oxford, 1964.

M. G. SHEBL Helwan Univ. Cairo, Egypt.
Dr. József ÚJ H-1521, Budapest