ON HOPF BIFURCATION OF RAYLEIGH'S EQUATION

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Summary

Rayleigh's differential equation exhibits a supercritical Hopf bifurcation. The bifurcating closed orbits correspond to fixed points of the Poincaré map. In this paper explicit estimates of the domain of the Poincaré map are given and these estimates are checked by a computer. The results may help the estimate of the possible amplitude of the stable oscillation of this important nonlinear system.

Introduction

Rayleigh's differential equation

$$\frac{d^2u}{dt^2}+\frac{du}{dt}\left(\left(\frac{du}{dt}\right)^2-\mu\right)+u=0, \quad \mu\in R \tag{1}$$

is an important model used extensively in the theory of non-linear oscillations (see e. g. Minorsky [3]). A system equivalent to the second order scalar equation (1) is

$$\dot{x}_1=x_2, \quad \dot{x}_2=-x_1+\mu x_2-x_2^3 \tag{2}$$

where dot denotes differentiation with respect to $t\in R$. This system satisfies all the conditions of the classical Hopf bifurcation theorem [1] (or see Marsden-McCracken [2]). That is to say, the origin $x=(x_1, x_2)=0$ is an equilibrium of system (2) for all $\mu\in R$. For $\mu<0$ it is asymptotically stable, for $\mu>0$ it is unstable. A $\delta>0$ and a smooth function $\mu:(-\delta, \delta)\mapsto R$ exist such that $\mu(0)=0$, $\mu'(0)=0$ and for all $x_1^0\in(-\delta, \delta)$ the solution of

$$\dot{x}_1=x_2, \quad \dot{x}_2=-x_1+\mu(x_1^0)x_2-x_2^3$$

corresponding to the initial condition $(x_1^0, 0)$ is periodic, and in a three dimensional neighbourhood of $(x_1, x_2, \mu)=(0, 0, 0)$ the corresponding trajectories are the only closed paths of system (2). Moreover, applying the method of

Negrini-Salvadori [4] a Lyapunov function

$$F(x_1, x_2) = x_1^2 + x_2^2 + \frac{3}{4} x_1^3 x_2 + \frac{5}{4} x_1 x_2^3$$

can be determined which is, obviously, positive definite in a neighbourhood of the origin $x=0$ and its derivative with respect to system (2) with $\mu=0$ is negative definite:

$$\dot{F}(x_1, x_2) = -\frac{3}{4}(x_1^2 + x_2^2)^2 + \Theta(|x|^4).$$

This implies that $x=0$ is a 3-attractor (a vague attractor) of system (2) with $\mu=0$ and, as a consequence, $\mu(x_1)>0$, $x_1 \neq 0$, and the bifurcating closed paths are orbitally asymptotically stable.

In this paper we are giving an explicit extimate of the domain of the Poincaré map attached to the problem. This estimate may enable us to estimate either the domain or the range of the function μ, i. e. the values of the bifurcation parameter μ and the initial conditions x_1^0 for which still bifurcating closed paths exist. (It is, of course, known that (1) has a non-constant periodic solution for all $\mu \neq 0$, since, (1) is equivalent to van der Pol's equation if $\mu \neq 0$). Our method seems to be fairly general and possibly applicable in treating bifurcations of other two dimensional systems. However, the estimates gained by this exact method are rather cc nservative compared to results gained by computer experiments. In section 2 the domain of the Poincaré map is estimated from below, and in section 3 the results gained by a computer are presented.

The Estimate of the Poincaré Map

First of all, system (2) will be transformed by the polar transtormation $x_1 = r\cos\Theta$, $x_2 = r\sin\Theta$ into

$$\dot{r} = r\sin^2\Theta(\mu - r^2\sin^2\Theta)$$

$$\dot{\Theta} = -1 - \frac{r^2}{2}\sin 2\Theta \sin^2\Theta + \frac{\mu}{2}\sin 2\Theta. \qquad (3)$$

The solution of (3) assuming the initial values (r, Θ) at $t=0$ will be denoted by $(\tilde{r}(t, r, \Theta, \mu), \tilde{\Theta}(t, r, \Theta, \mu))$. It will be assumed always that $r \geq 0$. If the initial values are $(0, \Theta)$ then, clearly, $\tilde{r}(t, 0, \Theta, \mu) \equiv 0$ and the function $\tilde{\Theta}(t, 0, \Theta, \mu)$ satisfies

$$\dot{\tilde{\Theta}}(t, 0, \Theta, \mu) \equiv -1 + \frac{\mu}{2}\sin 2\tilde{\Theta}(t, 0, \Theta, \mu). \qquad (4)$$

In particular, $\tilde{\Theta}(t, 0, 0, 0) = -t$ and $\tilde{\Theta}(2\pi, 0, 0, 0) = -2\pi$. As a consequence, it is clear that for small enough μ and initial condition $(r, 0)$ the integral curves of the solutions of (3) will cut the $\Theta = -2\pi$ plane (in t, r, Θ space) at some moment $t = T(\mu, r) > 0$, see Fig. 1.

In view of the polar transformation the initial condition $(x_1, x_2) = (r, 0)$ attached to system (2) corresponds to the initial condition $(r, 0)$ attached to (3).

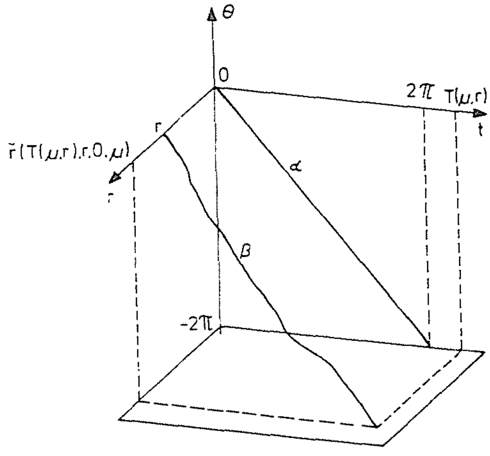

Fig. 1. Integral curves of the solutions of (3)
α: $\mu = 0$, $(r, \Theta) = (0, 0)$;
β: μ small, r small, $\Theta = 0$

If $(\varphi_1(t, x_1, x_2, \mu), \varphi_2(t, x_1, x_2, \mu))$ denotes the solution of (2) satisfying the initial conditions

$$(\varphi_1(0, x_1, x_2, \mu), \varphi_2(0, x_1, x_2, \mu)) = (x_1, x_2) \text{ then}$$

$$\varphi_1(T(\mu, r), r, 0, \mu) = \tilde{r}(T(\mu, r), r, 0, \mu) \cos(-2\pi) =$$

$$= \tilde{r}(T(\mu, r), r, 0, \mu) \geq 0,$$

$$\varphi_2(T(\mu, r), r, 0, \mu) = \tilde{r}(T(\mu, r), r, 0, \mu) \sin(-2\pi) = 0.$$

This means that the path corresponding to the solution $(\varphi_1(t, r, 0, \mu), \varphi_2(t, r, 0, \mu))$ of (2) intersects the positive x_1 axis at moment $t = T(\mu, r)$ the first time after the intersection $(r, 0)$ at $t = 0$. See Fig. 2. The Poincaré map, the domain of which we want to estimate, is the mapping defined by

$$(\mu, r) \mapsto \varphi_1(T(\mu, r), r, 0, \mu). \tag{5}$$

In order to get an estimate for the domain of this mapping we have to establish an estimate for the solutions of (3).

LEMMA. The solution $(\tilde{r}(t, r, 0, \mu), \tilde{\Theta}(t, r, 0, \mu))$ of (3) corresponding to the initial condition $(r, 0)$ $r > 0$ and to the parameter value $\mu \geq 0$ is defined in $0 \leq t < \infty$, and satisfies the inequality

$$0 < \tilde{r}(t, r, 0, \mu) \leq r(1 + t\mu^2/2r^2)^{1/2}. \tag{6}$$

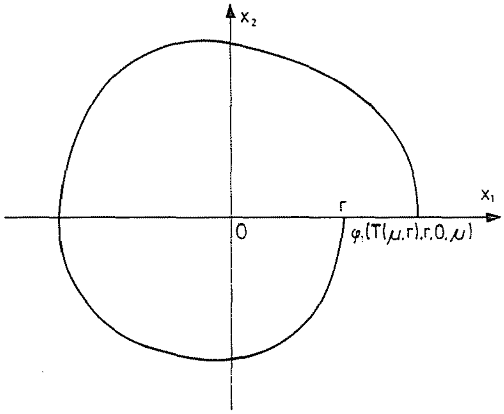

Fig. 2. Poincaré map attached to (2) around the origin

PROOF. If $r > 0$ the corresponding path cannot cross the axis $r = 0$ since the latter is itself a trajectory, thus, $\tilde{r}(t, r, 0, \mu) > 0$ in its existence domain. Taking this into account we estimate the maximum of the function standing on the right hand side of the first equation of (3) for fixed $r > 0$ and $\mu \geq 0$:

$$f_{\mu,r}(\Theta) := r \sin^2 \Theta \quad (\mu - r^2 \sin^2 \Theta).$$

After a somewhat lengthy but elementary calculation we get

$$f_{\mu,r}(\Theta) \leq \mu^2/4r, \quad r > 0, \quad \mu \geq 0.$$

Here the right hand side is the actual value of the maximum if $0 \leq \mu \leq 2r^2$ and it is an upper estimate in case $\mu > 2r^2$. Thus, the following inequality holds

$$\dot{\tilde{r}}(t, r, 0, \mu) \leq \mu^2/4\tilde{r}(t, r, 0, \mu), \quad r > 0, \quad \mu \geq 0.$$

Solving this differential inequality we get (6) for $t \geq 0$ in the existence domain of the solution. The right hand side of (6) is bounded in every bounded interval. Thus, $\tilde{r}$ is bounded in every bounded interval, and in view of the second equation of (3), the same is true for $\dot{\tilde{\Theta}}$ and, as a consequence, for $\tilde{\Theta}$. This implies that the solution is defined in $[0, \infty)$, and this completes the proof.

As it was shown at the beginning of this section, the integral curve of (3) corresponding to the initial condition (0, 0) and to the parameter value $\mu=0$ cuts the $\Theta=-2\pi$ plane at $t=2\pi$. Therefore, we are going to fix an interval larger than $[0, 2\pi]$ to ensure a crossing of the same plane by the integral curve corresponding to the initial condition $(r, 0)$, $r\geq 0$ and the parameter value $\mu\geq 0$.

THEOREM. Let us choose a constant $a>1$ and let μ and r satisfy the inequalities

$$0\leq\mu\leq 2(a-1)/a, \quad 0\leq r\leq(2(a-1)/a-a\pi\mu^2)^{1/2}; \tag{7}$$

then the integral curve of the solution $(\tilde{r}(t,r,0,\mu), \tilde{\Theta}(t,r,0,\mu))$ cuts the plane $\Theta=-2\pi$ at some moment $T(\mu, r)\in(0, 2a\pi]$.

PROOF. We get from the second inequality of (7) that

$$r^2(1+t\mu^2/2r^2)\leq r^2(1+2a\pi\mu^2/2r^2)\leq 2(a-1)/a$$

for $t\in[0, 2a\pi]$, $r>0$. Hence applying (6) we get that

$$\tilde{r}^2(t, r, 0, \mu)\leq 2(a-1)/a$$

for $t\in[0, 2a\pi]$, $r>0$, $\mu\geq 0$. However, the last inequality is trivially true also for $r=0$. The first inequality of (7) and the one we have just obtained imply (the writing out of the arguments will be omitted)

$$\dot{\tilde{\Theta}}(t, r, 0, \mu)=-1+\frac{1}{2}(\mu-r^2\sin^2\tilde{\Theta})\sin 2\tilde{\Theta}\leq$$

$$\leq -1+\frac{1}{2}\left|\mu-\tilde{r}^2\sin^2\tilde{\Theta}\right|\leq -1+\frac{a-1}{a}=-\frac{1}{a}.$$

Integrating from 0 to $2a\pi$ we get $\tilde{\Theta}(2a\pi, r, 0, \mu)\leq -2\pi$. Thus, for some $0<T(\mu, r)\leq 2a\pi$ we have $\tilde{\Theta}(T(\mu, r), r, 0, \mu)=-2\pi$ and this was to be proved.

COROLLARY. For any fixed $a>1$ the set D_a defined by (7) is a subset of the domain of the Poincaré map (5).

Numerical calculations show that the optimal set D_a is obtained if $a=2.5$ is chosen. Introducing the notation

$$F_a(\mu)=(2(a-1)/a-a\pi\mu^2)^{1/2}$$

Figure 3 shows the set $D_{2.5}$ and the graph of the function μ discussed in the Introduction. The latter has been gained by numerical methods.

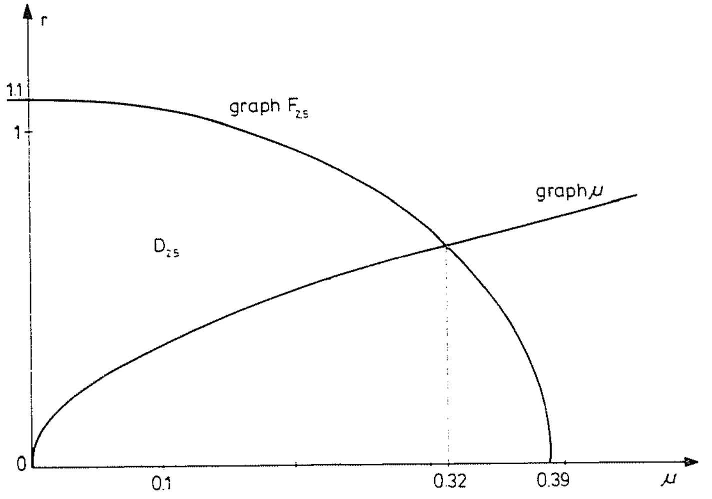

Fig. 3. Estimate of the domain of the Poincaré map

Numerical Results

The fixed points of the Poincaré map defined in (5) can be determined by computer. To each $r>0$ small enough the value $\mu(r)$ of the function μ can be uniquely determined for which $r=\varphi_1(T(\mu(r),r),\ r,\ 0,\ \mu(r))$ holds. The corresponding solution $(\varphi_1(t,r,0,\mu(r)),\ \varphi_2(t,r,0,\mu(r))$ of (2) (where $\mu=\mu(r)$) is periodic with period $T(\mu(r),r)$.

We have used a second generation computer of type ODRA-1204 and a CIL digital plotter to get graphical results. A Runge-Kutta type method was applied to solve system (2). We could bring down the relative error of the Poincaré map below 10^{-5}, keeping, at the same time, computer time within reasonable limits. We have determined the values of the function μ and the corresponding periods $T(\mu(r),r)$ first within the limits of the domain established theoretically (Fig. 3) then the values of r were increased considerably and the closed paths remained to exist, though became more and more deformed. The results are shown in Table 1. It can be seen from the fourth column that for small values of r we have $\mu(r) \approx 0.75r^2$.

Table 1

Initial value (r), corresponding bifurcation parameter value $\mu(r)$, period $T(\mu(r), r)$ and $\mu(r)/r^2$ for Rayleigh's equation

r	$\mu(r)$	$T(\mu(r), r)$	$\mu(r)/r^2$
.1	.007 503	6.283 207	0.7503
.2	.029 999	6.283 538	0.7500
.3	.067 451	6.284 970	0.7495
.4	.119 699	6.288 809	0.7481
.5	.186 359	6.296 808	0.7454
.6	.266 640	6.311 050	0.7407
.7	.359 269	6.333 690	0.7332
.8	.462 407	6.366 648	0.7225
.9	.573 786	6.411 260	0.7084
1.0	.690 961	6.468 034	0.6910
1.1	.811 566	6.536 639	0.6707
1.2	.933 595	6.616 046	0.6483
1.3	1.055 498	6.704 795	0.6246
1.4	1.176 237	6.801 289	0.6001
1.5	1.295 205	6.903 970	0.5756
2.0	1.857 111	7.469 813	0.4643
2.5	2.370 079	8.065 658	0.3792
3.0	2.846 108	8.660 461	0.3162
3.5	3.293 772	9.245 180	0.2689
4.0	3.718 595	9.817 202	0.2324
4.5	4.124 565	10.376 058	0.2037
5.0	4.514 543	10.922 108	0.1806
6.0	5.254 925	11.978 473	0.1460
7.0	5.952 409	12.992 099	0.1215
8.0	6.615 321	13.968 404	0.1034
9.0	7.249 579	14.911 949	0.0895
10.0	7.859 517	15.826 517	0.0786

Figure 4 shows some of the bifurcating closed paths from $r=0.1$ to $r=1.5$. Figure 5 shows the graphs of the functions $r \mapsto \mu(r)$ and $r \mapsto T(\mu(r), r)$. One can see from Figure 6 how strongly attractive is the closed path corresponding to $r=1$, $\mu(1)=0.69096$ yet.

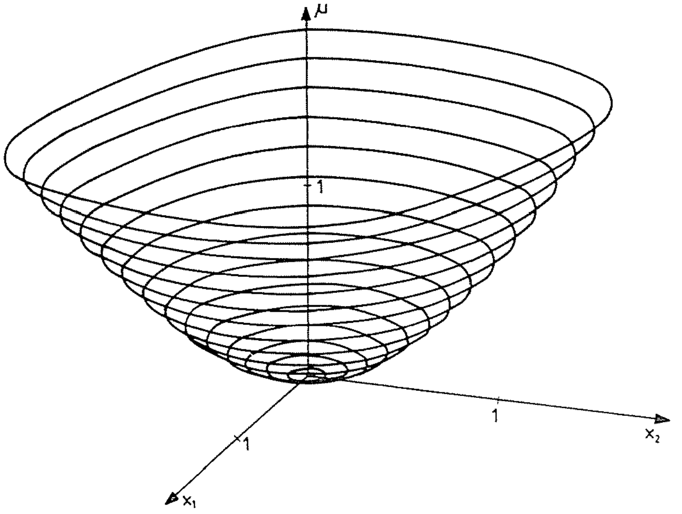

Fig. 4. Bifurcating closed paths of Rayleigh's equation

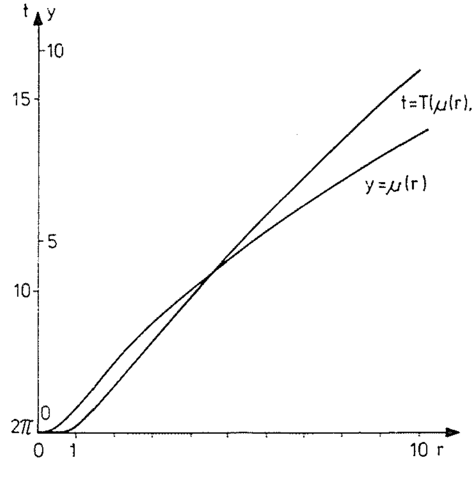

Fig. 5. Graphs of the functions $r \mapsto \mu(r)$ and $r \mapsto T(\mu(r), r)$

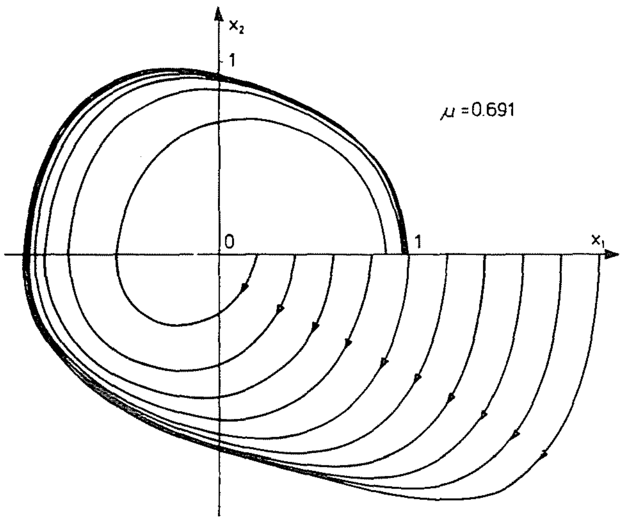

Fig. 6. The attractive closed path of Rayleigh's equation for $\mu = 0.69096$

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