RELIABILITY EVALUATION OF MULTISTATE SYSTEMS WITH NON-OVERLAP HOMOGEN RECTANGULAR COVERING (NOHRC)

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Abstract

On the basis of the lattice theory NOHRC concept is introduced and its application for evaluating multistate system reliability is presented. By this way, at the systems of simple structure the algorithm proposed seems simpler than other methods from point of view of engineering use.

Introduction

Principle of Maximal Rectangular Covering (MRC) is usually used in the reliability evaluation of multistate systems, mainly in the case of high complexity. By means of MRC principle extraction of prime implicants and prime implicates, minimum upper vectors and maximal lower vectors can be detected which play a very important role in the reliability evaluation. But, by systems of simpler structure, on the basis of the lattice theory, one can use the so-called Non-Overlap Homogen Rectangular Covering (i.e. the rectangulars used to construct the lattice expression of the system do not overlap each other in the Karnaugh map representing the given system) and one can give a discrete function describing this system in a form of lattice expression which differs from the well known normal forms [1—3]. Remember, by means of a convenient procedure [4], [5] a system of high complexity may be decomposed into the equivalent simple minor systems from which one can determine the reliability of the given system.

In detail, principle of NOHRC is a procedure expressing the discrete function described the given system in the lattice expression form. The aim of the lattice ex-

pression, in general, is the determination of any value of the function when values of all its variables are given. Using the NOHRC principle the determination of the minimum upper vectors and the maximal lower vectors can be avoided. So, by a simple probability calculation reliability of the system may be easily determined.

Symbols and notations

$\vee$ = disjunction, a binary operation in the lattice theory, means supremum in lattice operation which is equivalent to the maximum in discrete functions.

$\wedge$ = (or not symbol)=conjunction, a binary operation in lattice theory, means infimum in lattice operation which is equivalent to the minimum in discrete functions.

$\subset$ = is subset of set

$\in$ = is (are) element(s) of

$\#$ = cardinality of a set

$(\cdot)$ = set of the integer numbers

$1, \nu$ = $1, 2, 3, \ldots, \nu$

c_i = component i of the system

x_i = state variable of component c_i

S_i = set of states of the component c_i (in the present discrete model S_i is finite)

S = set of states of the system (in the present discrete model S is finite)

C_i = subset of S_i

C = subset of S

n = number of system components

n_i = state number of the component c_i, consequently $n_i = \# S_i = 0, 1, 2, 3, \ldots, n_i - 1$

m = state number of the system, consequently, $m = \# S = 1, 2, 3, \ldots, m-1$

$l_k \in S$ = a constant

$\{C, f\}$ = system

x = $(x_0, x_1, \ldots, x_{n-1})$ state variable vector

j_i = state (level) j of the component c_i, $j_i = \overline{0, (n_i - 1)}$

k = state (level) k of the system, $k = \overline{0, (m-1)}$

$P_j(x_i)$ = $P(x_i = j_i)$: probability that component c_i is at state j (or at level j)

$\mathbf{P}(x_i)$ = $(P(x_i=0), P(x_i=1), \ldots, P(x_i=n_i-1))$: state probability vector of component c_i

$P_k(f)$ = $P(f=k)$: probability that system is at state k (or at level k)

$\mathbf{P}(f)$ = $(P(f=0), P(f=1), \ldots, P(f=m-1))$: state probability vector of the system.

The NOHRC principle

In this section we give the concept of NOHRC principle and show how to describe a discrete function by lattice expression using this principle. We explain it via some simple examples.

It is well known that there are many ways to represent a discrete function. But the more favourable and elegant tools are the lattice expression forms whose essential elements are the lattice exponentiations

$$x_i^{(C_i)} = \begin{cases} n_i - 1 & \text{iff} \quad x_i \in C_i, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Lattice expression of a discrete function consists of the lattice terms as follows

$$l_k \wedge x_i^{(C_i)} \tag{2}$$

where $l_k \in S = \{0, 1, 2, \ldots, m-1\}$ is the corresponding constant discussed later.

Opposite to the normal form which can be derived from the value table or the Karnaugh map of the function by means of MRC, in the case of the application of NOHRC the covering rectangulars do not overlap each other and in any rectangular the values of function are identical. For the sake of better understanding let us consider an illustrative example as follows.

There is a function

$$f(x_0, x_1) = f: \{0, 1, 2\} \to \{0, 1, 2, 3\}$$

given in form of Karnaugh map

Solution. From the given conditions one may state that

$$n = 2$$
$$n_1 = n_2 = 3$$
$$m = 4$$
$$S_1 = 0, 1, 2$$
$$S = 0, 1, 2$$
$$S = 0, 1, 2, 3$$

After the NOHRC principle we have its lattice expression form as follows

$$f(x_0, x_1) = 3 \wedge x_0^{(1,2)} \wedge x_1 \vee 2 \wedge x_0^{(1,2)} \wedge x_1^{(2)} \vee 2 \wedge x_0^{(0)} \wedge x_1^{(2)} \vee 1 \wedge x_0^{(1,2)} \wedge x_1^{(0)} \vee 1 \wedge x_0^{(0)} \wedge x_1^{(1)},$$

if instead of $\wedge$ no symbol is used the given function is

$$f(x_0, x_1) = 3x_0^{(1,2)} x_1^{(2)} \vee 2x_0^{(1,2)} x_1^{(2)} \vee 2x_0^{(0)} x_1^{(2)} \vee 1x_0^{(1,2)} x_1^{(0)} \vee 1x_0^{(0)} x_1^{(1)}. \tag{3}$$

Whereas its normal form [1] is

$$f(x_0, x_1) = 3x_0^{(1,2)}x_1^{(2)} \vee 2x_0^{(1,2)}x_1^{(1,2)} \vee 2x_1^{(2)} \vee 1x_0^{(1,2)} \vee 1x_1^{(1,2)}. \quad (4)$$

Checking the function value by some variable value and comparing with the result derived from (4) we can be sure of the rightness of (3). Indeed, if $x_0=1$ and $x_1=2$ then from expression (3)

$$f(1, 2) = 3\,3\,3 \vee 2\,3\,3 \vee 2\,0\,3 \vee 1\,3\,0 \vee 1\,0\,0 = 3,$$

whereas from (4)

$$f(1, 2) = 3\,3\,3 \vee 2\,3\,3 \vee 2\,0 \vee 1\,3 \vee 1\,3 = 3.$$

We see, the results are identical and indicated by the circle in Fig. 1. The situation is the same at any value of the variables x_0 and x_1. For example, by (3)

$$f(2, 0) = 3\,3\,0 \vee 2\,3\,0 \vee 2\,0\,0 \vee 1\,3\,3 \vee 1\,0\,0 = 1.$$

Thus, it is true (see the dashed circle in Fig. 1).

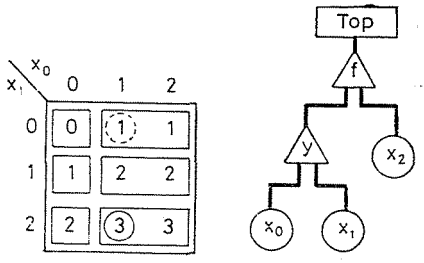

Fig. 1

Note that in the case of two-variable function the rectangulars are two dimensional (or planar). When the variable number of a function is $n>2$, in principle, dimension of its "hyperrectangular" is n itself. For retaining the pictorial property it is better to limit the dimension of function in $n\leqq 3$ with the help of decomposition mentioned above.

Probability of state (level) and reliability evaluation

The advantage of the discrete model concept is that over the system reliability one can evaluate its probability of different states or performance levels of the system. Otherwise, in a certain way reliability of the system can be determined by means of the probability of its states (or levels). This state becomes evident when failure is represented by a certain level, besides one use the so-called homogen s-coherent concept [6].

So, if probability of the component c_i $(i=\overline{0, (n-1)})$ at level $j(j=\overline{0, (n_i-1)})$ is known, or in other words probability vector c_i may be given, probability of the system at any state (level) k can be determined. For instance, if state probability vectors of system components shown in Fig. 1 are

$$\mathbf{P}(x_0) = (0.01,0.50,0.49)$$

$$\mathbf{P}(x_1) = (0.05,0.45,0.50)$$

then the probability that the system is at sate (level) $l_k \in \{0, 1, 2, 3\}$ may be received as follows

$$P_0(f) = P(f=0) = P(x_0=0)P(x_1=0) = 0.0.01 \cdot 0.05 = \underline{0.0005}$$

$$P_1(f) = P(f=1) = P(x_0=0)P(x_1=1)+P(x_1=0)[P(x_0=1)+P(x_0=2)]$$
$$= 0.01 \cdot 0.45+0.05[0.50+0.49] = \underline{0.054}$$

$$P_2(f) = P(f=2) = P(x_0=0)P(x_1=2)+P(x_1=1)[P(x_0=1)+P(x_0=2)]$$
$$= 0.01 \cdot 0.50+0.45[0.50+0.49] = \underline{0.4505}$$

$$P_3(f) = P(f=3) = P(x_1=2)[P(x_0=1)+P(x_0=2)]$$
$$= 0.50[0.50+0.49] = \underline{0.495}$$

Therefore the state probability vector of the system is

$$\mathbf{P}(f) = (0.0005, 0.0540, 0.4505, 0.4950).$$

In general, when the state probability vector of all components of the system is known, for calculating the probability that system is at level k one can use the algorithm as follows

1) write the lattice expression of the given system by means of the NOHRC principle,

2) $l_k \in \{0, 1, 2, \ldots, m-1\}$ is the state (level) k under consideration of the system,

3) calculate the probability that system is at level k from the component state probability vectors by the computer program flowchart shown in Fig. 2 for the case of two-variable functions.

Of course, one can extend this flowchart to the systems of three-variable function, and so on, without difficulty. Indeed, for the case of the functions of the higher variable number, in principle, the procedure may be continued but the calculation is more time consuming. It is better to use a convenient decomposition.

Example for 3-variable function. Let us evaluate the probability vector of the system (with its structure functions y and f) shown in Fig. 3 and the state probability

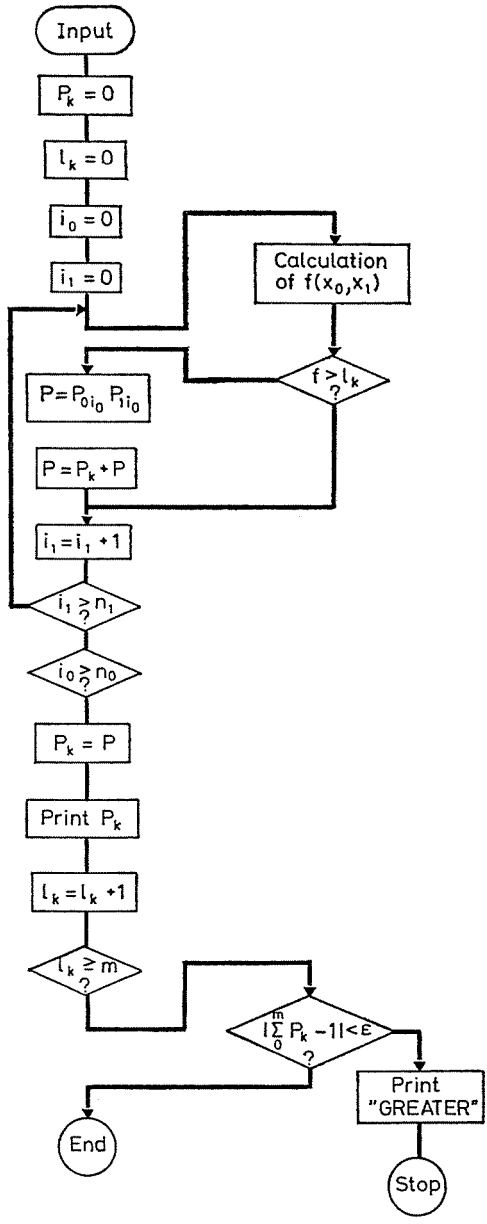
Input
$P_k = 0$
$l_k = 0$
$i_0 = 0$
$i_1 = 0$
Calculation of $f(x_0, x_1)$
$f > l_k$?
$P = P_{0i_0} P_{1i_0}$
$P = P_k * P$
$i_1 = i_1 + 1$
$i_1 \geq n_1$?
$i_0 \geq n_0$?
$P_k = P$
Print P_k
$l_k = l_k + 1$
$l_k \geq m$?
$|\sum_0^m P_k - 1| < \varepsilon$?
Print "GREATER"
End
Stop

Fig. 2

vectors for system components are

$$\mathbf{P}(x_0) = (0.01,0.50,0.49)$$
$$\mathbf{P}(x_1) = (0.05,0.45,0.50)$$
$$\mathbf{P}(x_2) = (0.10,0.55,0.35)$$

$y(x_0,x_1)$:

x_1 \ x_0	0	1
0	0	1
1	2	2

$f(y,x_2)$:

y \ x_2	0	1
0	0	0
1	1	1
2	2	2

Fig. 3

Solution 1. Allowing the procedure described above with perceiving the

$$f(x_0, x_1, x_2) = f(y(x_0, x_1), x_2)$$

first, one have to calculate the $P(y)$ then the $P(f) = P(f(y, x_2))$.

Namely

$$P(y = 0) = P(x_0 = 0)[P(x_1 = 0) + P(x_1 = 1)] + P(x_0 = 1)\,P(x_1 = 0)$$
$$= 0.01\,[0.05 + 0.45] + 0.5 \cdot 0.05 = \underline{0.03.}$$

Similarly

$$P(y = 1) = P(x_0 = 1)\,P(x_1 = 0) + P(x_0 = 1)\,P(x_1 = 1) + P(x_0 = 0)\,P(x_1 = 1)$$
$$P(y = 1) = \underline{0.2545,}$$

and
$$P(y = 2) = P(x_0 = 2)[P(x_1 = 1) + P(x_1 = 2)] + P(x_0 = 1)\,P(x_1 = 2)$$
$$P(y = 2) = \underline{0.7155.}$$

Therefore

$$P(y(x_0, x_1)) = (0.0300,0.2545,0.7155).$$

Remember that both

$$\sum_{j=0}^{n_i-1} P_j(x_i) = 1$$

and

$$\sum_{k=0}^{m-1} P_k(y) = 1.$$

So, we see that the results above are right.

Continue this procedure for receiving the top event, i.e. calculate the $P(f(x_0, x_1, x_2))=P(f(y, x_0))$ with

$$\mathbf{P}(x_2) = (0.10,0.55,0.35)$$

$$\mathbf{P}(y) = (0.0300,0.2545,0.7155)$$

by the same algorithm. Namely

$$P(f(y, x_2) = 0) = P(y = 0)[P(x_2 = 0)+P(x_2 = 1)+P(x_2 = 2)]+$$
$$+P(x_2 = 0)[P(y = 1)+P(y = 2)] = \underline{0.127\,000},$$

$$P(f(y, x_2) = 1) = P(y = 1)[P(x_2 = 1)+P(x_2 = 2)]+$$
$$+P(y = 2)P(x_2 = 1) = \underline{0.622\,575},$$

$$P(f(y, x_2) = 2) = P(y = 2)P(x_2 = 2) = \underline{0.250\,425}.$$

Therefore

$$Pf(y, x_2) = (0.127000,0.622575,0.250425).$$

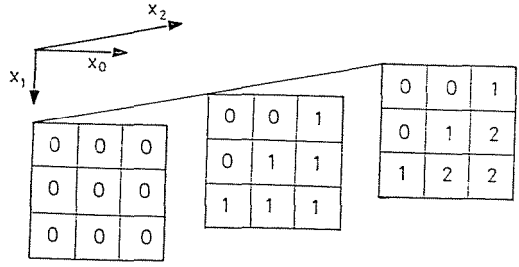

Fig. 4

Solution 2. In other side, by the direct function $f(x_0, x_1, x_2)$ after the equivalent Karnaugh map of 3-dimension [3] shown in Fig. 4 calculation of course may be directly performed.

$$P_0(f(x_0, x_1, x_2)) = P(x_2 = 0)[P(x_0 = 0)+P(x_0 = 1)+P(x_0 = 2)]\times$$
$$\times[P(x_1 = 0)+P(x_1 = 1)+P(x_1 = 2)+[P(x_2 = 1)+[P(\lambda_2 = 2)]\times$$
$$\times\{P(x_0 = 0)[P(x_1 = 0)+P(x_1 = 1)]+P(x_0 = 1)P(x_1 = 0)\} = \underline{0.127}.$$

It is evident that the two results (from two different ways) are totally identical. This agreement may be experienced in both state (level) of the system (top event)

fx_0, x_1, x_2. That is in detail

$$P_1(f(x_0, x_1, x_2)) = P(x_2 = 1)\{P(x_0 = 2)[P(x_1 = 0)+P(x_1 = 1)+P(x_1 = 2)]+$$
$$+P(x_0 = 1)[P(x_1 = 1)+P(x_1 = 2)]+P(x_0 = 0)P(x_1 = 2)\}+$$
$$+P(x_2 = 2)[P(x_0 = 2)P(x_1 = 0)+P(x_0 = 1)P(x_1 = 1)+P(x_0 = 0)P(x_1 = 2)]$$
$$= \underline{0.622575},$$
$$P_2(f(x_0, x_1, x_2)) = P(x_2 = 2)\{P(x_0 = 2)[P(x_1 = 1)+P(x_1 = 2)]+$$
$$+P(x_0 = 1)P(x_1 = 2)\} = \underline{0.250425}.$$

So the final result is

$$\mathbf{P}(f(x_0, x_1, x_2)) = (0.127000,0.622575,0.250425).$$

It is necessary to remark that from the structure functions (i.e. the $y(x_0, x_1)$ and the $f(y, x_2)$) the probability that system is at any state k can be determined on the basis of the state probability vector of all system components. So, for the system the result in form of a state probability vector totally agrees with the one derived from the direct function $f(x_0, x_1, x_2)$ without decomposition in form $f(y(x_0, x_1, x_2))$. Furthermore from the state probability vector of the system the reliability of that can be easily evaluated as follows.

If the state (level) 0 means the impossibility of system operation and state (level) 1 corresponds to the acceptable functioning one then system reliability is

$$R = \sum_{k=1}^{m} P_k(f). \tag{5}$$

For instance in our example

$$R = 0.622575+0.250425 = \underline{0.873}.$$

Finally, as to the decomposition of 3-dimension functions the solution of the problem is definite if and only if one of the two structure functions is accordingly defined.

Conclusion

In the case of the relatively simple structure system the reliability evaluation may be effectually carried out by means of the NOHRC principle and so, finding the minimum upper vectors (or maximal lower vectors) as well as other more difficult methods can be disregarded.

System reliability evaluation based on discrete model concept seems very usable. Since between the evaluation of the performance level (or the availability) and that of

the reliability there exists a close relationship constructed by (5) which is logical and evident by their concept.

From point of view of mathematics lattice expression form is very advantageous to describe a multistate system and together with the NOHRC principle an algorithm can be favourably performed by computer for calculating the system reliability. The algorithm has been described by the computer program flowchart as shown in Fig. 2.

Some illustrative examples are presented for the purpose of showing and explaining the application of the algorithm. They are simple, very exact and not time consuming.

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