THE STABILITY OF A SPECIAL DIFFERENTIAL EQUATION UNDER ZERO EQUILIBRIUM STATE

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Abstract

This paper presents new stability criteria for the asymptotical stability of a first order nonlinear differential equation (1.1). Furthermore, stability charts and an electrical application of this differential equation are also presented.

1. Introduction

In the scientific field we often encounter stability problems. For example in physics, chemistry, biology, fluid mechanics, mechanical vibrations and engineering. Since many physical laws and relations are mathematically represented in the form of a differential equation, differential equations are of fundamental importance in the investigation of stability problems. Among these various problems of stability, research was carried out on the stability of a special differential equation of a system under zero equilibrium state.

The aim of this paper is to establish new stability criteria for the asymptotic stability of the system (1.1) under zero equilibrium state. As a result, stability regions are established for this system (see Figs 1, 2 and 3). We compare the well-known Coppel criterion for linear differential equations with the stability criteria of Eq. (1.1). This is represented by Figs 5 and 6. Finally, appropriate linear transformations and an important application in connection with an electrical circuit are shown.

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Let us consider the first order nonlinear differential equation

$$\dot{x} = A(t, x)x \tag{1.1}$$

where $A = \begin{bmatrix} -\rho & 1 \\ -1 & -\pi \end{bmatrix}$, $x \in \mathbf{R}^2$, $\rho = \rho(t, x)$, $\pi = \pi(t, x)$, $t \geq 0$.

Before discussing the new results (see sections 3, 4 and 5), let us review the main theorems applied here in section 2. In fact some of these theorems and their proofs can be found in [3, 6].

2. The applied main theorems

Let us denote by μ the real valued function, defined by *W.* Coppel [7] in connection with matrices and applied for the estimation of the lower and upper bounds of solutions of linear differential equations. This function depends upon the norm applied. The definition $\mu(A)=\lim_{h\to+0}\frac{\|I+hA\|-1}{h}$, where I is $n\times n$ unit matrix and the matrix norm $\|\cdot\|$ is defined as $\|A\|=\sup_{\|x\|=1}\|Ax\|$, holds for every $n\times n$ matrix of A. In [8], it is noted that the best possible bounds can be obtained by applying an appropriate linear transformation of the differential equation. Moreover, in the case of real and autonomous systems, the appropriate linear transformation is real and does not depend on the time if μ is induced by the Euclidean norm. The Coppel criterion allows the appropriate linear transformations to be used for a nonlinear system as well [6].

Suppose system (1.1) is a linear and autonomous one, for example,

$$\dot{x}=A_0x, \tag{2.1}$$

where $A_0=\begin{bmatrix}-r & 1\\ -1 & -p\end{bmatrix}$, p, r are positive constants.

The solution $x\equiv 0$ of equation (2.1) is asymptotically stable if, and only if, the roots of the characteristic equation have negative real parts.

Now, let A be an arbitrary constant matrix. In [3], it has been shown that if A is stable, then there exists an S transformation matrix such that the inequality

$$\mu(SAS^{-1})<0 \tag{2.2}$$

must be valid. In general, for any positive definite P there always exists a positive definite V such that

$$VA+A^TV=-P. \tag{2.3}$$

Suppose $P=2I$, where I is the unit matrix, then

$$VA+A^TV=-2I. \tag{2.4}$$

Since $VA+A^TV$ is symmetric and VA is not symmetric, we deduce that

$$VA=Z-I, \tag{2.5}$$

where Z is the skew — symmetric matrix, i.e. $Z^T = -Z$. Hence, we obtain the explicit equation for V as

$$V=(Z-I)A^{-1}. \tag{2.6}$$

Furthermore, suppose

$$V=S^TS, \tag{2.7}$$

and letting

$$S=DM, \tag{2.8}$$

where $D=\begin{bmatrix}\alpha_1 & 0\\ 1 & \alpha_2\end{bmatrix}$, $M=\frac{1}{\sqrt{1+r_0^2}}\begin{bmatrix}r_0 & -1\\ 1 & r_0\end{bmatrix}$, $M^T=M^{-1}$ and $\alpha_1^2=\lambda_1$, $\alpha_2^2 =\lambda_2$. λ_1, λ_2 are the eigenvalues (i.e. $\lambda_1 \geq \lambda_2 > 0$) and the column vectors of M are the corresponding eigenvectors of V. One can easily see that the transformation matrix S can be constructed for the case of (2.2). Obviously:

$$\|S\|=\|D\|=\alpha_1 \tag{2.9}$$

$$\|S^{-1}\|=\|D^{-1}\|=\frac{1}{\alpha_2}. \tag{2.10}$$

Let

$$A=\begin{bmatrix}\alpha & \delta\\ \gamma & \beta\end{bmatrix}, \tag{2.11}$$

and

$$Z=\begin{bmatrix}0 & x\\ -x & 0\end{bmatrix}. \tag{2.12}$$

Substitution of (2.11) and (2.12) in (2.6), gives

$$V=\begin{bmatrix}a & b\\ b & c\end{bmatrix}. \tag{2.13}$$

The Eq. (2.6) is equivalent to four scalar equations. By this reasoning, the unknowns a, b, c and x may be uniquely determined and we have

$$V=\frac{1}{(\alpha+\beta)\det(A)}\begin{bmatrix}-\det(A)-\beta^2-\gamma^2 & \alpha\gamma+\delta\beta\\ \alpha\gamma+\delta\beta & -\det(A)-\alpha^2-\delta^2\end{bmatrix} \tag{2.14}$$

From (2.14), the elements of matrices D and M are obtained, respectively:

$$D=\begin{bmatrix}\sqrt{\lambda_1} & 0\\ 0 & \sqrt{\lambda_2}\end{bmatrix}, \tag{2.15}$$

$$M=\frac{1}{\sqrt{1+r_0^2}}\begin{bmatrix}r_0 & -1\\ 1 & r_0\end{bmatrix}, \tag{2.16}$$

where

$$\lambda_1 = c + br_0, \tag{2.17}$$

$$\lambda_2 = a - br_0, \tag{2.18}$$

and

$$r_0 = \frac{a-c}{2b} + \sqrt{\left(\frac{a-c}{2b}\right)^2 + 1}. \tag{2.19}$$

Since S is the appropriate transformation matrix needed, we can write

$$S^T SA + A^T S^T S = -2I, \tag{2.20}$$

and

$$SAS^{-1} + (SAS^{-1})^T = -2S^{T-1}S^{-1} = -2D^{-1}MM^TD^{-1} = -2D^{-2}. \tag{2.21}$$

From this, we deduce that

$$\mu(SAS^{-1}) = -\frac{1}{\lambda_2} < 0. \tag{2.22}$$

Now, let us study a more general case. Assuming A to be an arbitrary $n \times n$ constant matrix, let us recall some basic concepts about μ [3, 7, 8].

$$\|\mu(A)\| \le \|A\|, \tag{2.23}$$

$$\mu(A^T) = \mu(A), \tag{2.24}$$

$$\mu(cA) = c\mu(A), \quad c \ge 0, \tag{2.25}$$

$$\mu(I) = 1, \tag{2.26}$$

$$\mu(A + \lambda I) = \mu(A) + \lambda, \tag{2.27}$$

$$\mu(A+B) \le \mu(A) + \mu(B). \tag{2.28}$$

$\lambda_1, \ldots, \lambda_n$ denote by the eigenvalues of A. Assuming that $\lambda(A)$ and $v(A)$ are maximum $\operatorname{Re}\lambda_i$ and minimum $\operatorname{Re}\lambda_i$ respectively, where $1 \le i \le n$, then

$$\lambda(A) \leqq \mu(A). \tag{2.29}$$

Since

$$\lambda(-A) = -v(A),$$

then (2.29) implies

$$-\mu(-A) \le v(A). \tag{2.30}$$

Furthermore, if the norm $\|\cdot\| = \|\cdot\|_v$ applied in the definition of μ (see [7]) is defined by a positive Hermitian matrix V, i.e. $\|x\|_V = \sqrt{x^* Vx}$ then for $\mu_{\|\cdot\|_V}$ (or written μ_V) we have

$$\mu_V(A) = \sup_{\|x\|=1} \operatorname{Re} x^* VAx = \lambda\left(\frac{A^*V + VA}{2}\right), \tag{2.31}$$

where * denotes the conjugate transpose. Obviously $\|\cdot\|_I$ is the usual Euclidean norm. We shall use this norm in section 3.

Let t_0 be a fixed real number and consider the nonlinear differential equation

$$\dot{x}=A(t,x)x, \qquad t\geq t_0, \tag{2.32}$$

where $A(t,x)$ is a continuous matrix function defined for $t\geq t_0$ and $x\in\mathbf{R}^n$. Then, for any solution $x(t)$ of Eq. (2.32), we have

$$\|x(t_0)\|\exp\left(-\int_{t_0}^{t}\mu(-A(\tau,x(\tau)))\mathrm{d}\tau\right)\leq$$

$$\|x(t)\|\leq\|x(t_0)\|\exp\left(\int_{t_0}^{t}\mu(A(\tau,x(\tau)))\mathrm{d}\tau\right) \qquad t\geq t_0. \tag{2.33}$$

In [6], a relationship between this inequality and the transformation of the variables is developed in case of a linear and autonomous system. For example, let us consider the first order linear homogeneous differential equation having constant coefficients

$$\dot{z}=Az. \tag{2.34}$$

If the variable z is appropriately transformed to a new variable

$$\omega=Sz, \tag{2.35}$$

then the differential equation

$$\dot{\omega}=SAS^{-1}\omega, \tag{2.36}$$

for the variable ω is obtained.

By applying Eq. (2.33) to the Eq. (2.36), an exponential estimation can be given for the norm $\|\omega(t)\|$. If $\mu(SAS^{-1})\leq\mu_0<0$, then $\lim_{t\to\infty}\|\omega(t)\|=0$. In fact, we have

$$\|\omega(t)\|\leqq\|\omega(t_0)\|e^{\mu_0(t-t_0)}. \tag{2.37}$$

Moreover, a similar exponential estimation can be obtained for the norm $\|z(t)\|$ as well.

Remark 1. From inequality (2.33) it follows that the asymptotic stability of the zero solution of (2.32) is implied by the condition

$$\mu(A(\tau,x(\tau))<\delta<0, \qquad \tau\geq t_0. \tag{2.38}$$

More precisely, the converse of this statement is not true even for the linear and autonomous cases. This is shown by the following:

example 1. Let A be defined as follows:

$$A=\begin{bmatrix} a & 2b \\ 0 & c \end{bmatrix} \quad \text{where} \quad a<0,\ c<0,\ b^2>ac. \tag{2.39}$$

It is easy to see that A is asymptotically stable. However $\mu_I(A)>0$, but in the autonomous case the converse of our statement is true if the variable x is appropriately transformed. It is true that if A is asymptotically stable, then $\mu(C)<0$, where $\dot{z}=C(t,x)z$ is the appropriately transformed equation.

Substitution of $z=Q^{-1}(t)x$ where $Q(t)$ is a continuously differentiable nonsingular matrix function defined for $t\geq t_0$ results that (2.32) can be written as

$$\dot{z}=(\dot{Q}^{-1}(t)Q(t)+Q^{-1}(t)A(t,x)Q(t))z=C(t,x)z. \tag{2.40}$$

Applying inequality (2.33) for (2.40), we obtain

$$\|x(t_0)\|\,\|Q(t_0)\|^{-1}\|Q^{-1}(t)\|^{-1}\exp\left(-\int_{t_0}^{t}\mu(-C(\tau,x(\tau)))\mathrm{d}\tau\right)\leqq\|x(t)\|\leqq$$
$$\leqq\|x(t_0)\|\,\|Q^{-1}(t_0)\|\,\|Q(t)\|\exp\left(\int_{t_0}^{t}\mu(C(\tau,x(\tau)))\mathrm{d}\tau\right),\quad t\geq t_0 \tag{2.41}$$

Assuming that $Q(t)$ is constant, we have $C(t,x)=Q^{-1}A(t,x)Q$ and the inequality (2.41) turns as the form

$$\|x(t_0)\|\,\|Q\|^{-1}\|Q^{-1}\|^{-1}\exp\left(-\int_{t_0}^{t}(-Q^{-1}A(\tau,x(\tau))Q)\mathrm{d}\tau\right)\leqq\|x(t)\|\leqq$$
$$\leqq\|x(t_0)\|\,\|Q^{-1}\|\,\|Q\|\exp\left(\int_{t_0}^{t}\mu(Q^{-1}A(\tau,x(\tau))Q)\mathrm{d}\tau\right),\ \tau\geq t_0. \tag{2.42}$$

Similarly, the nonlinear Eq. (1.1) can be transformed by applying the results above. The main idea of this transformation is to construct S for the linear and autonomous Eq. (2.1) which is considered similar to the nonlinear Eq. (1.1). Since the matrices A_0 in (2.1) and A in (1.1) have similar structures, one can expect that S is an appropriate transformation matrix, indeed [6].

3. Stability criteria for equation (1.1)

Consider the following matrix

$$VA_0=\begin{bmatrix} -1 & c \\ -c & -1 \end{bmatrix}, \tag{3.1}$$

where

$$A_0=\begin{bmatrix} -r & 1 \\ -1 & -p \end{bmatrix},$$

and p, r are positive constants which can be determined later. We assume V to be a symmetric matrix and by this condition the constant c can be determined. Hence, we have

$$V=\frac{1}{pr+1}\begin{bmatrix} p+c & 1-cr \\ pc-1 & c+r \end{bmatrix}. \tag{3.2}$$

From the symmetry condition, we have

$$1-cr=pc-1,$$

$$c(p+r)=2,$$

$$c=\frac{2}{p+r}. \tag{3.3}$$

Substitution of (3.3) in (3.2), gives

$$V=\frac{1}{pr+1}\begin{bmatrix} p+\dfrac{2}{p+r} & \dfrac{p-r}{p+r} \\ \dfrac{p-r}{p+r} & r+\dfrac{2}{p+r} \end{bmatrix}. \tag{3.4}$$

To simplify the writing, we introduce the following notations for the elements of matrix V:

$$a=\frac{1}{pr+1}\left(p+\frac{2}{p+r}\right),\quad b=\frac{1}{pr+1}\left(\frac{p-r}{p+r}\right),\quad c=\frac{1}{pr+1}\left(r+\frac{2}{p+r}\right) \tag{3.5}$$

and matrix V turns as

$$V=\begin{bmatrix} a & b \\ b & c \end{bmatrix}. \tag{3.6}$$

The characteristic equation

$$\begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix}=0 \tag{3.7}$$

or

$$\lambda^2-(a+c)\lambda+ac-b^2=0,$$

has roots

$$\lambda_{1,2}=\frac{a+c}{2}\pm\sqrt{\left(\frac{a+c}{2}\right)^2+b^2}. \tag{3.8}$$

Corollary 1. The eigenvectors of V are

$$S_1=\begin{bmatrix} r_0 \\ 1 \end{bmatrix}, \qquad S_2=\begin{bmatrix} -1 \\ r_0 \end{bmatrix}, \tag{3.9}$$

where

$$r_0=\frac{a-\lambda_2}{b}=\frac{a-c}{2b}+\sqrt{\left(\frac{a+c}{2}\right)^2+b^2}, \qquad b\neq 0. \tag{3.10}$$

Proof. We prove this corollary by using the definition of the eigenvector:

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} r_0 \\ 1 \end{bmatrix}=\begin{bmatrix} ar_0+b \\ br_0+c \end{bmatrix}=\begin{bmatrix} \lambda_1 r_0 \\ \lambda_1 \end{bmatrix}. \tag{3.11}$$

From this definition, of course, we write

$$ar_0+b=\lambda_1 r_0 \tag{3.12}$$

and

$$br_0+c=\lambda_1 \tag{3.13}$$

substitution of r_0 in Eq. (3.12), gives

$$a\left(\frac{a-\lambda_2}{b}\right)+b=\lambda_1\left(\frac{a-\lambda_2}{b}\right),$$

$$\frac{a^2-a\lambda_2+b^2}{b}=\frac{a\lambda_1-\lambda_1\lambda_2}{b},$$

$$a^2-a\lambda_2+b^2=a\lambda_1-\lambda_1\lambda_2,$$

$$a^2+b^2+\lambda_1\lambda_2=a(\lambda_1+\lambda_2). \tag{3.14}$$

But from (3.7) we write the sum and product of the roots respectively:

$$\lambda_1+\lambda_2=a+c, \tag{3.15}$$

$$\lambda_1\lambda_2=ac-b^2. \tag{3.16}$$

Substitution of (3.15) and (3.16) in (3.14), yields

$$a^2+b^2+ac-b^2=a(a+c),$$

$$a^2+ac=a(a+c),$$

$$a(a+c)=a(a+c). \tag{3.17}$$

From this, we conclude that our corollary is valid. Since S_1, S_2 are perpendicular, it follows that $S_2=\begin{bmatrix} -1 \\ r_0 \end{bmatrix}$!

Finally, from these assumptions we obtain expressions for the determination of the eigenvalues λ_1, λ_2 and the unit eigenvectors $\mathring{S}_1, \mathring{S}_2$ respectively:

$$\lambda_{1,2} = \frac{1}{2(pr+1)(p+r)}\left((p+r)^2+4 \pm |p-r|\sqrt{(p+r)^2+4}\right) \tag{3.18}$$

$$\mathring{S}_1 = \frac{1}{\sqrt{1+r_0^2}}\begin{bmatrix} r_0 \\ 1 \end{bmatrix}, \qquad \mathring{S}_2 = \frac{1}{\sqrt{1+r_0^2}}\begin{bmatrix} -1 \\ r_0 \end{bmatrix}. \tag{3.19}$$

where

$$r_0 = \frac{p+r}{2} + \sqrt{\left(\frac{p+r}{2}\right)^2 + 1}, \quad p, r \tag{3.20}$$

are positive constants.

Hence, we deduce the coefficient matrix function Q in the following way:

$$Q = DM^T AMD^{-1}, \tag{3.21}$$

where

$$D = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix}, \qquad M = \frac{1}{\sqrt{1+r_0^2}}\begin{bmatrix} r_0 & -1 \\ 1 & r_0 \end{bmatrix}, \qquad A = \begin{bmatrix} -\rho & 1 \\ -1 & -\pi \end{bmatrix}, \tag{3.22}$$

M^T is the transpose of matrix M and D^{-1} is the inverse of matrix D.

Development and substitution of these matrices, yields

$$Q = \frac{1}{1+r_0^2}\begin{bmatrix} -(r_0^2\rho+\pi) & (r_0\rho+r_0^2-\pi r_0+1)\sqrt{\dfrac{\lambda_1}{\lambda_2}} \\ (r_0\rho-r_0^2-r_0\pi-1)\sqrt{\dfrac{\lambda_2}{\lambda_1}} & -(\rho+r_0^2\pi) \end{bmatrix}. \tag{3.23}$$

Introducing the notations $\alpha, \beta, \gamma, \delta$, the elements of Q matrix can be expressed in the simplified form as

$$Q = \begin{bmatrix} \alpha & \delta \\ \gamma & \beta \end{bmatrix}. \tag{3.24}$$

This matrix is written as the sum of a symmetric matrix Q_S and a skew — symmetric matrix Q_A:

$$Q = Q_S + Q_A. \tag{3.25}$$

Corollary 2. It is true that

$$x^T Q_A x = 0, \tag{3.26}$$

where

$$x=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad x^T=[x_1, x_2], \qquad Q_A=\begin{bmatrix} 0 & -(\gamma-\delta) \\ (\gamma-\delta) & 0 \end{bmatrix}. \tag{3.27}$$

Proof:

$$\begin{aligned} x^T Q_A x &= [x_1, x_2]\begin{bmatrix} 0 & -(\gamma-\delta) \\ (\gamma-\delta) & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= [x_1, x_2]\begin{bmatrix} -x_2(\gamma-\delta) \\ -x_1(\gamma-\delta) \end{bmatrix} \\ &= (\gamma-\delta)\,[x_1x_2 - x_1x_2] \\ &= 0. \end{aligned}$$

Therefore, an obvious consequence of corollary 2 shows that $\mu(Q) = \mu(Q_S)$.

Corollary 3. The necessary and sufficient condition of

$$x^T Q_S x < 0, \tag{3.28}$$

or

$$\mu(Q_S) < 0,$$

where

$$x \neq 0, \quad Q_S=\begin{bmatrix} \alpha & \dfrac{\gamma+\delta}{2} \\ \dfrac{\gamma+\delta}{2} & \beta \end{bmatrix}, \tag{3.29}$$

is that the following conditions must hold:

$$\alpha+\beta<0, \tag{3.30}$$

$$\alpha\beta-\left(\frac{\gamma+\delta}{2}\right)^2>0. \tag{3.31}$$

Proof: Consider that the eigenvalues of Q_S must be positive. From corollary 2 and 3, it follows that $\mu(Q_S)<0$, if the conditions (3.30) and (3.31) are true.

By comparing Eq. (3.23) and (3.24), we obtain expressions for the functions ρ and π. We introduce the further notations $a, b, c, d, e, A, B, C, D, E, F, G$ and then verify the validity of the conditions (3.30) and (3.31) respectively:

$$\alpha=\left(\frac{-r_0^2}{1+r_0^2}\right)\rho+\left(\frac{-1}{1+r_0^2}\right)\pi=a\rho+b\pi, \tag{3.32}$$

$$\beta=\left(\frac{-1}{1+r_0^2}\right)\rho+\left(\frac{-r_0^2}{1+r_0^2}\right)\pi=b\rho+a\pi, \tag{3.33}$$

$$\gamma=\left(\frac{r_0}{1+r_0^2}\sqrt{\frac{\lambda_2}{\lambda_1}}\right)\rho+\left(\frac{-r_0}{1+r_0^2}\sqrt{\frac{\lambda_2}{\lambda_1}}\right)\pi-\sqrt{\frac{\lambda_2}{\lambda_1}}, \tag{3.34}$$

$$\delta=\left(\frac{r_0}{1+r_0^2}\sqrt{\frac{\lambda_1}{\lambda_2}}\right)\rho+\left(\frac{-r_0}{1+r_0^2}\sqrt{\frac{\lambda_1}{\lambda_2}}\right)\pi+\sqrt{\frac{\lambda_1}{\lambda_2}}, \tag{3.35}$$

$$\frac{\gamma+\delta}{2}=\left(\frac{r_0}{2(1+r_0^2)}\left(\frac{\lambda_1+\lambda_2}{\sqrt{\lambda_1\lambda_2}}\right)\right)\rho+\left(\frac{-r_0}{2(1+r_0^2)}\left(\frac{\lambda_1+\lambda_2}{\sqrt{\lambda_1\lambda_2}}\right)\right)\pi+\frac{1}{2}\left(\frac{\lambda_1-\lambda_2}{\sqrt{\lambda_1\lambda_2}}\right). \tag{3.36}$$

Letting

$$c=\frac{r_0}{2(1+r_0^2)}\left(\frac{\lambda_1+\lambda_2}{\sqrt{\lambda_1\lambda_2}}\right),\quad d=\frac{-r_0}{2(1+r_0^2)}\left(\frac{\lambda_1+\lambda_2}{\sqrt{\lambda_1\lambda_2}}\right),\quad e=\frac{1}{2}\left(\frac{\lambda_1-\lambda_2}{\sqrt{\lambda_1\lambda_2}}\right), \tag{3.37}$$

we have

$$\frac{\gamma+\delta}{2}=c\rho+d\pi+e \tag{3.38}$$

Substituting for α, β, γ, δ in condition (3.31), we have

$$\alpha\beta-\left(\frac{\gamma+\delta}{2}\right)^2=(ab-c^2)\rho^2+((a^2+b^2)-2cd)\rho\pi+(ab-d^2)\pi^2+$$
$$+(-2ce)\rho+(-2de)\pi+e^2, \tag{3.39}$$

and letting

$$A=ab-c^2,\qquad B=(a^2+b^2)-2cd,\qquad C=ab-d^2,$$
$$D=-2ce,\qquad E=-2de,\qquad F=e^2,$$

the following inequality is obtained

$$\alpha+\beta=(a+b)\rho+(a+b)\pi=(a+b)(\rho+\pi), \tag{3.41}$$

By similar reasoning, we write condition (3.30) as

$$\alpha\beta-\left(\frac{\gamma+\delta}{2}\right)^2=A\rho^2+B\rho\pi+C\pi^2+D\rho+E\pi+F>0. \tag{3.40}$$

and letting $G=a+b$, we obtain the inequality

$$\alpha+\beta=G(\rho+\pi)<0. \tag{3.42}$$

From this, one can see that from inequality (3.40) follows inequality (3.42) too. The significance of these conditions can be shown by substituting some values for the parameters p, r (e.g. (p, r): (4, 1), (4, 0.1), (4, 0.01) and the functions obtained are drawn on the (ρ, π) coordinate system. These are shown in Fig. 1, 2, and 3. The shaded regions of these figures represent the stability criteria of the nonlinear Eq. (1.1).

As a result of this section, we have the following theorem. The stability charts shown above can be used in the following way: if we could find positive constants for the parameters p, r so that the values of the functions $\rho(t, x)$ and $\pi(t, x)$ fall in the shaded region (and not tending to the boundary) of

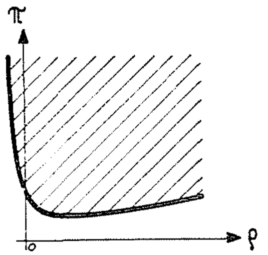

Fig. 1

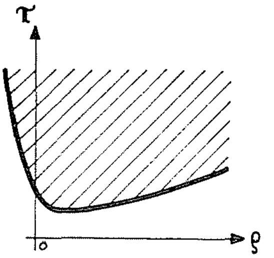

Fig. 2

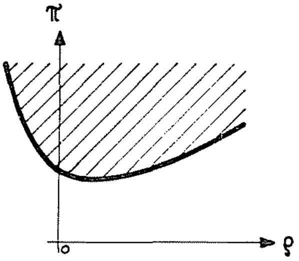

Fig. 3

the stability charts corresponding to these constants, then the zero equilibrium state of the nonlinear Eq. (1.1) is asymptotically stable. Moreover, every solution of the nonlinear Eq. (1.1) tends to zero:

$$\lim_{t \to \infty} \|x(t)\| = 0 \tag{3.43}$$

4. Application

To illustrate a practical example of this theorem, let us consider an application in connection with an electrical circuit, where C, L, R_1, and R_2 are constants and R_3 is a nonlinear resistance (see Fig. 4). In [1, 4], possibilities of more important applications of this theorem can be seen especially in connection with mechanical vibrations.

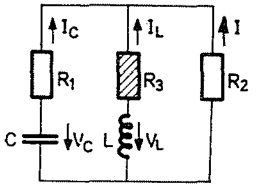

Fig. 4

C —capacitance ($I_C = C\dot{V}_C$)
L —inductance ($V_L = L\dot{I}_L$)
R_1, R_2 — resistances ($R_1, R_2 > 0$)
R_3 — nonlinear and or time dependent resistance
I — current
V — voltage

By applying Kirchhoff's laws, we obtain the differential equation of the electric circuit of the form

$$\begin{bmatrix} \dot{I}_L \\ \dot{V}_C \end{bmatrix} = \frac{1}{R_1+R_2} \begin{bmatrix} \dfrac{-(R_1R_2+R_1R_3+R_2R_3)}{L} & \dfrac{R_2}{L} \\ -\dfrac{R_2}{C} & -\dfrac{1}{C} \end{bmatrix} \begin{bmatrix} I_L \\ V_c \end{bmatrix} \tag{4.1}$$

This equation can be transformed to the form of Eq. (1.1). This transformation is shown as follows: denoting *a, b, c, d* for the elements of the matrix in (4.1), we write Eq. (4.1) as

$$\dot{z} = Kz, \tag{4.2}$$

where

$$K = \frac{1}{R_1+R_2} \begin{bmatrix} \dfrac{-(R_1R_2+R_1R_3+R_2R_3)}{L} & \dfrac{R_2}{L} \\ -\dfrac{R_2}{C} & -\dfrac{1}{C} \end{bmatrix} = \begin{bmatrix} -a & c \\ -d & -b \end{bmatrix},$$

$$Z = \begin{bmatrix} I_L \\ V_c \end{bmatrix}, \qquad \dot{z} = \begin{bmatrix} \dot{I}_L \\ \dot{V}_c \end{bmatrix}. \tag{4.3}$$

Furthermore, let

$$\omega = Sz \tag{4.4}$$

where

$$S=\begin{bmatrix} \frac{1}{\alpha} & 0 \\ 0 & \frac{1}{\beta} \end{bmatrix}, \qquad Z=\begin{bmatrix} I_L \\ V_c \end{bmatrix}, \tag{4.5}$$

α and β are constants.

By applying an appropriate linear transformation for the variable ω, we have:

$$\dot{\omega}=SKS^{-1}\omega=\begin{bmatrix} -a & \frac{c\beta}{\alpha} \\ -\frac{d\alpha}{\beta} & -b \end{bmatrix}\begin{bmatrix} \frac{I_L}{\alpha} \\ \frac{V_c}{\beta} \end{bmatrix}. \tag{4.6}$$

Assuming the above matrix to be symmetric, then we write

$$-\frac{d\alpha}{\beta}=\frac{c\beta}{\alpha} \tag{4.7}$$

If

$$d\tfrac{\alpha}{\beta}=c\tfrac{\beta}{\alpha}=-q,$$

then

$$d\tfrac{\alpha}{\beta}=-q \qquad \text{implies } \frac{\beta}{\alpha}=-\frac{d}{q} \tag{4.8}$$

and

$$c\tfrac{\beta}{\alpha}=-q \qquad \text{implies } \frac{\beta}{\alpha}=-\frac{q}{c}. \tag{4.9}$$

Dividing and multiplying (4.8) by (4.9), respectively we have the following:

$$q=\sqrt{cd}, \tag{4.10}$$

$$\frac{\alpha}{\beta}=\sqrt{\frac{c}{d}}, \tag{4.11}$$

and suppose $\beta=1$, then $\alpha=\sqrt{\frac{c}{d}}$.

However, substitution of (4.10) in (4.6) gives

$$\dot{\omega}=\begin{bmatrix} -a & q \\ -q & -b \end{bmatrix}\begin{bmatrix} \frac{I_L}{\alpha} \\ \frac{V_c}{\beta} \end{bmatrix}, \tag{4.12}$$

and factorization of q, yields

$$\dot{\omega}=q\begin{bmatrix}-\frac{a}{q} & 1\\ -1 & -\frac{b}{q}\end{bmatrix}\begin{bmatrix}\frac{I_L}{\alpha}\\ \frac{V_c}{\beta}\end{bmatrix}. \tag{4.13}$$

Furthermore, we introduce another transformation.

If $\tau=\gamma t$, where γ is a constant and τ is the transformed time, then

$$\frac{d\omega}{d\tau}=\frac{d\omega}{dt}\cdot\frac{dt}{d\tau}, \tag{4.14}$$

where

$$\frac{d\omega}{dt}=\dot{\omega}=q\begin{bmatrix}-\frac{a}{q} & 1\\ -1 & -\frac{b}{q}\end{bmatrix}\begin{bmatrix}\frac{I_L}{\alpha}\\ \frac{V_c}{\beta}\end{bmatrix}, \qquad \frac{dt}{d\tau}=\frac{1}{\gamma}. \tag{4.15}$$

Substitution of (4.15) in (4.14), gives

$$\frac{d\omega}{d\tau}=\frac{q}{\gamma}\begin{bmatrix}-\frac{a}{q} & 1\\ -1 & -\frac{b}{q}\end{bmatrix}\begin{bmatrix}\frac{I_L}{\alpha}\\ \frac{V_c}{\beta}\end{bmatrix}. \tag{4.16}$$

From this, the required result follows on, letting

$$\gamma=q, \qquad \frac{a}{q}=\rho, \qquad \frac{b}{q}=\pi, \qquad \omega=\begin{bmatrix}\frac{I_L}{\alpha}\\ \frac{V_c}{\beta}\end{bmatrix}=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

and $\omega(t(\tau))=x(\tau)$.

Hence, we deduce

$$\frac{d\omega}{d\tau}=\begin{bmatrix}-\rho & 1\\ -1 & -\pi\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}. \tag{4.17}$$

Observe that the original Eq. (1.1) is obtained.

The system (4.1) under equilibrium state (i.e. $I_L=V_c=0$) is asymptotically stable if C, L, R_1 and R_2 are positive constants. On the other hand, the stability criteria allow that asymptotic stability can hold when R_3 (i.e. ρ and π) is a variable and lies within the stability region. But surprisingly, it has been shown in this work that for R_3 and ρ being negative variables, asymptotic stability can also hold, if R_3, ρ remain within the stability regions (see Figs 1, 2 and 3). In other words, the negative nonlinear resistance R_3 keeps on adding energy into the system yet the total energy in the system decreases and the equilibrium point is asymptotically stable if R_3 remains within the stability regions established above.

5. Remarks

From the estimation point of view of the stability criteria, we consider the following assumptions. When ρ and π are constants (i.e. $\rho = r$, $\pi = p$) the solution $x \equiv 0$ of Equation (1.1) is asymptotically stable if, and only if,

$$r + p > 0, \tag{5.1}$$

$$rp > -1. \tag{5.2}$$

Namely, we write equation (1.1) as

$$\dot{x} = \begin{bmatrix} -r & 1 \\ -1 & -p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{5.3}$$

The characteristic equation

$$\begin{bmatrix} -r-\lambda & 1 \\ -1 & -p-\lambda \end{bmatrix} = 0 \tag{5.4}$$

or

$$\lambda^2 + (r+p)\lambda + pr + 1 = 0$$

has roots

$$\lambda_{1,2} = \frac{-(r+p) \pm \sqrt{(r+p)^2 - 4(pr+1)}}{2}. \tag{5.5}$$

The solution $x \equiv 0$ of Eq. (5.3) is asymptotically stable if, and only if, the roots λ_1, λ_2 of the characteristic Eq. (5.4) have negative real parts. This can only hold if the two conditions (5.1) and (5.2) are valid. Thus the domain of these conditions is shown in Fig. 5. On the other hand, if ρ and π are variables, then the well-known stability criterion of asymptotic stability $\mu(A) < 0$ is obtained. Similarly, in [7], we can see that $\mu(A) = \max \cdot (-\rho, -\pi)$; which implies that the functions $\rho(t, x)$ and $\pi(t, x)$ have positive values for every value of the time $t \geq t_0$ and x. This is represented by the shaded region in Fig. 6.

However, it has been shown in this paper that in special cases, when ρ is negative for a nonlinear system, the asymptotic stability (Figs 5 and 6) can hold as well (see Figs 1, 2 and 3).

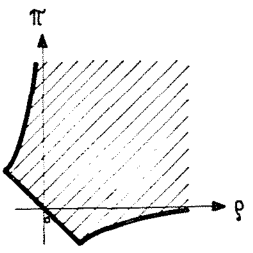

Fig. 5

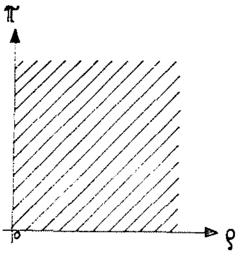

Fig. 6

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