INTEGRAL MANIFOLDS AND SOME OPTIMAL CONTROL PROBLEMS

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Abstract

A method of integral manifolds is applied to study singularly perturbed differential systems. The use of this method permits us to solve a problem of decomposition of singularly perturbed systems. The applications of the method are illustrated on examples.

Introduction

The purpose of this paper is to study the problem of singularly perturbed systems decomposition by the method of integral manifolds [1, 2].

Throughout this paper E^n denotes the real n-dimensional Euclidean space and $|\cdot|$ the Euclidean norm on this space.

The following system of differential equations is analyzed:

$$\dot{\chi} = f(t, \chi, y, \varepsilon), \qquad \varepsilon\dot{y} = g(t, \chi, y, \varepsilon) \tag{1}$$

where χ and f vary in E^m, y and g vary in E^n, $t \in R$, ε is the small positive parameter. Such systems appear in some problems of mechanics [3, 4] and control [5–8].

The object of our investigation is to obtain a transformation allowing to reduce (1) to system of form

$$\dot{u} = F(t, u, \varepsilon) \tag{2}$$

$$\varepsilon\dot{v} = G(t, u, v, \varepsilon) \tag{3}$$

and to discuss some applications in stability, boundary value and control problems.

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Integral manifolds

First we recall the definition of an integral manifold for the equation $\dot{\chi} = X(t, \chi)$, where $\chi \in E^n$. A set $S \subset R \times E^n$ is said to be an integral manifold if for $(t_0, \chi_0) \in S$, the solution $(t, \chi(t))$, $\chi(t_0) = \chi_0$ is in S for $t \in R$. If $(t, \chi(t)) \in S$ only at a finite interval, then we say that S is a local integral manifold.

Let us suppose that (1) satisfies the following hypotheses.

(i) Equation $g(t, \chi, y, 0) = 0$ has the isolated solution $y = h_0(t, \chi)$ for $t \in R$, $\chi \in E^m$. The function h_0 and its first and second partial derivatives with respect to all variables are uniformly continuous and bounded for $t \in R$, $\chi \in E^m$.

(ii) Functions f, g and their first and second partial derivatives with respect to all variables are uniformly continuous and bounded for $t \in R$, $\chi \in E^m$, $|y - h_0(t, \chi)| \leq \rho$, $0 \leq \varepsilon \leq \varepsilon_0$.

(iii) The eigenvalues $\lambda_i = \lambda_i(t, \chi)$, $i = 1, \ldots, n$ of the matrix $\frac{\partial g}{\partial y}(t, \chi, h_0, 0)$ satisfy the inequality $\mathrm{Re}\lambda_i \leq -2\beta < 0$, $t \in R$, $\chi \in E^n$.

Under such assumptions the system (1) has the integral manifold $y = h(t, \chi, \varepsilon)$. The flow on this manifold is governed by the m-dimensional system

$$\dot{\chi} = f(t, \chi, h(t, \chi, \varepsilon), \varepsilon). \tag{1.1}$$

Function h is continuously differentiable and $h(t, \chi, 0) = h_0$ [1, 2].

If f and g are sufficiently smooth with respect to all variables, then h may be represented as asymptotic expansion $h = h_0(t, \chi) + \varepsilon h_1(t, \chi) + \varepsilon^2 \ldots$ The coefficients of this expansion can be found from the equation

$$\varepsilon \frac{\partial h}{\partial t} + \varepsilon \frac{\partial h}{\partial \chi} f(t, \chi, h, \varepsilon) = g(t, \chi, h, \varepsilon) \tag{1.2}$$

by algebraic operations [3, 4].

Let us introduce new variables u, z, w, where u satisfies (1.1), $z = y - h(t, \chi, \varepsilon)$, $w = \chi - u$ and consider the auxiliary differential system

$$\begin{aligned} \dot{u} &= f(t, u, h(t, u, \varepsilon), \varepsilon) \\ \dot{w} &= f_1(t, u, w, z, \varepsilon) \\ \varepsilon \dot{z} &= Z(t, u, w, z, \varepsilon) \end{aligned} \tag{1.3}$$

where

$$f_1 = f(t, u+w, z+h(t, u+w, \varepsilon), \varepsilon) - f(t, u, h(t, u, \varepsilon), \varepsilon)$$

$$Z = g(t, u+w, z+h(t, u+w, \varepsilon), \varepsilon) - g(t, u+w, h(t, u+w, \varepsilon), \varepsilon) -$$

$$-\varepsilon \frac{\partial h}{\partial \chi}(t, u+w, \varepsilon)\left[f(t, u+w, z+h(t, u+w, \varepsilon), \varepsilon) - f(t, u+w, h(t, u+w, \varepsilon), \varepsilon)\right].$$

This last system has the integral manifold $w=\varepsilon H(t, u, z, \varepsilon)$, where function H satisfies the inequalities

$$|H(t, u, z, \varepsilon)| \leq a|z| \tag{1.4}$$

$$\begin{aligned} &|H(t, u, z, \varepsilon)-H(t, \bar{u}, z, \varepsilon)| \leq b|z| \cdot |u-\bar{u}| \\ &|H(t, u, z, \varepsilon)-H(t, u, \bar{z}, \varepsilon)| \leq c|z-\bar{z}| \end{aligned} \tag{1.5}$$

with the positive constants a, b, c for $t \in R$, $u \in E^m$, $|z| \leq \rho_1 \leq \rho$, $0 < \varepsilon \leq \varepsilon_1 \leq \varepsilon_0$.

The proof of this statement is similar to the proof of the existence of "stable manifold" in [9]. The flow on this manifold is governed by the $(m+n)$-dimensional system (2), (3) where

$$F(t, u, \varepsilon)=f(t, u, h(t, u, \varepsilon), \varepsilon),\ G(t, u, v, \varepsilon)=Z(t, u, \varepsilon H(t, u, v, \varepsilon), v, \varepsilon)$$

Note that $G(t, u, 0, \varepsilon)=0$.

Let $\chi=\chi(t)$, $y=y(t)$ be the solution of (1) and $|y_0-h(t_0, \chi_0, \varepsilon)| \leq \rho_1$, where $\chi_0=\chi(t_0)$, $y_0=y(t_0)$. Then

$$\begin{aligned} &\chi=u+\varepsilon H(t, u, v, \varepsilon) \\ &y=v+h(t, \chi, \varepsilon)=v+h(t, u+\varepsilon H(t, u, v, \varepsilon), \varepsilon) \end{aligned} \tag{1.6}$$

where $u=u(t)$, $v=v(t)$ is the solution of (2), (3), $v_0=v(t_0)=y_0-h(t_0, \chi_0, \varepsilon)$ and $u_0=u(t_0)$ can be found as asymptotic expansion $u_0=u_0^0+\varepsilon u_0^1+\dots$ from the equation

$$\chi_0=u_0+\varepsilon H(t_0, u_0, v_0, \varepsilon). \tag{1.7}$$

It is easy to see that $u_0^0=\chi_0$, $u_0^1=-H(t_0, \chi_0, y_0-h_0(t_0, \chi_0), 0)$.

The next result, however, shows that, in principle, function H can be approximated to any degree of accuracy with respect to ε.

Let $D(\varepsilon H)=\varepsilon \dfrac{\partial H}{\partial t}+\varepsilon \dfrac{\partial H}{\partial u} F(t, u, \varepsilon)+\dfrac{\partial H}{\partial v} Z(t, u, \varepsilon H, v, \varepsilon)-f_1(t, u, \varepsilon H, v, \varepsilon)$.

If $D(\varepsilon \bar{H})=O(\varepsilon^{\kappa+1})$, where κ is a positive integer, then $|H-\bar{H}|=O(\varepsilon^\kappa)$.

In many cases H can be found as asymptotic expansion $\varepsilon H=\varepsilon H_0(t, u, v)+\varepsilon^2\dots$ from the equation $D(\varepsilon H)=0$ or

$$\varepsilon \frac{\partial H}{\partial t}+\frac{\partial H}{\partial u} F(t, u, \varepsilon)+\frac{\partial H}{\partial v} Z(t, u, \varepsilon H, v, \varepsilon)=f_1(t, u, \varepsilon H, v, \varepsilon) \tag{1.8}$$

Note that if the hypotheses (i)–(iii) hold only at a bounded domain with respect to t and χ, then $y=h(t, \chi, \varepsilon)$ and $w=\varepsilon H(t, u, z, \varepsilon)$ are local integral manifolds.

The stability problem

Let $u=u(t)$, $v=v(t)$ be any solution of (2), (3) such that $u(t_0)=u_0$, $v(t_0)=v_0$, $|v_0|\leq\rho_1$. Hypothesis (ii) implies

$$|v(t)|\leq Ke^{-\frac{\beta}{\varepsilon}(t-t_0)} \quad t\geq t_0 \tag{2.1}$$

where K is a positive constant, $0<\varepsilon\leq\varepsilon_1$.

It is well known that for any solution $\chi=\chi(t)$, $y=y(t)$, $\chi(t_0)=\chi_0$, $y(t_0)=y_0$ of (1) with sufficiently small $|y_0-h(t_0,\chi_0,\varepsilon)|$ there is a solution $u=u(t)$, $u(t_0)=u_0$ of (2) such that

$$\chi=u(t)+\varphi_1(t), \qquad y(t)=h(t,u(t),\varepsilon)+\varphi_2(t) \tag{2.2}$$

where $\varphi_i(t)=O(e^{-\frac{\beta}{\varepsilon}(t-t_0)})$ as $t-t_0\to\infty$ [2].

Now we have the exact expressions for φ_1 and φ_2

$$\varphi_1=\varepsilon H(t,u(t),v(t),\varepsilon)$$
$$\varphi_2=h(t,u(t)+\varepsilon H(t,u(t),v(t),\varepsilon),\varepsilon)-h(t,u(t),\varepsilon)+v(t)$$

and the equation (1.7) for u_0.

The representation (2.2) tells us that (2) contains all the necessary information needed to determine the asymptotic behaviour of the solutions of (1).

Let $u(t)$ be a solution of (2). Then $\chi=u(t)$, $y=h(t,u(t),\varepsilon)$ is a solution of (1). If $u(t)$ as a solution of (2) is stable (asymptotically stable, unstable), then $(u(t), h(t,u(t),\varepsilon))$ as a solution of (1) is stable (asymptotically stable, unstable) [2].

Initial and boundary value problems

If we have the initial condition $\chi(t_0)=\chi_0$, $y(t_0)=y_0$ for (1) then for (2), (3) we obtain the following initial condition $u(t_0)=u_0$, $v(t_0)=v_0$, where $v_0=y_0-h(t_0,\chi_0,\varepsilon)$ and u_0 is the solution of the equation

$$u_0=\chi_0-\varepsilon H(t_0,u_0,v_0,\varepsilon). \tag{3.1}$$

If we have a boundary value problem for (1), then for (2), (3) we obtain coupled boundary conditions, which can be decoupled in some cases, when such values as $e^{-\frac{1}{\varepsilon}}$ can be neglected.

Everywhere below we let $O(e^{-\frac{1}{\varepsilon}})=0$ in boundary conditions.

Example 3.1. Consider the system

$$\begin{aligned}\dot{\chi}&=y\\ \varepsilon\dot{y}&=A(t,\chi)y+f(t,\chi)\end{aligned} \tag{3.2}$$

where $\chi,\ y \in E^n$, matrix-function A and vector-function f are smooth and bounded, eigenvalues of A have negative real parts. This system has the integral manifold $y = h(t, \chi, \varepsilon)$. From the equation $\varepsilon \frac{\partial h}{\partial t} + \varepsilon \frac{\partial h}{\partial \chi} h = Ah + f$, which is analogous to (1.2), we obtain $h = h_0(t, \chi) + \varepsilon h_1(t, \chi) + \varepsilon^2 \ldots$ where $h_0 = -A^{-1} f$, $h_1 = A^{-1}\left(\frac{\partial h_0}{\partial t} + \frac{\partial h_0}{\partial \chi} h_0\right)$. In this case (2) is $\dot{u} = h(t, u, \varepsilon)$ and the system (1.3) is

$$\dot{u} = h(t, u, \varepsilon)$$

$$\dot{w} = h(t, u + w, \varepsilon) - h(t, u, \varepsilon) + z$$

$$\varepsilon \dot{z} = \left[A(t, u + w) - \varepsilon \frac{\partial h}{\partial \chi}(t, u + w, \varepsilon) \right] z.$$

This last system has the integral manifold $w = \varepsilon H(t, u, z, \varepsilon)$ and H can be found from the equation

$$\varepsilon \frac{\partial H}{\partial t} + \varepsilon \frac{\partial H}{\partial u} h(t, u, \varepsilon) + \frac{\partial H}{\partial z} \left[A(t, u + \varepsilon H) - \varepsilon \frac{\partial h}{\partial \chi}(t, u + \varepsilon H, \varepsilon) \right] z =$$

$$= z + h(t, u + \varepsilon H, \varepsilon) - h(t, u, \varepsilon)$$

as asymptotic expansion $\varepsilon H = \varepsilon A^{-1}(t, u) z + \varepsilon^2 \ldots$.

Thus we have the representation

$$\chi = u + \varepsilon A^{-1}(t, u) v + \varepsilon^2 \ldots, \qquad y = v + h_0(t, \chi) + \varepsilon h_1(t, \chi) + \varepsilon^2 \ldots,$$

and the equations

$$\dot{u} = h_0(t, u) + \varepsilon h_1(t, u) + \varepsilon^2 \ldots \tag{3.3}$$

$$\varepsilon \dot{v} = \left[A(t, u + \varepsilon A^{-1}(t, u) v) - \varepsilon \frac{\partial h_0}{\partial \chi}(t, u) + \varepsilon^2 \ldots \right] v. \tag{3.4}$$

If we have the initial condition $\chi(t_0) = \chi_0$, $y(t_0) = y_0$ for (3.2) then for (3.3) we obtain the initial condition $u(t_0) = \chi_0 - A^{-1}(t_0, \chi_0)\,[y_0 - A^{-1}(t_0, \chi_0) f(t_0, \chi_0)] + \varepsilon^2 \ldots$ and for (3.4) we obtain $v(t_0) = y_0 - h(t_0, \chi_0, \varepsilon)$.

If we have the boundary condition

$$\chi(0) + y(0) = 0, \quad \chi(0) + \chi(1) + y(1) = 0$$

for (3.2) then for (3.3) we obtain the boundary condition

$$u(0) + u(1) + h_0(1, u(1)) + \varepsilon\{h_1(1, u(1)) - A^{-1}(0, u(0))\,[u(0) + h_0(0, u(0))]\} + \varepsilon^2 \ldots = 0.$$

For (3.4) we obtain the initial condition $v(0)=v_0^0+\varepsilon v_1^0+\varepsilon^2\ldots$, where

$$v_0^0=-u(0)-h_0(0,u(0))$$

$$v_0^1=-A^{-1}(0,u(0))v_0^0-\frac{\partial h_0}{\partial\chi}(0,u(0))A^{-1}(0,u(0))v_0^0-h_1(0,u(0)).$$

Linear state regulator problem

Let us consider the problem of minimization of the functional

$$I_\varepsilon=\frac{1}{2}\chi'(1)F\chi(1)+\frac{1}{2}\int_0^1[\chi'(t)Q(t)\chi(t)+u'(t)R(t)u(t)]\,\mathrm{d}t \tag{4.1}$$

under the restrictions

$$\begin{aligned}\dot y&=A_1(t)y+A_2(t)z+B_1(t)u, && y(0)=y_0, && y\in E^m\\ \varepsilon\dot z&=A_3(t)y+A_4(t)z+\varepsilon B_2(t)u, && z(0)=z_0, && z\in E^n\end{aligned} \tag{4.2}$$

where $u\in E^k$

$$\chi=\begin{pmatrix}y\\z\end{pmatrix},\qquad Q=Q'=\begin{pmatrix}Q_1&Q_2\\Q_2'&Q_3\end{pmatrix}\geq 0,\qquad F(\varepsilon)=F'(\varepsilon)=\begin{pmatrix}F_1&\varepsilon F_2\\\varepsilon F_2'&\varepsilon F_3\end{pmatrix}\geq 0,$$

$$R=R'>0.$$

It is well known (see [7]), that this problem has a linear feedback solution given by

$$u=-R^{-1}\begin{pmatrix}B_1\\B_2\end{pmatrix}'\begin{pmatrix}K_1&\varepsilon K_2\\\varepsilon K_2'&\varepsilon K_3\end{pmatrix}\chi$$

where K_1, K_2, K_3 is the solution of the initial value problem for the system

$$\dot K_1=-K_1A_1-A_1'K_1-K_2A_3-A_3'K_2'+K_1S_1K_1-Q_1+\varepsilon K_1S_2K_2'+$$
$$+\varepsilon K_2S_2'K_1+\varepsilon^2\ldots=f(t,K_1,K_2,K_3,\varepsilon)$$
$$\varepsilon\dot K_2=-K_1A_2-K_2A_4-A_3'K_3-Q_2-\varepsilon A_1'K_2+\varepsilon K_1S_1K_2+$$
$$+\varepsilon K_1S_2K_3+\varepsilon^2\ldots=g_1(t,K_1,K_2,K_3,\varepsilon)$$
$$\varepsilon\dot K_3=-K_3A_4-A_4'K_3-Q_3-\varepsilon K_2'A_2-\varepsilon A_2'K_2+\varepsilon^2\ldots=g_2(t,K_1,K_2,K_3,\varepsilon)$$
$$K_1(1)=F_1,\qquad K_2(1)=F_2,\qquad K_3(1)=F_3$$
$$S_1=B_1R^{-1}B_1',\qquad S_2=B_1R^{-1}B_2',\qquad S_3=B_2R^{-1}B_2'.$$

In this case we have the representation (1.6) in the form

$$K_1 = U + \varepsilon H(t, U, V_1, V_2, \varepsilon), \qquad K_2 = V_1 + P_1(t, K_1, \varepsilon),$$
$$K_3 = V_2 + P_2(t, K_1, \varepsilon)$$

where U is the solution of the equation

$$\dot{U} = f(t, U, P_1(t, U, \varepsilon), P_2(t, U, \varepsilon), \varepsilon) \tag{4.3}$$

and V_1, V_2 satisfy the system

$$\varepsilon \dot{V}_1 = g_3(t, U, V_1, V_2, \varepsilon), \qquad \varepsilon \dot{V}_2 = g_4(t, U, V_1, V_2, \varepsilon) \tag{4.4}$$

where

$$g_{i+2} = g_i(t, U + \varepsilon H, V_1 + P_1, V_2 + P_2, \varepsilon) - g_i(t, V + \varepsilon H, P_1, P_2, \varepsilon) -$$
$$- \frac{\partial P_i}{\partial K_1}(t, U + \varepsilon H, \varepsilon)[f(t, U + \varepsilon H, V_1 + P_1, V_2 + P_2, \varepsilon)$$
$$- f(t, U + \varepsilon H, P_1, P_2, \varepsilon)]$$
$$H = H(t, U, V_1, V_2, \varepsilon), \qquad P_i = P_i(t, V + \varepsilon H, \varepsilon), \qquad i = 1, 2.$$

Matrix-functions H, P_1, P_2 can be represented as asymptotic expansions

$$P_i = P_i^0(t, K_1) + \varepsilon P_i^1(t, K_1) + \varepsilon^2 \ldots, \qquad i = 1, 2,$$
$$\varepsilon H = \varepsilon H_1(t, U, V_1, V_2) + \varepsilon^2 \ldots$$

where P_2^0 is the solution of the equation $-P_2^0 A_4 - A_4' P_2^0 - Q_3 = 0$ and $P_1^0 = -(K_1 A_2 + A_3' P_2^0 + Q_2) A_4^{-1}$; P_2^1 is the solution of the equation $\dot{P}_2^0 = -(P_1^0)' A_2 - A_2' P_1^0 - P_2^1 A_4 - A_4' P_1^1$ and

$$P_1^1 = \left\{ \frac{\partial}{\partial t} P_1^0 + [-K_1 A_1 - A_1' K_1 - P_1^0 A_3 - A_3' P_1^0 + K_1 S_1 K_1 - Q_1] A_2 A_4^{-1} - \right.$$
$$\left. A_1' P_1^0 - A_3' P_2^1 + K_1 S_1 P_1^0 + K_1 S_2 P_2^0 \right\} A_4^{-1}.$$

For H_1 we obtain an expression

$$H_1 = V_1 A_4^{-1} A_3 + (A_4^{-1} A_3)' V_1 + (A_4^{-1} A_3)' V_2 A_4^{-1} A_3.$$

Finally, for (4.3) we have the initial condition

$$U(1) = F_1 - \varepsilon H_1(1, F_1, F_2 - P_1^0(1, F_1), F_3 - P_2^0(1, F_1))$$

and for (4.4) we have the initial condition

$$V_1(1) = F_2 - P_1(1, F_1, \varepsilon), \qquad V_2(1) = F_3 - P_2(1, F_1, \varepsilon).$$

Systems with two small parameters

Consider the system

$$\dot{\chi}_0 = f_0(t, \chi_0, \chi_1, \chi_2, \varepsilon, \mu)$$
$$\varepsilon\dot{\chi}_1 = f_1(t, \chi_0, \chi_1, \chi_2, \varepsilon, \mu) \tag{5.1}$$
$$\varepsilon\mu\dot{\chi}_2 = f_2(t, \chi_0, \chi_1, \chi_2, \varepsilon, \mu)$$

where χ_i and f_i vary in E^{n_i} $i=0, 1, 2$, ε and μ are small positive parameters. Let us suppose that (5.1) satisfies the following hypotheses.

(i) The equation $f_2(t, \chi_0, \chi_1, \chi_2, 0, 0)=0$ has the isolated solution $\chi_2 = h_{20}(t, \chi_0, \chi_1)$ for $t \in R$, $\chi_0 \in E^{n_0}$, $\chi_1 \in E^{n_1}$ and the function h_{20} and its first and second partial derivatives with respect to all variables are uniformly continuous and bounded for $t \in R$, $\chi_i \in E^{n_i}$, $i=0, 1$.

(ii) Functions f_i, $i=0, 1, 2$ and their first and second partial derivatives with respect to all variables are uniformly continuous and bounded for $t \in R$, $\chi_i \in E^{n_i}$, $i=0, 1$, $|\chi_2 - h_{20}(t, \chi_0, \chi_1)| \leq \rho$, $0 \leq \varepsilon \leq \varepsilon_0$, $0 \leq \mu \leq \mu_0$.

(iii) The eigenvalues $\lambda_i = \lambda_i(t, \chi_0, \chi_1)$, $i=1, \ldots, n_2$ of the matrix

$\frac{\partial f_2}{\partial \chi_2}(t, \chi_0, \chi_1, h_{20}(t, \chi_0, \chi_1), 0, 0)$ satisfy the inequality $\mathrm{Re}\lambda_i \leq -2\beta < 0$, $t \in R$, $\chi_i \in E^{n_i}$, $i=0, 1$.

These hypotheses are analogous to (i)—(iii) and as in Section 1 there exist functions $h_2 = h_2(t, \chi_0, \chi_1, \varepsilon, \mu)$, $H_i = H_i(t, y_0, y_1, y_2, \varepsilon, \mu)$, $i=1, 2$ such that

$$\chi_0 = y_0 + \varepsilon\mu H_0(t, y_0, y_1, y_2, \varepsilon, \mu)$$
$$\chi_1 = y_1 + \mu H_1(t, y_0, y_1, y_2, \varepsilon, \mu)$$
$$\chi_2 = y_2 + h_2(t, \chi_0, \chi_1, \varepsilon, \mu).$$

The function h_2 defines the integral manifold $\chi_2 = h_2(t, \chi_0, \chi_1, \varepsilon, \mu)$ of (5.1). If f_i, $i=0, 1, 2$ are sufficiently smooth with respect to all variables then h_2 can be found as asymptotic expansion $h_2 = h_{20}(t, \chi_0, \chi_1, \varepsilon) + \mu h_{21}(t, \chi_0, \chi_1, \varepsilon) + \mu^2 \ldots$ from the equation

$$\varepsilon\mu \frac{\partial h_2}{\partial t} + \varepsilon\mu \frac{\partial h_2}{\partial \chi_0} f_0 + \mu \frac{\partial h_2}{\partial \chi_1} f_1 = f_2$$
$$f_i = f_i(t, \chi_0, \chi_1, h_2, \varepsilon, \mu), \quad i=0, 1, 2.$$

The functions H_0, H_1 define the integral manifold $w_0 = \varepsilon\mu H_0(t, y_0, y_1,$

$y_2, \varepsilon, \mu)$, $w_1 = \mu H_1(t, y_0, y_2, \varepsilon, \mu)$ of the system

$$\dot{y}_0 = F_0(t, y_0, y_1, \varepsilon, \mu)$$
$$\varepsilon \dot{y}_1 = F_1(t, y_0, y_1, \varepsilon, \mu)$$
$$\dot{w}_0 = f_3(t, y_0, y_1, w_0, w_1, z_2, \varepsilon, \mu)$$
$$\varepsilon \dot{w}_1 = f_4(t, y_0, y_1, w_0, w_1, z_2, \varepsilon, \mu)$$
$$\varepsilon\mu \dot{z}_2 = Z_2(t, y_0, y_1, w_0, w_1, z_2, \varepsilon, \mu)$$

where

$$F_i = f_i(t, y_0, y_1, h_2(t, y_0, y_1, \varepsilon, \mu), \varepsilon, \mu)$$
$$f_{i+3} = f_i(t, y_0 + w_0, y_1 + w_1, z_2 + h_2(t, y_0 + w_0, y_1 + w_1, \varepsilon, \mu)) -$$
$$- f_i(t, y_0, y_1, h_2(t, y_0, y_1, \varepsilon, \mu)), \qquad i = 0, 1.$$
$$Z_2 = \Delta f_2 - \varepsilon\mu \frac{\partial h_2}{\partial \chi_0} \Delta f_0 - \mu \frac{\partial h_2}{\partial \chi_1} \Delta f_1$$
$$\Delta f_i = f_i(t, y_0 + w_0, y_1 + w_1, z_2 + h_2, \varepsilon, \mu) - f_i(t, y_0 + w_0, y_1 + w_1, h_2, \varepsilon, \mu)$$
$$h_2 = h_2(t, y_0 + w_0, y_1 + w_1, \varepsilon, \mu), \qquad i = 0, 1, 2$$

and can be found as asymptotic expansions $H_i = H_{i0}(t, y_0, y_1, y_2, \varepsilon) + \mu H_{i1}(t, y_0, y_1, y_2, \varepsilon) + \mu^2 \dots$ from the equations

$$\varepsilon\mu \frac{\partial H_i}{\partial t} + \varepsilon\mu \frac{\partial H_i}{\partial y_0} F_0 + \mu \frac{\partial H_i}{\partial y_1} F_1 + \frac{\partial H_i}{\partial y_2} Z_2 = f_{i+3}$$

where

$$H_i = H_i(t, y_0, y_1, y_2, \varepsilon, \mu), \quad F_i = F_i(t, y_0, y_1, \varepsilon, \mu)$$
$$Z_2 = Z_2(t, y_0, y_1, \varepsilon\mu H_0, \mu H_1, y_2, \varepsilon, \mu)$$
$$f_{i+3} = f_{i+3}(t, y_0, y_1, \varepsilon\mu H_0, \mu H_1, y_2, \varepsilon, \mu), \qquad i = 0, 1.$$

The variables y_0, y_1, y_2 satisfy the equations

$$\dot{y}_0 = F_0(t, y_0, y_1, \varepsilon, \mu) \tag{5.2}$$
$$\varepsilon \dot{y}_1 = F_1(t, y_0, y_1, \varepsilon, \mu) \tag{5.3}$$
$$\varepsilon\mu \dot{y}_2 = G_2(t, y_0, y_1, y_2, \varepsilon, \mu) \tag{5.4}$$

where

$$G_2 = Z_2(t, y_0, y_1, \varepsilon\mu H_0, \ \mu H_1, y_2, \varepsilon, \mu), \quad H_i = H_2(t, y_0, y_1, y_2, \varepsilon, \mu).$$

If we have the initial condition $\chi_i(t_0) = \chi_i^0$, $i = 0, 1, 2$ for (5.1) then for (5.2)—(5.4) we obtain the initial condition $y_i(t_0) = y_i^0$ where $y_2^0 = \chi_2^0 - h_2(t_0, \chi_0^0, \chi_1^0, \varepsilon, \mu)$ and y_0^0, y_1^0 can be found from the equations

$$\begin{aligned} \chi_0^0 &= y_0^0 + \varepsilon\mu H_0(t, y_0^0, y_1^0, y_2^0, \varepsilon, \mu) \\ \chi_1^0 &= y_1^0 + \mu H_1(t, y_0^0, y_1^0, y_2^0, \varepsilon, \mu). \end{aligned} \tag{5.5}$$

As earlier, the stability problem for (5.1) is equivalent to the stability problem for (5.2), (5.3) and for sufficiently small $|y_2^0|$ we obtain $|y_2(t)| \leq K_2 |y_2^0| e^{-\frac{\beta}{\varepsilon\mu}(t-t_0)}$, $t \geq t_0$.

Now let us suppose that the system (5.2), (5.3) satisfies the following hypotheses.

(i) The equation $F_1(t, y_0, y_1, 0, 0) = 0$ has the isolated solution $y_1 = h_{10}(t, y_0)$ for $t \in R$, $y_0 \in E^{n_0}$. The function h_{10} and its first and second partial derivatives with respect to all variables are uniformly continuous and bounded for $t \in R$, $y_0 \in E^{n_0}$.

(ii) The functions F_i, $i = 0, 1$ and their first and second partial derivatives with respect to all variables are uniformly continuous and bounded for $t \in R$, $y_0 \in E^{n_0}$, $|y_1 - h_{10}(t, y_0)| \leq \rho_1$, $0 \leq \varepsilon \leq \varepsilon_1 \leq \varepsilon_0$, $0 \leq \mu \leq \mu_1 \leq \mu_0$.

(iii) The eigenvalues $\lambda_i \rightleftharpoons \lambda_i(t, y_0)$, $i = 1, \ldots, n_1$ of the matrix $\frac{\partial F_1}{\partial y_1}(t, y_0, h_{10}(t, y_0), 0, 0)$ satisfy the inequality $\mathrm{Re}\lambda_i \leq -2\alpha < 0$, $t \in R$, $y_0 \in E^{n_0}$.

Under such assumptions which are analogous to (i)–(iii), the systems (5.2), (5.3) can be decomposed by the transformation

$$y_0 = u + \varepsilon H(t, u, v, \varepsilon, \mu), \quad y_1 = v + h(t, y_0, \varepsilon, \mu).$$

The functions h and H can be found as asymptotic expansions $h = h_0(t, y_0, \mu) + \varepsilon h_1(t, y_0, \mu) + \varepsilon^2 \ldots$, $H = H_0(t, u, v, \mu) + \varepsilon H_1(t, u, v, \mu) + \varepsilon^2 \ldots$ from equations which are analogous to (1.2) and (1.8).

Finally, we obtain the system

$$\dot{u} = F(t, u, \varepsilon, \mu)$$

$$\varepsilon\dot{v} = G(t, u, v, \varepsilon, \mu)$$

$$\varepsilon\mu\dot{y}_2 = G_1(t, u, v, y_2, \varepsilon, \mu)$$

where

$$F = F_0(t, u, h(t, u, \varepsilon, \mu), \varepsilon, \mu)$$

$$G = F_1(t, u + \varepsilon H, v + h(t, u + \varepsilon H, \varepsilon, \mu), \varepsilon, \mu) -$$

$$- F_1(t, u + \varepsilon H, h(t, u + \varepsilon H, \varepsilon, \mu), \varepsilon, \mu) -$$

$$- \varepsilon \frac{\partial h}{\partial y_0}(t, u + \varepsilon H, \varepsilon, \mu)\,[F_0(t, u + \varepsilon H, v + h(t, u + \varepsilon H, \varepsilon, \mu), \varepsilon, \mu) -$$

$$- F_0(t, u + \varepsilon H, h(t, u + \varepsilon H, \varepsilon, \mu), \varepsilon, \mu)]$$

$$G_1 = G_2(t, u + \varepsilon H, v + h(t, u + \varepsilon H, \varepsilon, \mu), y_2, \varepsilon, \mu),$$

$$H = H(t, u, v, \varepsilon, \mu).$$

It is easy to obtain the initial conditions for this system. Moreover, the stability problem for (5.1) is equivalent to the stability problem for the reduced system

$$\dot{u}=F(t, u, \varepsilon, \mu).$$

Linear systems

Consider the following linear singularly perturbed system

$$\chi=A_{11}\chi+A_{12}y+f_1, \qquad \varepsilon\dot{y}=A_{21}\chi+A_{22}y+f_2 \tag{6.1}$$

where χ and $f_1=f_1(t, \varepsilon)$ vary in E^m, y and $f_2=f_2(t, \varepsilon)$ vary in E^n, $A_{ij}=A_{ij}(t, \varepsilon)$ $(i, j=1, 2)$ are matrix-functions, $t \in R$, ε is the small positive parameter.

Let us suppose that f_i and A_{ij} $(i, j=1, 2)$ are bounded and smooth functions of t and ε and the eigenvalues $\lambda_i=\lambda_i(t)$ of $A_{22}(t, 0)$ satisfy the inequality $\mathrm{Re}\lambda_i \leq -2\beta<0$.

Under such assumptions there exists a transformation

$$\chi=u+\varepsilon H(t, \varepsilon)v, \qquad y=v+P(t, \varepsilon)\chi+p(t, \varepsilon) \tag{6.2}$$

which is the analogue of (1.6) for the linear case. The new variables u and v satisfy the equations

$$\dot{u}=(A_{11}+A_{12}P)u+f_1+A_{12}p \tag{6.3}$$

$$\varepsilon\dot{v}=(A_{22}-\varepsilon PA_{12})v. \tag{6.4}$$

The matrices P, H and the vector-function p can be found as asymptotic expansions $P=P_0(t)+\varepsilon P_1(t)+\varepsilon^2\ldots$, $H=H_0(t)+\varepsilon H_1(t)+\varepsilon^2\ldots$, $P=P_0(t)+\varepsilon P_1(t)+\varepsilon^2\ldots$ from the following equations

$$\varepsilon\dot{P}+\varepsilon P(A_{11}+A_{12}P)=A_{21}+A_{22}P \tag{6.5}$$

$$\varepsilon\dot{H}+H(A_{22}-\varepsilon PA_{12})=\varepsilon(A_{11}+A_{12}P)H+A_{12} \tag{6.6}$$

$$\varepsilon\dot{p}+\varepsilon Pf_1=(A_{22}-\varepsilon PA_{12})p+f_2. \tag{6.7}$$

It is a straightforward exercise to obtain an expression for P_i, H_i, p_i from these equations.

As for nonlinear systems, the stability of (6.3) is equivalent to the stability of (6.1), and the transformation (6.2) permits us to decouple initial and boundary value problems.

Now let us apply the transformation (6.2) to study the Ito equations [10]. Consider the system

$$d\chi=[A_{11}\chi+A_{12}y+f_1(t,\omega)]\,dt+G_1(t,\omega)\,dw$$
$$\varepsilon dy=[A_{21}\chi+A_{22}y+f_2(t,\omega)]\,dt+G_2(t,\omega)\,dw \tag{6.8}$$

where $w(t,\omega)=(w_1(t,\omega),\ldots,w_{n+m+1}(t,\omega))$ with w_i Wiener processes such

$$E[w_j(t,\omega)-w_j(s,\omega)]=0, \quad \text{for all} \quad t,s\in[0,\infty), \quad \omega\in\Omega$$
$$E\{[w_j(t,\omega)-w_j(s,\omega)][w_i(t,\omega)-\omega_i(s,\omega)]\}=\delta_{ji}(t-s).$$

If F_s is the σ-algebra generated by $\{w(t)-w(\tau),\ \tau\le t\le s\}$ we assume $f_j(t,\cdot)$ and $G_j(t,\cdot)$ are measurable with respect to F_s; we assume also f_j and G_j are measurable on the product space $(0,\infty)\times\Omega$. Moreover

$$E[f_1^*(t,\omega)f_1(t,\omega)]+E[f_2^*(t,\omega)f_2(t,\omega)]\le\alpha<\infty$$
$$\operatorname{Tr}E[G_1^*(t,\omega)G_1(t,\omega)]+\operatorname{Tr}E[G_2^*(t,\omega)G_2(t,\omega)]\le\alpha_1<\infty$$

for all $t\in[0,\infty)$.

Under these assumptions by the transformation (6.2) with $p=0$ we obtain the reduced system

$$du=[A(t,\varepsilon)u+p_1(t,\varepsilon,\omega)]\,dt+F_1(t,\varepsilon,\omega)\,dw$$
$$\varepsilon\,dv=[B(t,\varepsilon)v+p_2(t,\varepsilon,\omega)]\,dt+F_2(t,\varepsilon,\omega)\,dw$$

$$A=A_{11}+A_{12}P,\quad B=A_{22}-\varepsilon PA_{12},\quad p_1=(I+\varepsilon HP)f_1-Hf_2,\quad p_2=f_2-\varepsilon Pf_1$$

$$F_1=(I+\varepsilon HP)G_1-HG_2,\qquad F_2=G_2-\varepsilon PG_1.$$

Linear systems with two small parameters

Consider the linear analogue of (5.1):

$$\dot\chi=A_{00}\chi_0+A_{01}\chi_1+A_{02}\chi_2$$
$$\varepsilon\dot\chi_1=A_{10}\chi_0+A_{11}\chi_1+A_{12}\chi_2$$
$$\varepsilon\mu\dot\chi_2=A_{20}\chi_0+A_{21}\chi_1+A_{22}\chi_2$$

where $t\in R$, $\chi_i\in E^{n_i}$, $A_{ij}=A_{ij}(t,\varepsilon,\mu)$ are smooth and bounded matrix-functions, ε and μ are small positive parameters.

Let us suppose that the eigenvalues $\lambda_i=\lambda_i(t)$ $(i=1,\ldots,n_2)$ of the matrix $A_{22}(t,0,0)$ satisfy the inequality $\operatorname{Re}\lambda_i\le-2\beta<0$.

Our first step is to use the transformation

$$\chi_0 = y_0 + \varepsilon\mu H_0(t, \varepsilon, \mu) y_2, \quad \chi_1 = y_1 + \mu H_1(t, \varepsilon, \mu)$$

$$\chi_2 = y_2 + P_0(t, \varepsilon, \mu)\chi_0 + P_1(t, \varepsilon, \mu)\chi_1$$

to obtain the system

$$\begin{aligned} \dot{y} &= B_{00} y_0 + B_{01} y_1 \\ \varepsilon \dot{y}_1 &= B_{10} y_0 + B_{11} y_1 \\ \varepsilon\mu \dot{y}_2 &= B_{22} y_2 \end{aligned} \tag{7.1}$$

where

$$B_{ij} = A_{ij} + A_{i2} P_j \quad (j, i = 0, 1), \quad B_{22} = A_{22} - \varepsilon\mu P_0 A_{02} - \mu P_1 A_{12}.$$

The matrix-functions P_0 and P_1 can be found as asymptotic expansions $P_0 = P_0^0(t, \varepsilon) + \mu P_0^1(t, \varepsilon) + \mu^2 \ldots$, $P_1 = P_1^0(t, \varepsilon) + \mu P_1^1(t, \varepsilon) + \mu^2 \ldots$ from the equations

$$\varepsilon\mu \dot{P}_0 + \varepsilon\mu P_0 (A_{00} + A_{02} P_0) + \mu P_1 (A_{10} + A_{12} P_0) = A_{20} + A_{22} P_0$$

$$\varepsilon\mu \dot{P}_1 + \varepsilon\mu P_0 (A_{01} + A_{02} P_1) + \mu P_1 (A_{11} + A_{12} P_1) = A_{21} + A_{22} P_1$$

and the matrix-functions H_0 and H_1 can be found as asymptotic expansions $H_0 = H_0^0(t, \varepsilon) + \mu H_0^1(t, \varepsilon) + \mu^2 \ldots$, $H_1 = H_1^0(t, \varepsilon) + \mu H_1^1(t, \varepsilon) + \mu^2 \ldots$ from the equations

$$\varepsilon\mu \dot{H}_0 + H_0 B_{22} = \varepsilon\mu B_{00} H_0 + \mu B_{01} H_1 + A_{02}$$

$$\varepsilon\mu \dot{H}_1 + H_1 B_{22} = \varepsilon\mu B_{10} H_0 + \mu B_{11} H_1 + A_{12}.$$

Now let us assume that the eigenvalues $\lambda_i = \lambda_i(t)$ $(i = 1, \ldots, n_1)$ of the matrix $B_{11}(t, 0, 0) = A_{11}(t, 0, 0) - A_{12}(t, 0, 0) A_{22}^{-1}(t, 0, 0) A_{21}(t, 0, 0)$ satisfy the inequality $\mathrm{Re}\lambda_i \leq -2\alpha < 0$, $t \in \mathrm{R}$.

Our second step is to reduce system (7.1) to the form

$$\begin{aligned} \dot{u} &= (B_{00} + B_{01} P) u \\ \varepsilon \dot{v} &= (B_{11} - \varepsilon P B_{01}) v \\ \varepsilon\mu \dot{y}_2 &= B_{22} y_2 \end{aligned}$$

by the transformation

$$y_0 = u + \varepsilon H(t, \varepsilon, \mu) v, \qquad y_1 = v + P(t, \varepsilon, \mu) y_0.$$

The matrix-functions H and P can be found as asymptotic expansions $P = P^0(t, \mu) + \varepsilon P^1(t, \mu) + \varepsilon^2 \ldots$, $H = H^0(t, \mu) + \varepsilon H^1(t, \mu) + \varepsilon^2 \ldots$ from the equations

$$\varepsilon \dot{P} + \varepsilon P (B_{00} + B_{01} P) = B_{10} + B_{11} P$$

$$\varepsilon \dot{H} + H (B_{11} - \varepsilon P B_{01}) = \varepsilon (B_{00} + B_{01} P) H + B_{01}$$

which are analogous to (6.5), (6.6).

Nonlinear singularly perturbed regulator problem

In many optimal control problems the eigenvalues λ_i of the matrix $\frac{\partial g}{\partial y}$ (see the hypothesis (ii), Section 1) satisfy the inequalities

$$\mathrm{Re}\lambda_i \leq -2\beta < 0, \quad i=1, \ldots, k; \quad \mathrm{Re}\,\lambda_i \geq 2\beta > 0, \quad i=k+1, \ldots, n. \tag{8.1}$$

Let us suppose, for simplicity, that system (1) can be represented as

$$\begin{aligned} \dot{\chi} &= f(t, \chi, y_1, y_2, \varepsilon) \\ \varepsilon \dot{y}_1 &= A(t, \chi) y_1 + g_1(t, \chi, y_1, y_2, \varepsilon) \\ \varepsilon \dot{y}_2 &= B(t, \chi) y_2 + g_2(t, \chi, y_1, y_2, \varepsilon) \end{aligned} \tag{8.2}$$

where $y_1 \in E^k$, $y_2 \in E^{n-k}$, the eigenvalues of A satisfy (8.1) for $i=1, \ldots, k$ and the eigenvalues of B satisfy (8.1) for $i=k+1, \ldots, n$. Then (8.2) like the system (5.1) can be reduced to the form

$$\begin{aligned} \dot{u} &= F(t, u, \varepsilon) \\ \varepsilon \dot{v}_1 &= A(t, u) v_1 + G_1(t, u, v_1, \varepsilon) \\ \varepsilon \dot{v}_2 &= B(t, u) v_2 + G_2(t, u, v_1, v_2, \varepsilon). \end{aligned}$$

Example 8.1. Let we have the problem of minimization of the functional

$$I_\varepsilon = \frac{1}{2}(\chi^2(1) + y^2(1)) + \frac{1}{2}\int_0^1 (\chi^2(t) + y^2(t) + u^2(t))\, dt$$

under the restrictions

$$\dot{\chi} = y, \quad \varepsilon \dot{y} = f(\chi) + u, \quad \chi(0) = \chi_0, \quad y(0) = y_0.$$

The boundary value problem by the maximum principle can be represented in the form (see [6])

$$\dot{\chi} = y, \quad \dot{p} = -\chi - q f'(\chi), \quad \varepsilon \dot{y} = f(\chi) - q, \quad \varepsilon \dot{q} = -p - y \quad \left(f' = \frac{df}{d\chi}\right)$$

$$\chi(0) = \chi_0, \quad y(0) = y_0, \chi(1) = p(1), \quad y(1) = \varepsilon q(1)$$

and $u = -q$.

This boundary value problem can be reduced to the boundary value problem

$$\dot{u}_1 = -u_2(1 - \varepsilon f') + \varepsilon^2 \ldots, \quad \dot{u}_2 = -(u_1 + f' f)(1 - \varepsilon f') + \varepsilon^2 \ldots \quad (f = f(u_1))$$

$$u_1(0) - \varepsilon u_2(0) = \chi_0 + \varepsilon y_0 + \varepsilon^2 \ldots, \quad u_1(1) = u_2(1)[1 + \varepsilon(f'(u_1(1)) + 1)] + \varepsilon^2 \ldots$$

and two initial value problems

$$\varepsilon\dot{v}_1 = -[1+\varepsilon f'(u_1)+\varepsilon^2\ldots]v_1, \qquad v_1(0)=[1+\varepsilon f'(\chi_0)]y_0+u_2(0)+\varepsilon^2\ldots$$

$$\varepsilon\dot{v}_2 = [1+\varepsilon f'(u_1)+\varepsilon^2\ldots]v_2, \qquad v_2(1)= -u_2(1)(1-\varepsilon)-\varepsilon f(u_1(1))+\varepsilon^2\ldots$$

by the final transformation

$$\chi = u+\varepsilon(v_1+v_2)+\varepsilon^2\ldots, \qquad p=u_2+\varepsilon f'(u_1)(v_1-v_2)+\varepsilon^2\ldots$$

$$y=v_1-v_2-p(1-\varepsilon f'(\chi))+\varepsilon^2\ldots, \quad q=v_1+v_2+f(\chi)-\varepsilon(\chi+f'(\chi)f(\chi))+\varepsilon^2\ldots.$$

It should be observed that the existence theorem of the integral manifold $y=h(t,\chi,\varepsilon)$ for (1) was obtained in [11], analogous result for linear systems was obtained in [12]. The method of approximating integral manifolds for linear and nonlinear systems and for systems with several small parameters was essentially used in [3, 4, 13, 14]. Different aspects of the decomposition of singularly perturbed systems were studied in [15].

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