MULTISTATE SYSTEM RELIABILITY ANALYSIS BY BINARY VARIABLES

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Abstract

This paper presents an application of a discrete function class, i.e. the monotone s-coherent discrete functions and an extending concept [6] for the use of binary variables in multistate system investigation. Two propositions are proven for the purpose of the generalization procedure. Some illustrative examples are shown which prove that its application is simple and very suitable for the calculation of multistate system reliability.

Introduction

In the engineering domain many reliability investigations are not suitable if one assumes that the system and its components are either functioning or have failed. That is, mathematically we consider the system in question as a binary system. This concept not always proves the real situation [1, 2, 3, 5, 6, 7] because not only the function or the failure of systems but their performance levels interest us also. For instance, an electrical plant having 5 identical generators may function without breakdown even if some of the generators have failed (or are out of service). Of course, the best function quality of this system may be obtained when all generators function normally or are at full power. Corresponding to requirement one has to state the threshold level (usually expressed by an integer number) of the system. After the statement of system levels it is evident that the level number of components and of the system is higher than 2, i.e. their variables stop to be binary, they become multi-valued. The other example is encountered when we have to consider the three-state device systems. A fluid flow valve and electronic diode are typical examples of three-state devices. Either of these components may fail catastrophically in either the open or closed (shorted) mode. A given device will then have a probability of failure in the open mode and a probability failure in the closed or shorted mode. Because a three-state device cannot fail simultaneously in both the open and the closed modes, the failures are mutually exclusive events. The failure of any such device is considered independent of all the others.

In the following section we will use an extending method based on the binary variables to study the multistate systems of a special generation [6].

Notations, nomenclatures and definitions

Because the method in question is derived from the binary system theory and for the sake of better understanding we want to show in the following the analogy between the two concepts (i.e. binary and multistate).

For binary systems

a. Notations

X_i = state of component i, $X_i \in \{0, 1\} = \{\text{failed, functioning}\}$.
$P_i = \Pr\{X_i = 1\}$ (probability of event $X_i = 1$)
$\boldsymbol{X} = (X_1, \ldots, X_i, \ldots, X_n)$, vector of component states.
n = number of system components.
$\varphi(X)$ = state of system, called system structure function

$$\varphi(\boldsymbol{X}) = \begin{cases} 1 & \text{if system is functioning} \\ 0 & \text{if system has failed} \end{cases}$$

b. Nomenclatures and definitions

Monotone: a system for which

$$\varphi(\boldsymbol{X}) = \begin{cases} 0 & \text{if } \boldsymbol{X} = \boldsymbol{0} \\ 1 & \text{if } \boldsymbol{X} = \boldsymbol{1} \end{cases}$$

and $\varphi(\boldsymbol{X})$ is nondecreasing in $\boldsymbol{X}$.

Relevant: component i is relevant if there exists a set of component states $\{X_v \backslash X_i\}$, $v = \overline{1, n}$; $i \in \{1, 2, \ldots, n\}$

(where $\overline{1, n} = 1, 2, \ldots, n$) such that

$$\varphi(X_1, \ldots, X_{i-1}, 0, X_{i+1}, \ldots, X_n) = 0$$

and

$$\varphi(X_1, \ldots, X_{i-1}, 1, X_{i+1}, \ldots, X_n) = 1.$$

Series configuration: a system for which

$$\varphi(\boldsymbol{X}) = \begin{cases} \prod_{i=1}^{n} X_i = \min(X_i) = 1 & \textit{if all components function} \\ 0 & \text{otherwise} \end{cases}$$

Parallel configuration: a system for which

$$\varphi(\mathbb{X})=\begin{cases}1-\prod_{i=1}^{n}(1-X_i)=\max(X_i)=0 & \text{if all components fail}\\ 1 & \text{otherwise}\end{cases}$$

k-out-of-n : G : a system for which

$$\varphi(\mathbb{X})=\begin{cases}1 & \text{if at least } k \text{ of its } n \text{ elements function}\\ 0 & \text{otherwise}\end{cases}$$

k-out-of-n : F : a system for which

$$\varphi(\mathbb{X})=\begin{cases}0 & \text{iff at least } k \text{ of its } n \text{ elements are failed}\\ 1 & \text{otherwise}\end{cases}$$

Reliability: Probability that the system functions without repair or replacement $= \Pr\{\varphi(\mathbb{X})=1\}=E\{\varphi(\mathbb{X})\}$.

Mincut set: a minimal set of components such that if all components in the set have failed then the system fails.

Minpath set: minimal set of components, such that if all components in the set function then the system functions.

For multistate systems

a. Notations

X_i = state of component i, $X_i \in \{0, 1, \dots, N\}$.
N_i = best state of component i.
$\mathbb{X}=(X_1, \dots, X_n)$, vector of component states.
n = number of system components
X_{ij} = indicator, meaning that

$$X_{ij}=\begin{cases}1 & \text{if } X_i \geq j\\ 0 & \text{otherwise}\end{cases}$$

$P_{ij}=\Pr\{X_{ij}=1\}$
$\varphi(\mathbb{X})$ = state of the system, called the system structure function; $\varphi(\mathbb{X}) \in \{0, 1, \dots, M\}$
M = best state of the system.

$$\varphi^k(\mathbb{X})=\begin{cases}1 & \text{if } \varphi(\mathbb{X}) \geq k\\ 0 & \text{otherwise}\end{cases}$$

i.e., $\varphi^k(\mathbb{X})$ is a binary monotone structure function which gives only two values: $\{1, 0\}$.

b. Nomenclatures and definitions

Monotone: a system for which

$$\varphi(\boldsymbol{X})=\begin{cases}0 & \text{if } \boldsymbol{X}=\boldsymbol{0}\\ M & \text{if } \boldsymbol{X}=(N_1,\ldots,N_n)\end{cases}$$

and $\varphi(\boldsymbol{X})$ is nondecreasing in $\boldsymbol{X}$.

Relevant: component i is relevant if there exists a set of component states $\{X_v \backslash X_i\}$, $v=\overline{1,n}$, $i\in\{1,2,\ldots,n\}$ such that

$$\varphi(X_1,\ldots,X_{i-1},0,X_{i+1},\ldots,X_n)<\varphi(X_1,\ldots,X_{i-1},N_i,X_{n+1},\ldots,X_n)$$

Series configuration: a system for which

$$\varphi(\boldsymbol{X})=\min(X_i)$$

Parallel configuration: a system for which

$$\varphi(\boldsymbol{X})=\max(X_i)$$

k-out-of-n: G: a system for which

$$\varphi(\boldsymbol{X})=\max(j\text{: at least } k \text{ components are above or at level } j)$$

Series configuration at level j: a system for which

$$\varphi(\boldsymbol{X})=j \quad \text{iff} \min(X_i)=j$$

Parallel configuration at level j: a system for which

$$\varphi(\boldsymbol{X})=j \quad \text{iff} \max(X_i)=j$$

k-out-of-n: G: at level j: a system for which

$$\varphi(\boldsymbol{X})=j \text{ iff at least } k \text{ components are at or above level } j$$

Reliability at level k: probability that the system state is k or higher $=\Pr\{\varphi(\boldsymbol{X})\geq k\}=\Pr\{\varphi^k(\boldsymbol{X})\}=E\{\varphi^k(\boldsymbol{X})\}$ (because of the monotone character of the system).

After these it is necessary to define the concept of the s-coherent system as soon as one of the s-coherent structures functions.

A system is called s-coherent iff

1) every one of its components plays a role in the system,

2) any state change of any element influences the system,

3) the direction of the state change is consistent with that of the change of the system.

It is evident that these three conditions of the s-coherent system also mean that, if any component is not s-coherent then the system is not s-coherent, too.

This definition holds for both binary and multistate systems.

Note that a binary *s*-coherent structure function is a binary monotone structure function for which all components are relevant. Corresponding to the monotone and *s*-coherent properties of multistate system structure function $\Phi(X)$ which may be received by means of two propositions, shown in the next section, connected to binary variables.

Determination of component state and structure function (system state) in multistate case by means of binary variables

For determination of the component state in a multistate case the following proposition will be used:

Proposition 1. From the definition of indicator X_{ij} the state of multistate component i, X_i, may be derived from a sum of binary variables as follows

$$X_i = \sum_{j=1}^{N_i} X_{ij} \tag{1}$$

Proof. From the definition of X_{ij} given above, i.e.

$$X_{ij} = \begin{cases} 0 & \text{if} \quad X_i < j \\ 1 & \text{if} \quad X_i \geq j \end{cases}$$

the proposition is evident. We assume namely, $X_i = j$ and take an index $j_0 \leq j$. The proof is to check, allowing (1) whether X_i equals j or not? For this purpose, we first, consider the truth of relations as follows

$$X_{ij_0} = \begin{cases} 0 & \text{if} \quad X_i = j < j_0 & \text{(a)} \\ 1 & \text{if} \quad X_i = j \geq j_0. & \text{(b)} \end{cases}$$

We see that the relation (b) is true, because the given value $j_0 \leq j$, therefore this means that

$$X_{ij_0} = 1, \quad \forall j_0 \leq j = X_i \tag{2}$$

Besides, the expression (1) is equivalent to two corresponding partial sums, that is

$$X_i = \sum_{v=1}^{j} X_{iv} + \sum_{v=j+1}^{N_i} X_i \tag{3}$$

From (2) we find that

$$\sum_{v=1}^{j} X_{iv} = j \tag{4}$$

and we can prove that in the right-hand-side of (3) the second term is zero. That is, from

$$\underset{(v=j+1)}{X_{iv}} \left\{ = \begin{matrix} 0 & \text{if} & X_i = j < v \\ 1 & \text{if} & X_i = j \geq v \end{matrix} \right. \quad \begin{matrix} \text{(c)} \\ \text{(d)} \end{matrix}$$

relation (c) is true. This fact means that

$$X_i = 0, \quad \forall v > j \tag{5}$$

from which the mentioned second term really equals zero. Because the given value X_i (i.e. j) is optional, the proof is completed. □

Consequently

$$X_{i0} = 0, \qquad X_{iN_i} = 1 \tag{6}$$

and

$$\max \left\{ X_i = \sum_{j=1}^{N_i} X_{ij} \right\} = N_i \tag{7}$$

We see, the result is totally identical with the definition of the best state of the component. This fact means that the generalization procedure based on the binary variables is logically true.

Proposition 2. From the definition of $\varphi^k(X)$ the state of the multistate system is

$$\varphi(X) = \sum_{k=1}^{M} \varphi^k(X) \tag{8}$$

Proof. By the definition of $\varphi^k(X)$ and with the procedure used to prove Proposition 1, Proposition 2 is evident. □

Block-diagram and event tree

For multistate systems the reliability at level k is determined by constructing the block diagram corresponding to $\varphi^k(X)$ and then calculating $\Pr\{\varphi^k(X) = 1\}$. If the components are s-independent and if, for each component i, at most one binary variable X_{ij} appears at most once in the block diagram, then reliability can be calculated as though the binary components were s-independent. Otherwise, conditional probability expansions are necessary.

Remember that a block diagram is a logic diagram composed of series and parallel configurations. However, a block diagram is not limited to systems which have only series and parallel combination of components since it is well known that a system can be represented in terms of its minpath sets or mincut sets [8]. Each block in a block diagram represents a binary random variable.

The series configuration replaces n blocks in series with

$$\prod_{i=1}^{n} X_i$$

while the parallel one replaces n blocks in parallel with

$$1-\prod_{i=1}^{n}(1-X_i).$$

As for the event tree, from the block diagram and with the well-known correspondances existing between series configuration and AND gate ($\triangle$), or between the parallel configuration and OR gate ($\triangle$) [3] the corresponding event tree of $\varphi^k(\boldsymbol{X})$ can be constructed and from it the system reliability at level k may be determined.

Illustrative examples

Example 1: There is a monotone s-coherent multistate system consisting of two four-state components X_1 and X_2, i.e.,

$$\boldsymbol{X}(X_1, X_2) \rightarrow n=2$$

and

$$X_1 \in \{0, 1, 2, 3\} \rightarrow N_1 = 3$$
$$X_2 \in \{0, 1, 2, 3\} \rightarrow N_2 = 3$$

and

$$\varphi(\boldsymbol{X}) = \varphi(X_1, X_2) \in \{0, 1, 2, 3\} \rightarrow M = 3$$

with its representation in value table [4] as follows

X_2 \ X_1	0	1	2	3
0	0	0	1	2
1	0	1	2	2
2	1	2	2	3
3	2	2	3	3

Determine the structure functions $\varphi^k(\boldsymbol{X})$, their block diagram and event tree (success) and system reliability at level 1.

Before determining the value of $\varphi(\boldsymbol{X})$ check concretely some values of X_i after (1).

First, let X_1 be 2. By means of (1) we have

$$X_1 = \sum_{j=1}^{3} X_{1j} = X_{11} + X_{12}$$

Because

$$X_{11}=\begin{cases}0 & \text{if} \quad X_1<1?\\ 1 & \text{if} \quad X_1\geq 1?\end{cases}\Bigg|_{X_1=2}$$

therefore

$$X_{11}=1.$$

Similarly

$$X_{12}=\begin{cases}0 & \text{if} \quad X_1<2?\\ 1 & \text{if} \quad X_1\geq 2?\end{cases}\Bigg|_{X_1=2}$$

from which

$$X_{12}=1.$$

We may see that if $j=3$ then $X_{13}=0$, namely

$$X_{13}=\begin{cases}0 & \text{if} \quad X_1<3?\\ 1 & \text{if} \quad X_1\geq 3?\end{cases}\Bigg|_{X_1=2}$$

and thus X_{13} is zero! After all

$$X_1=X_{11}+X_{12}+X_{13}=1+1+0=2,$$

the result thus really is true.

Similarly, controlling the value of X_2 is possible in the same way.

Now, we determine the structure functions $\varphi^k(X)$ from the given values of $\varphi(\boldsymbol{X})$ (in the value table). We receive the following results:

$$\varphi(0,0)=\varphi(1,0)=\varphi(0,1)=0$$

$$\varphi(2,0)=\varphi(1,1)=\varphi(0,2)=1$$

$$\varphi(0,3)=\varphi(1,3)=\varphi(1,2)=\varphi(2,2)=\varphi(2,1)=\varphi(3,1)=\varphi(3,0)=$$
$$=\varphi(3,0)=2$$

$$\varphi(3,2)=\varphi(2,3)=\varphi(3,3)=3$$

from which

$$\varphi^1(\boldsymbol{X})=\varphi^1(X_{12},X_{11}X_{21},X_{22})$$

$$\varphi^2(\boldsymbol{X})=\varphi^2(X_{23},X_{11}X_{23},X_{11}X_{22},X_{12}X_{22},X_{12}X_{21},X_{13}X_{21},X_{13})$$

$$\varphi^3(\boldsymbol{X})=\varphi^3(X_{13}X_{22},X_{12}X_{23},X_{13}X_{23})$$

The corresponding block-diagrams and event trees are shown in Fig. 1. (It is well known that the fault tree may be derived from the success event tree as a complement or negation of it).

Because both the state levels and the components appear in the block-

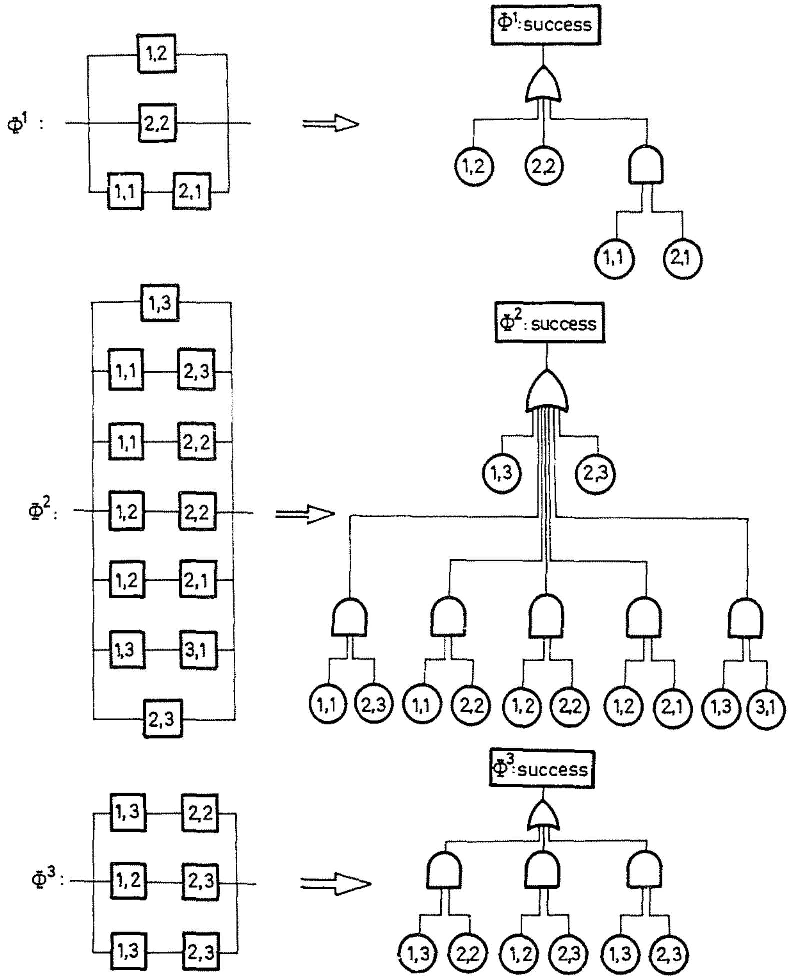

Fig. 1

diagram of $\varphi(\boldsymbol{X})$ the conditional probability expansion will be used:

$$E\{\varphi^1(\boldsymbol{X})\} = P_{12}E\{\varphi^1(\boldsymbol{X})/X_{12}=1\} + (1-P_{12})E\{\varphi^1(\boldsymbol{X})/X_{12}=0\}$$

$$= P_{12} + (1-P_{12})P_{22} + (1-P_{12})(1-P_{22})E\{\varphi^1(\boldsymbol{X})/X_{12}=0, X_{22} = 0\}$$

$$= P_{12} + (1-P_{12})P_{22} + (1-P_{12})(1-P_{22}) \times (P_{11}-P_{12}) \times (P_{21} - P_{22}).$$

Example 2: Consider an oil supply consisting of three identical gear-pumps working in a parallel configuration connecting with a common oil filter.

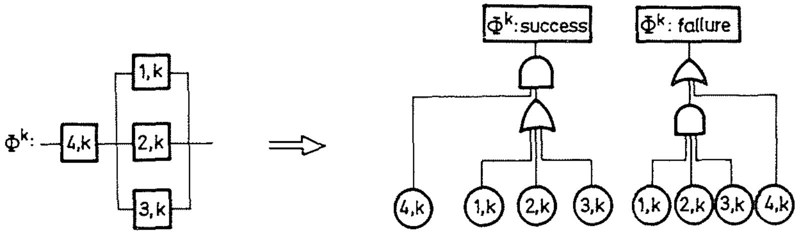

Fig. 2

Corresponding to revolution per minute (RPM) the volumetrical flow may supply 0%, 25%, 50%, 75% and 100% of the maximal performance (corresponding to states 0, 1, 2, 3 and 4). Pipelines are assumed to be perfect. The system is acceptable if one of the three pumps functions perfectly, assuming *s*-independent components.

With the given conditions, both the block diagram and the event tree at level k are easily constructed (see Fig. 2) where blocks 1, 2, 3 represent the pumps and block 4 represents the oil filter.

After the block diagram corresponding to $\varphi^k(X)$ the result is

$$\Pr\{\varphi^k(\mathrm{X})=1\}=P_{4k}[1-(1-P_{1k})(1-P_{2k})(1-P_{3k})]$$

Conclusion and discussion

As we see in the introduction, in the engineering domain multistate systems are usually encountered because of different concepts. In other words the causes of a system being multistate system are not the same, each occurs through system performance or efficiency and the other differing occurrence in different positions. There are many other concepts which may be discussed in detail.

By means of binary variables and at simpler systems the multistate concept may be handled without special difficulties. Also, with two simple propositions and generalization steps from a binary system into a multi-valued system and a set of generalized definitions, multistate components, configurations and systems are defined.

We may see that corresponding to the state level in question the block diagram and event (or fault) tree are easily constructed, from which system reliability at the mentioned level may be calculated. It is very interesting to observe that from a given system the structure function, viz. the block diagram and event tree may have different configurations corresponding to the state level of interest.

The truth is that at more complicate system structures the effectiveness of the method is no more evident, furthermore it may be lost in the case of certain system complexity.

In general in spite of the s-independence existing between the elements of the system, instead of conditionless probability, condition probability expansion is used in reliability calculations because with a binary state representation, redundancy and state level, as conditions, occur in the block diagram or in the event tree at the level in question.

Remember that this method may only be used for the monotone s-coherent systems, but it is well known that in general both engineering systems always satisfy these conditions. Thus, this method is very useful in the technical domain.

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