# ANALYSIS OF THE GEOMETRICAL CONDITION OF CONTACT

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## Abstract

Hertz's solution for contact problems is valid for the contact between bodies with rigid-body or elastic displacements along the symmetry axis; furthermore, certain assumptions are met.

Geometrical relationships for the contact between bodies in arbitrary rigid-body motion are written and analyzed for elastic displacements of arbitrary direction, and an initial gap. The presented example of application refers to contacting teeth of cylindrical gears.

## Introduction

Contact state between two structural members may be analyzed by means of contact theories known from the literature, under more or less approximative assumptions. The best known of them is Hertz's theory [1] for relatively simple contact problems, under the following assumptions:

a) the theory of small deformations is valid;
b) about the contact region, surfaces of contacting solids can be replaced by second-order surfaces;
c) the distributed force system between bodies is of normal direction;
d) the contacting bodies are of a homogeneous, isotropic material obeying Hooke's law;
e) body surfaces are perfectly smooth;
f) displacements of contacting points can be calculated from the problem of elastic half-space loaded by a distributed force system;
g) elastic displacements of points of the contact region are coincident with the direction of "rigid-body" approach of the solids;
h) the contact pressure distribution is proportional to ordinates of a half-ellipsoid.

In recent decades, several contact theories arose (surveyed e. g. in [2], [5]), involving even less assumptions compared to Hertz's contact theory.

In analyzing the contact problems, mostly two relationships (a stress and a geometry type) are applied. The stress assumption refers to the force system distributed over the interface; the other relates rigid-body and elastic displacements of the contacting bodies with the initial gap. This paper will only be concerned with this latter geometry relationship.

Among geometry assumptions in Hertz's theory, b) could be omitted by several authors (e. g. [2], [4], etc.) by means of various numerical procedures. While f) involves in-plane contact region. Examples for omitting this assumption are found, among others, in [3] and [8] for two-dimensional cases, for translational rigid-body displacements.

In the followings, interpretation of terms and symbols is illustrated on hand of the Hertz problem, then the geometry equation of contact will be written for the tridimensional rigid-body displacement, taking also translation and angular rotation into consideration. The presented example illustrates relationships for contacting bodies in mechanics such as gears.

## "One-dimensional" geometry relationship for the Hertzian contact problem

"One-dimensional" is to say that rigid-body displacements of the two bodies, and elastic displacements of the contact points are parallel to the (common) symmetry axis of the bodies, hence they can be described by scalar magnitudes.

To present the geometry relationship for contact, let us introduce the following coordinate systems:

- Let us connect orthogonal coordinate systems $\hat{x}_1$, $\hat{y}_1$, $\hat{z}_1$, and $\hat{x}_2$, $\hat{y}_2$, $\hat{z}_2$ denoted $R1$, and $R2$, resp., to points of both bodies rather far from the contact region, where elastic displacements due to contact pressure distribution are negligible. Assume coordinate axes so that axes $\hat{y}_1$ and $\hat{y}_2$ point in directions opposite to the contact region for both bodies;
- let orthogonal coordinate system $M$ be fixed to a given point in space. This reference system illustrates the rigid-body displacement of the bodies, that is, other than elastic displacements of points of the two bodies far from the contact region.

Point displacements can be described by vectors with components to be produced in coordinate systems $M$, $R1$ and $R2$. This problem being a "one-dimensional" one, it is more convenient to involve vector moduli as did Hertz. Bodies in the initial, point-like contact condition are seen in Fig. 1a. Let $h_1+h_2$ be the distance between solids $P_1$ and $P_2$, to contact later, where $h_1$ and $h_2$ are moduli of position vectors of points $P_1$ and $P_2$ in $M$, respectively. Figure 1b shows an imaginary position of the two points where they are subject to a contact pressure distribution under load, but having not yet performed the rigid-body displacement needed to be in contact. Let $\hat{u}_1$ and $\hat{u}_2$ be moduli of elastic displacements due to contact pressure distribution arising in coordinate systems $R1$ and $R2$, respectively. Real positions of the two bodies under load (Fig. 1c) arise from rigid-body displacements $m_1$ and $m_2$ until their points meet. These displacements can be determined by measuring displacements of $R1$ and $R2$ in $M$. Points contacting upon $m_1$ and $m_2$ are clearly spaced at zero, hence in the imaginary position according to Fig. 1b they are at a distance $m_1+m_2$, since according to the system of Hertzian assumptions, bodies are only able to translational rigid-body displacement. Accordingly, absolute displacement values in coordinate systems $R$ are the same as in $M$, thus $u_1=\hat{u}_1$, $u_2=\hat{u}_2$ ($u_1$ and $u_2$ are elastic displacement values in $M$).

Accordingly, with notations in Fig. 1:

$$h_1+h_2+u_1+u_2=m_1+m_2 \tag{1}$$

In the one-dimensional case, computation with absolute values leads to obvious, simple relationships. But how to describe points displacements for bodies where rigid-body approach cannot be described with a single

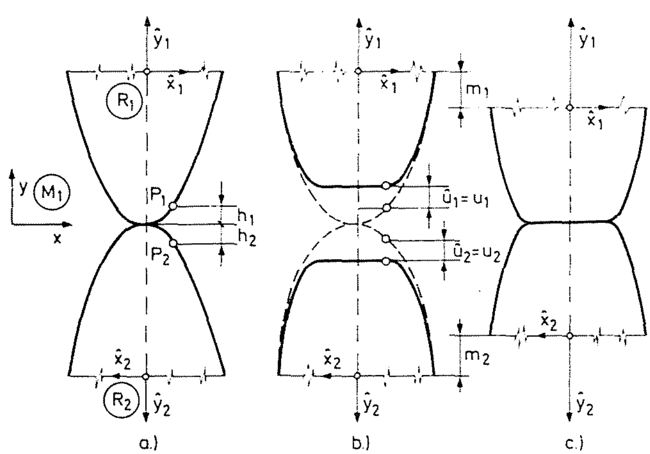



*Fig. 1.* Interpretation of initial gap, elastic and rigid-body displacements

direction? Namely, cases can be realized where contacting bodies approach by complex displacements, such as the common case in mechanical engineering (e. g. that of gears) where bodies engage by revolution. To pass the system of Hertzian assumptions is possible by applying vectorial description. Obviously, since displacements are considered in different reference systems, their relative displacements and position changes have to be exactly observed.

## General geometrical relationship of contact

Let us apply the relationship current in continuum mechanics that an arbitrary magnitude can be produced in a given coordinate system if it is known in an other, also given coordinate system, and so is the relative displacement of the two systems [7]. Let displacements of points of a body be restricted to. Let coordinate system $R$ be that where point displacements are observed, and $M$ where they are to be produced. In the following, magnitudes in $R$ will be capped by a circumflex (1) (Fig. 2).

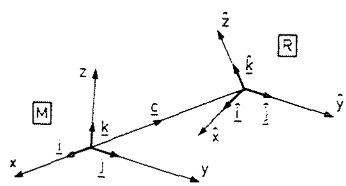



*Fig. 2.* Relationship between coordinate systems $M$ and $R$

Consider a point of the tested body, with place vectors $\mathbf{r}(x, y, z)$, and $\hat{\mathbf{r}}(\hat{x}, \hat{y}, \hat{z})$, in $M$, and $R$, resp., where $x, y, z$, and $\hat{x}, \hat{y}, \hat{z}$ are orthogonal coordinates in $M$, and in $R$. According to the theory of linear transformations, the two vectors are related as:

$$\mathbf{r}(x, y, z) = \boldsymbol{B}\hat{\mathbf{r}}(\hat{x}, \hat{y}, \hat{z}) + \mathbf{c}(c_x, c_y, c_z)\,, \tag{2}$$

where $\boldsymbol{B}$ is the rotation matrix, and $\mathbf{c}$ the shift vector. Elements of $\boldsymbol{B}$ are direction cosinuses of basis vectors of system $R$:

$$\boldsymbol{B} = \begin{bmatrix} \cos\alpha_{\hat{x}x} & \cos\alpha_{\hat{x}y} & \cos\alpha_{\hat{x}z} \\ \cos\alpha_{\hat{y}x} & \cos\alpha_{\hat{y}y} & \cos\alpha_{\hat{y}z} \\ \cos\alpha_{\hat{z}x} & \cos\alpha_{\hat{z}y} & \cos\alpha_{\hat{z}z} \end{bmatrix},$$

where coordinate axes, that is, basis vectors include angles $\alpha$ with each of the axes of $M$. Coordinates of vector $\mathbf{c}$ coordinates of the origin of system $R$ in $M$.

Let the considered point move from the starting point $P$ to a point $P'$. In the meantime, in the general case, $R$ partly rotates, partly translates in $M$. Magnitudes after the change are $\mathbf{r} + \Delta\mathbf{r}$, $\hat{\mathbf{r}} + \Delta\hat{\mathbf{r}}$, $\boldsymbol{B} + \Delta\boldsymbol{B}$, and $\mathbf{c} + \Delta\mathbf{c}$. Now, position vectors are related as:

$$\mathbf{r} + \Delta\mathbf{r} = (\boldsymbol{B} + \Delta\boldsymbol{B})(\mathbf{r} + \Delta\mathbf{r}) + \mathbf{c} + \Delta\mathbf{c}\,. \tag{3}$$

The point displacement equals, of course, the difference between position vectors before and after the change. Deducing from each other equations for the two positions, and neglecting secondary infinitesimal magnitudes $\Delta \boldsymbol{B} \cdot \Delta \mathbf{r}$ (small displacements being concerned with) yields:

$$\Delta \mathbf{r} = \boldsymbol{B} \cdot \Delta \hat{\mathbf{r}} + \Delta \boldsymbol{B} \cdot \hat{\mathbf{r}} + \Delta \mathbf{c} \tag{4}$$

Continuum mechanical considerations make it obvious that the most general motion of a body can be decomposed to a rigid-body and an elastic displacement, in arbitrary sequence, so for the displacement of any point of the body, a rigid-body and an elastic part may be spoken of [10].

Assume the body to have a rigid part. Fixing a coordinate system to this part will cause the other points performing elastic displacements of the body there to displace, by just as much as the total elastic displacement. While this displacement in $M$ will be the total displacement of the body points. The rigid part, hence system $R$ performs a displacement in $M$, identical to the rigid-body displacement of the points.

Displacement of the considered point in $M$ is $\Delta \mathbf{r}$, the total displacement, its displacement in $R$ is $\Delta \hat{\mathbf{r}}$, the elastic displacement, $\Delta \boldsymbol{B} \cdot \hat{\mathbf{r}} + \Delta \mathbf{c}$ describes displacement of $R$ in $M$, the rigid-body part. To discuss also elastic displacement in $M$ requires transformation $\boldsymbol{B} \cdot \Delta \hat{\mathbf{r}}$. Conclusively:

— displacements in $M$:
  total displacement—$\Delta \mathbf{r}$
  elastic displacement—$\boldsymbol{B} \cdot \Delta \hat{\mathbf{r}}$
  rigid-body displacement—$\Delta \boldsymbol{B} \cdot \hat{\mathbf{r}} + \Delta \mathbf{c}$
— displacement in $R$ is elastic displacement $\Delta \hat{\mathbf{r}}$.

In the followings, notations

$$\mathbf{m} = \Delta \boldsymbol{B} \cdot \hat{\mathbf{r}} + \Delta \mathbf{c}\,, \tag{5}$$

$$\hat{\mathbf{u}} = \Delta \hat{\mathbf{r}} \tag{6}$$

$$\mathbf{u} = \boldsymbol{B} \cdot \Delta \hat{\mathbf{r}} \tag{7}$$

will be applied. Decomposition of the point displacement in shown in Fig. 3.

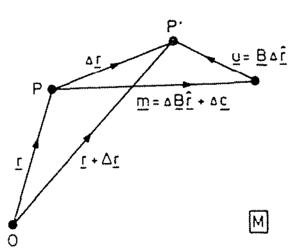



*Fig. 3.* Decomposition of the displacement of point $P$

It should be noted that in contact analyses relying on elastic half-space models, "infinitely distant" points or such "distant from the force application" and surroundings perform only rigid-body motions, and it is here that coordinate system $R$ is usually fixed. For contact procedures relying on inelastic half-space models, the elastic coordinate system has to be located in harmony with the support of the body.

In the analysis of contact between two bodies, it is convenient to assume three coordinate systems, where one is fixed to the rigid part of each body, and one to a stable point in space. With the actual notations, these reference systems are $R1$, $R2$, and $M$.

Displacements before, and after decomposition of an arbitrary point pair in contact are seen in Figs 4 and 5, respectively.

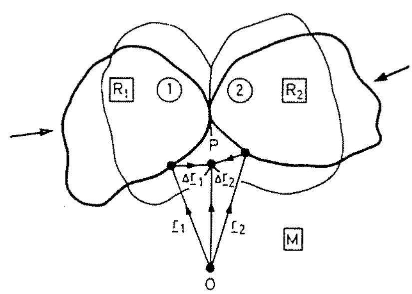



*Fig. 4.* Displacements of contacting points of each of two solids

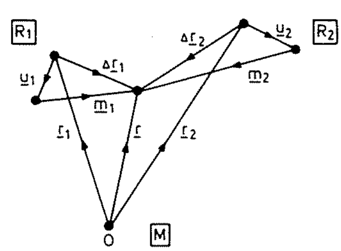



*Fig. 5.* Decomposition of displacements of contacting points

Vectorial equation of contact:

$$\mathbf{r}_1 + \Delta\mathbf{r}_1 = \mathbf{r} = \mathbf{r}_2 + \Delta\mathbf{r}_2 , \tag{8}$$

that is:

$$\mathbf{r}_1 + \mathbf{u}_1 + \mathbf{m}_1 = \mathbf{r}_2 + \mathbf{u}_2 + \mathbf{m}_2 . \tag{9}$$

In a more current form:

$$\mathbf{h} = \mathbf{u}_2 - \mathbf{u}_1 + \mathbf{m}_2 - \mathbf{m}_1 \tag{10}$$

where $\mathbf{h}=\mathbf{r}_1-\mathbf{r}_2$, the initial gap vector. In this form it expresses that the distance between the starting and the final contact positions of the points equals the difference between their elastic and rigid-body displacements.

It is simpler than the vectorial equation to reckon with its scalar product for a given direction, namely then a single scalar equation arises. Let $\mathbf{n}$ be the unit vector of the given direction, then

$$\mathbf{h}\cdot\mathbf{n}=\mathbf{u}_2\cdot\mathbf{n}-\mathbf{u}_1\cdot\mathbf{n}+\mathbf{m}_2\cdot\mathbf{n}-\mathbf{m}_1\cdot\mathbf{n}\,, \tag{11}$$

thus:

$$h^n=u_2^n-u_1^n+m_2^n-m_1^n \tag{12}$$

In the followings, $h$ will be called the initial gap vector, namely it describes the initial gap between the contacting bodies before deformation. Coordinates of this vector can be calculated from geometries and relative positions of the bodies.

Determination of elastic displacements $\mathbf{u}_1$ and $\mathbf{u}_2$ relies on analytic or numerical (e. g. coefficient matrix) methods. In non-Hertzian procedures, often iteration algorithms are applied (e. g. [1], [2], [9]). An increasing number of contact methods apply the finite element method.

In current contact analyses, bodies perform tangential advancement, described by vector $\Delta\mathbf{c}_2-\Delta\mathbf{c}_1$. Matrices $\Delta\boldsymbol{B}_1$ and $\Delta\boldsymbol{B}_2$ permit also to consider angular motions of solids.

Equation (12) may be produced mechanically. The necessary steps are recapitulated in a possible sequence as follows:

1) Definition of coordinate systems:
   a) determination of rigid parts of the bodies;
   b) origins $(O_M, O_{R1}, O_{R2})$;
   c) axes $(x_M, y_M, z_M; \hat{x}_{R1}, \hat{y}_{R1}, \hat{z}_{R1}; \hat{x}_{R2}, \hat{y}_{R2}, \hat{z}_{R2})$;
2) characteristics of pre-deformational positions of coordinate systems (superscript "b"):
   a) $R1$ in $M$: $\boldsymbol{B}_1^b$, $\mathbf{c}_1^b$;
   b) $R2$ in $M$: $\boldsymbol{B}_2^b$, $\mathbf{c}_2^b$;
3) characteristics of post-deformational positions of coordinate systems (superscript "a"):
   a) $R1$ in $M$: $\boldsymbol{B}_1^a$, $\mathbf{c}_1^a$;
   b) $R2$ in $M$: $\boldsymbol{B}_2^a$, $\mathbf{c}_2^a$;
4) change of coordinate system positions:
   a) $R1$: $\Delta\boldsymbol{B}_1=\boldsymbol{B}_1^a-\boldsymbol{B}_1^b$, $\Delta\mathbf{c}_1=\mathbf{c}_1^a-\mathbf{c}_1^b$;
   b) $R2$: $\Delta\boldsymbol{B}_2=\boldsymbol{B}_2^a-\boldsymbol{B}_2^b$, $\Delta\mathbf{c}_2=\mathbf{c}_2^a-\mathbf{c}_2^b$;
5) position vectors:
   a) in $R1$: $\hat{\mathbf{r}}_1$;
   b) in $R2$: $\hat{\mathbf{r}}_2$
   c) in $M$: $\hat{\mathbf{r}}_1=\boldsymbol{B}_1\mathbf{r}_1+\mathbf{c}_1$; $\hat{\mathbf{r}}_2=\boldsymbol{B}_2\mathbf{r}_2+\mathbf{c}_2$

6) gap vector
   a) $\mathbf{h}=\mathbf{r}_1-\mathbf{r}_2$
7) elastic displacements
   a) in $R1$: $\hat{\mathbf{u}}_1$
   b) in $R2$: $\hat{\mathbf{u}}_2$
   c) in $M$: $\mathbf{u}_1=\boldsymbol{B}_1\hat{\mathbf{u}}_1$; $\mathbf{u}_2=\boldsymbol{B}_2\hat{\mathbf{u}}_2$
8) rigid-body displacements
   a) $\mathbf{m}_1=\Delta\boldsymbol{B}_1\cdot\hat{\mathbf{r}}_1+\Delta\mathbf{c}_1$
   b) $\mathbf{m}_2=\Delta\boldsymbol{B}_2\cdot\hat{\mathbf{r}}_2+\Delta\mathbf{c}_2$
9) defining projectional directions, computation of scalar products
10) writing vectorial or scalar equations:

$$\mathbf{h}=\mathbf{u}_2-\mathbf{u}_1+\mathbf{m}_2-\mathbf{m}_1 \tag{13}$$

$$h^i=u_2^i-u_1^i+m_2^i-m_1^i \tag{14}$$

($i$ denoting scalar values for direction $i$).

## Contact geometry equation for cylindrical gears

Let us consider the engagement between two gears with straight teeth (Fig. 6).

The gears are assumed to be fastened to infinitely stiff shafts. Thus, rigid parts of both gears are the respective shafts.

Let origin $O_M$ of spatial coordinate system $M$ be axial to the shaft of gear 2 and coincide with origin $O_{R2}$ of system $R2$, and let origin $O_{R1}$ of coordinate system $R1$ on the axis of gear 1 be in the same front plane. Spacing $O_{R1}O_{R2}$ is shaft distance $a_W$. For the sake of clearness, rotation axis of gear 1 bearing origin of axes $y_1$ and $z_1$ has been omitted from Fig. 6.

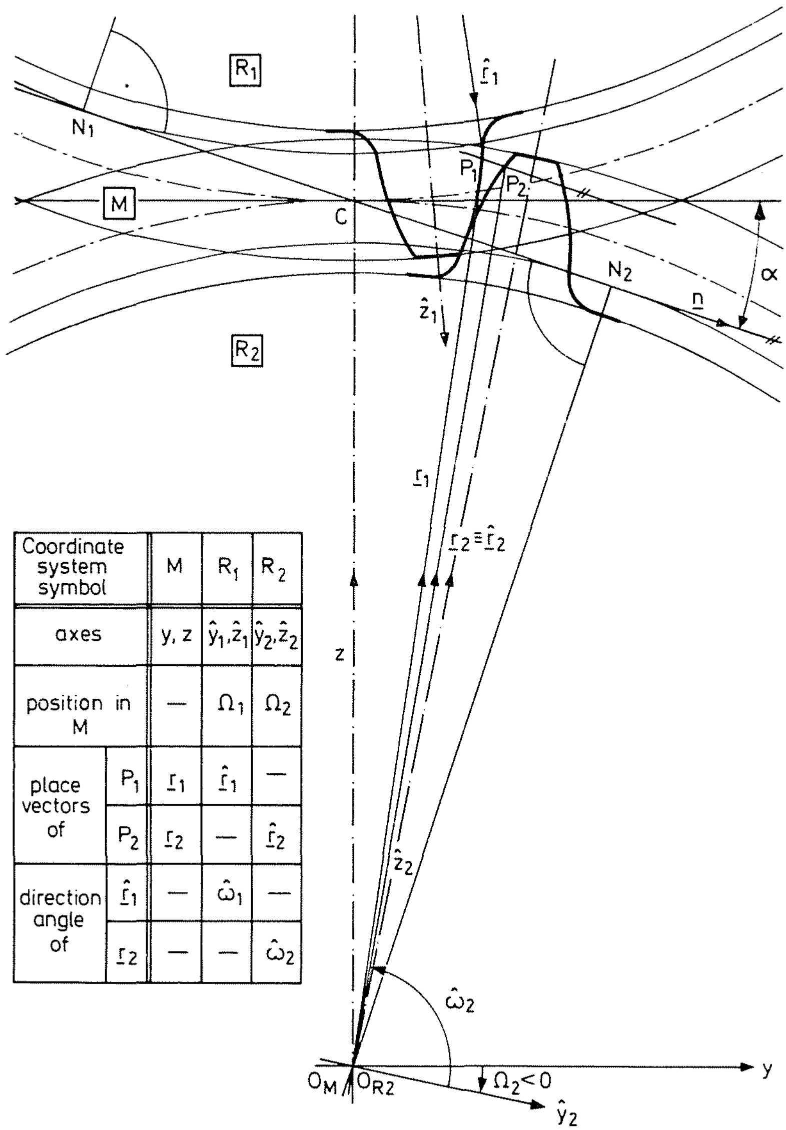
Let planes $(y, z)$ of all three coordinate systems lie in the front plane defined by $O_{R2}$. Now, axes $x$, $\hat{x}_1$, $\hat{x}_2$ are parallel, besides, $x$ and $\hat{x}_2$ are coincident. Positions of coordinate systems $R1$ and $R2$ in $M$ are defined by angles $\Omega_1$ and $\Omega_2$, respectively. Angles have been affected by signs conform to right-twisted coordinate systems.

In the followings, the approximation of a plane deformation state in contact will be applied [8]. So only components in directions $y$ and $z$ will be written.

Angles characterizing the pre-deformational position of coordinate system $R1$ are:

$$\alpha_{\hat{y}y}^1=180^\circ+\Omega_1\,; \qquad \alpha_{\hat{z}y}^1=270^\circ+\Omega_1$$

$$\alpha_{\hat{y}z}^1=90^\circ+\Omega_1\,; \qquad \alpha_{\hat{z}z}^1=180^\circ+\Omega_1\,,$$




| Coordinate system symbol | | M | $R_1$ | $R_2$ |
|---|---|---|---|---|
| axes | | y, z | $\hat{y}_1, \hat{z}_1$ | $\hat{y}_2, \hat{z}_2$ |
| position in M | | — | $\Omega_1$ | $\Omega_2$ |
| place vectors of | $P_1$ | $\underline{r}_1$ | $\hat{\underline{r}}_1$ | — |
| | $P_2$ | $\underline{r}_2$ | — | $\hat{\underline{r}}_2$ |
| direction angle of | $\hat{\underline{r}}_1$ | — | $\hat{\omega}_1$ | — |
| | $\underline{r}_2$ | — | — | $\hat{\omega}_2$ |

*Fig. 6.* Coordinate systems for interpreting displacements in cylindrical gear engagement

thereby matrix $\boldsymbol{B}_1^b$ becomes:

$$\boldsymbol{B}_1 = \begin{bmatrix} -\cos\Omega_1 & \sin\Omega_1 \\ -\sin\Omega_1 & -\cos\Omega_1 \end{bmatrix}$$

Vector $\mathbf{c}_1^b$ describing position of $O_{R1}$:

$$\mathbf{c}_1^e = \begin{bmatrix} \mathbf{0} \\ a_W \end{bmatrix}.$$

The same magnitudes for system $R2$ are:

$$\alpha^2_{\hat{y}y} = \Omega_2\,; \qquad \alpha^2_{\hat{z}y} = 90^\circ + \Omega_2$$

$$\alpha^2_{\hat{y}z} = 270^\circ - \Omega_2\,; \quad \alpha^2_{\hat{z}z} = \Omega_2$$

$$\boldsymbol{B}_2 = \begin{bmatrix} \cos \Omega_2 & -\sin \Omega_2 \\ \sin \Omega_2 & \cos \Omega_2 \end{bmatrix},$$

$$\mathbf{c}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

After deformation, coordinate system $R1$ turns about $\hat{x}_1$ by angle $\varepsilon_1$:

$$\boldsymbol{B}_1 = \begin{bmatrix} -\cos(\Omega_1 + \varepsilon_1) & \sin(\Omega_1 + \varepsilon_1) \\ -\sin(\Omega_1 + \varepsilon_1) & -\cos(\Omega_1 + \varepsilon_1) \end{bmatrix}.$$

$O_{R1}$ is not displaced: $\mathbf{c}_1^a = \mathbf{c}_1^b$ .

The new position of coordinate system $R2$ results from a rotation by angle $\varepsilon_2$:

$$\boldsymbol{B}_2 = \begin{bmatrix} \cos(\Omega_2 + \varepsilon_2) & -\sin(\Omega_2 + \varepsilon_2) \\ \sin(\Omega_2 + \varepsilon_2) & \cos(\Omega_2 + \varepsilon_2) \end{bmatrix},$$

and since neither $O_{R2}$ is displaced, $\mathbf{c}_2 = \mathbf{c}_2$.

Change of matrix $\boldsymbol{B}_1$ in course of deformation:

$$\Delta \boldsymbol{B}_1 = \boldsymbol{B}_1 - \boldsymbol{B}_1 = \begin{bmatrix} -\cos(\Omega_1 + \varepsilon_1) + \cos \Omega_1 & \sin(\Omega_1 + \varepsilon_1) - \sin \Omega_1 \\ -\sin(\Omega_1 + \varepsilon_1) + \sin \Omega_1 & -\cos(\Omega_1 + \varepsilon_1) + \cos \Omega_1 \end{bmatrix}$$

while that of $\boldsymbol{B}_2$:

$$\Delta \boldsymbol{B}_2 = \boldsymbol{B}_2 - \boldsymbol{B}_2 = \begin{bmatrix} \cos(\Omega_2 + \varepsilon_2) - \cos \Omega_2 & -\sin(\Omega_2 + \varepsilon_2) + \sin \Omega_2 \\ \sin(\Omega_2 + \varepsilon_2) - \sin \Omega_2 & \cos(\Omega_2 + \varepsilon_2) - \cos \Omega_2 \end{bmatrix}$$

Simplification of matrix elements may rely on trigonometric relationships, and on the fact that, since for small displacements, angles $\varepsilon_1$ and $\varepsilon_2$ very small, $\cos \varepsilon_1 \approx 1$, $\sin \varepsilon_1 \approx \varepsilon_1$, $\cos \varepsilon_2 \approx 1$, $\sin \varepsilon_2 \approx \varepsilon_2$. After substitutions:

$$\Delta \boldsymbol{B}_1 = \begin{bmatrix} \varepsilon_1 \sin \Omega_1 & \varepsilon_1 \cos \Omega_1 \\ -\varepsilon_1 \cos \Omega_1 & \varepsilon_1 \sin \Omega_1 \end{bmatrix}$$

$$\Delta \boldsymbol{B}_2 = \begin{bmatrix} -\varepsilon_2 \sin \Omega_2 & -\varepsilon_2 \cos \Omega_2 \\ \varepsilon_2 \cos \Omega_2 & -\varepsilon_2 \sin \Omega_2 \end{bmatrix}$$

Obviously, for shifting vectors: $\Delta \mathbf{c}_1 = \Delta \mathbf{c}_2 = 0$.

Assume points $P_1$ and $P_2$ in Fig. 6 to contact in course of gear loading. Initial point positions in coordinate systems $R1$ and $R2$ are described by angles $\hat{\omega}_1$ and $\hat{\omega}_2$, respectively. Now, lengths of position vectors in these systems are $R_1$ and $R_2$, that is:

$$\hat{\mathbf{r}}_1 = \begin{bmatrix} R_1 \begin{matrix} \cos \hat{\omega}_1 \\ \sin \hat{\omega}_1 \end{matrix} \end{bmatrix}; \qquad \hat{\mathbf{r}}_2 = \begin{bmatrix} R_2 \begin{matrix} \cos \hat{\omega}_2 \\ \sin \hat{\omega}_2 \end{matrix} \end{bmatrix}$$

Position vectors in $M$ are:

$$\mathbf{r}_1 = \boldsymbol{B}_1^e \hat{\mathbf{r}}_1 + \mathbf{c}_1^e = \begin{bmatrix} -\cos \Omega_1 & \sin \Omega_1 & \cos \hat{\omega}_1 & R_1 + & 0 \\ -\sin \Omega_1 & -\cos \Omega_1 & \sin \hat{\omega}_1 & & a_W \end{bmatrix}$$

$$\mathbf{r}_2 = \boldsymbol{B}_2^e \hat{\mathbf{r}}_2 + \mathbf{c}_2^e = \begin{bmatrix} \cos \Omega_2 & -\sin \Omega_2 & \cos \hat{\omega}_2 & R_2 + & 0 \\ \sin \Omega_2 & \cos \Omega_2 & \sin \hat{\omega}_2 & & 0 \end{bmatrix}$$

hence:

$$\mathbf{r}_1 = \begin{bmatrix} -R_1 \cos(\Omega_1 + \hat{\omega}_1) \\ a_W - R_1 \sin(\Omega_1 + \hat{\omega}_1) \end{bmatrix}; \qquad \mathbf{r}_2 = \begin{bmatrix} R_2 \begin{matrix} \cos(\Omega_2 + \hat{\omega}_2) \\ \sin(\Omega_2 + \hat{\omega}_2) \end{matrix} \end{bmatrix}$$

causing the gap vector to be:

$$\mathbf{h} = \mathbf{r}_1 - \mathbf{r}_2$$

that is:

$$\begin{matrix} h_y \\ h_z \end{matrix} = \begin{bmatrix} -R_1 \cos(\Omega_1 + \hat{\omega}_1) - R_2 \cos(\Omega_2 + \hat{\omega}_2) \\ a_W - R_1 \sin(\Omega_1 + \hat{\omega}_1) - R_2 \sin(\Omega_2 + \hat{\omega}_2) \end{bmatrix}.$$

Assume displacements $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$ in systems $R1$ and $R2$ to be known. Their elastic counterparts in $M$ are:

$$\mathbf{u}_1 = \boldsymbol{B}_1^e \hat{\mathbf{u}}_1 = \begin{bmatrix} -\cos \Omega_1 & \sin \Omega_1 & \hat{v}_1 \\ -\sin \Omega_1 & -\cos \Omega_1 & \hat{w}_1 \end{bmatrix}$$

and

$$\mathbf{u}_2 = \boldsymbol{B}_2^e \hat{\mathbf{u}}_2 = \begin{bmatrix} \cos \Omega_2 & -\sin \Omega_2 & \hat{v}_2 \\ \sin \Omega_2 & \cos \Omega_2 & \hat{w}_2 \end{bmatrix}$$

where:

$$\hat{\mathbf{u}}_1 = \begin{bmatrix} \hat{v}_1 \\ \hat{w}_1 \end{bmatrix}; \qquad \hat{\mathbf{u}}_2 = \begin{bmatrix} \hat{v}_2 \\ \hat{w}_2 \end{bmatrix}$$

After multiplications:

$$\mathbf{u}_1 = \begin{bmatrix} -\hat{v}_1 \cos \Omega_1 + \hat{w}_1 \sin \Omega_1 \\ -\hat{v}_1 \sin \Omega_1 - \hat{w}_1 \cos \Omega_1 \end{bmatrix}$$

$$\mathbf{u}_2 = \begin{bmatrix} \hat{v}_2 \cos \Omega_2 - \hat{w}_2 \sin \Omega_2 \\ \hat{v}_2 \sin \Omega_2 + \hat{w}_2 \cos \Omega_2 \end{bmatrix}.$$

Vectors of rigid-body displacements are:

$$\mathbf{m}_1 = \Delta \boldsymbol{B}_1 \hat{\mathbf{r}}_1 + \Delta \mathbf{c}_1 = R_1 \varepsilon_1 \begin{bmatrix} \sin \Omega_1 & \cos \Omega_1 \\ -\cos \Omega_1 & \sin \Omega_1 \end{bmatrix} \begin{bmatrix} \cos \hat{\omega}_1 \\ \sin \hat{\omega}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{m}_2 = \Delta \boldsymbol{B}_2 \hat{\mathbf{r}}_2 + \Delta \mathbf{c}_2 = R_2 \varepsilon_2 \begin{bmatrix} -\sin \Omega_2 & -\cos \Omega_2 \\ \cos \Omega_2 & -\sin \Omega_2 \end{bmatrix} \begin{bmatrix} \cos \hat{\omega}_2 \\ \sin \hat{\omega}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

hence:

$$\mathbf{m}_1 = R_1 \varepsilon_1 \begin{bmatrix} \sin (\Omega_1 + \hat{\omega}_1) \\ -\cos (\Omega_1 + \hat{\omega}_1) \end{bmatrix}$$

$$\mathbf{m}_2 = R_2 \varepsilon_2 \begin{bmatrix} -\sin (\Omega_2 + \hat{\omega}_2) \\ \cos (\Omega_2 + \hat{\omega}_2) \end{bmatrix}$$

Let us consider now scalar products of the written vectors normally to the teeth. From the engagement of involutes it is known that in any position of engagement, the tooth normal coincides with the line of action [11]. This line has, however, constant slope angle and position in $M$, hence its normal unit vector is constant. Direction of the line of action hence also of the normal unit vector is described by pressure angle $\alpha$. Thus, direction vector of the line of action is:

$$\mathbf{n} = \begin{bmatrix} \cos \alpha \\ -\sin \alpha \end{bmatrix}.$$

Because of space shortage, here only results will be given.

Expressions have been simplified by using trigonometrical identities.

$$\mathbf{h} \cdot \mathbf{n} = -[a_W \sin \alpha + R_1 \cos (\Omega_1 + \hat{\omega}_1 + \alpha) - R_2 \cos (\Omega_2 + \hat{\omega}_2 + \alpha)]$$

$$\mathbf{u}_1 \cdot \mathbf{n} = -\hat{v}_1 \cos (\Omega_1 + \alpha) + \hat{w}_1 \sin (\Omega_1 + \alpha)$$

$$\mathbf{u}_2 \cdot \mathbf{n} = \hat{v}_2 \cos (\Omega_2 + \alpha) - \hat{w}_2 \sin (\Omega_2 + \alpha)$$

$$\mathbf{m}_1 \cdot \mathbf{n} = -R_1 \varepsilon_1 \sin (\Omega_1 + \hat{\omega}_1 + \alpha)$$

$$\mathbf{m}_2 \cdot \mathbf{n} = R_2 \varepsilon_2 \sin (\Omega_2 + \hat{\omega}_2 + \alpha)$$

These relationships are needed to write geometrical contact Equs (13) and (14).

In conformity with the results, the contact condition between teeth of a pair of cylindrical gears can be analyzed by means of some analytical or numerical contact method taking exact rigid-body and elastic displacements into consideration.

## Conclusion

The general geometrical contact equation suits to represent geometrical relationships between bodies performing arbitrary motion (e. g. simultaneous translation and rotation) not fitting Hertzian assumptions.

The derived relationships suit to analyze the contact condition of bodies with arbitrary initial gap, performing arbitrary rigid-body motion and arbitrary elastic displacement.

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