THE STRESS FUNCTION OF PURE TORSION

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Received March 29, 1984

Summary

The problem of pure torsion of prismatic bars made of an elastic material is generally complicated and cannot be dealt with in a closed form. Here exceptional cases — partly known, partly new ones — are treated, where the stress function of the problem can be obtained in a closed form, being in this respect different from the general case.

Introduction

The stress function of pure torsion of prismatic bars of elastic material — as a rule — cannot be set up in a closed form. Thus, the exceptional cases should be of interest, where the stress function of pure torsion can be expressed in a closed form, in contrast to the general case. In the following — without any pretention of being complete — cases of this type are dealt with. Some of them are to be found in the special technical literature, others have not been published, yet.

In the following only symmetrical bar cross sections are dealt with. In certain cases, besides the form reduced to zero, stress functions in shapes directly suitable for calculation are also dealt with. In some figures besides the boundary line of the cross section, two stress lines are also indicated. The zero reduced equations of these stress lines only differ from the zero reduced equations of the stress function in additive constants K_1 and K_2 respectively.

The stress functions to be mentioned in the following can also be applied to hollow cross sections bordered either by two internal stress lines, or by the external rim and an internal stress line, or by the internal rim and an external stress line.

The author entered into direct relation with Professor Á. G. Pattantyús 50 years ago in connection with one of his papers dealing with the problem of torsion. Remembering it, he renders homage to his respected memory, with a study about torsion.

Cross sections bordered by an ellipse

Stress functions of cross sections belonging to this group [2] are in rectangular coordinate system 0 (x, y) of form

$$x^2+\frac{a^2}{b^2}y^2-a^2=0,$$

where a and b are the main radiuses of the ellipse (Fig. 1). The stress lines are also ellipses, similar to the external boundary curve of the cross section and are proportional to it.

In case of $a=b$ the elliptical cross section degenerates into a circular one.

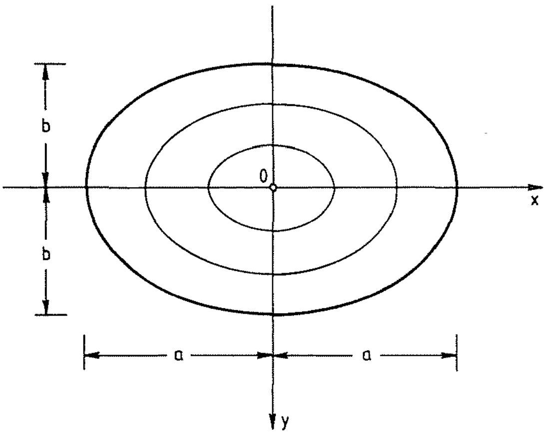

Fig. 1. Cross section bordered by an ellipse ($K_1=0.555a^2$, $K_2=0.889a^2$)

Cross sections bordered by two confocal ellipses

The hollow cross sections bordered by two ellipses having common focuses belong to this group. Their stress functions in an elliptic coordinate system 0 (ξ, η) [4] takes the form

$$\cosh 2\xi+\cos 2\eta-\frac{\cosh(\xi_0+\xi_1-2\xi)}{\cosh(\xi_1-\xi_0)}\cos 2\eta-\cosh 2\xi_1=0\,.$$

In this formula ξ, η are running coordinates $(0\leqq\xi<\infty, 0<|\eta|<\pi/2)$ and ξ_1 is the coordinate of the external ellipse and ξ_0 is the coordinate of the internal ellipse. The relation of coordinates ξ, η to coordinates x, y are expressed by formulas

$$x = c \cosh \xi \cos \eta, \qquad y = c \sinh \xi \sin \eta$$

where c means the distance of the focuses measured from the origin 0 of the coordinate system.

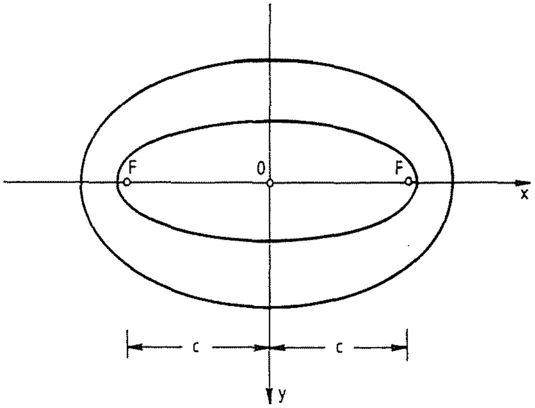

Fig. 2. Cross section bordered by two confocal ellipses ($\xi_1 = 0.805$, $\xi_0 = 0.434$)

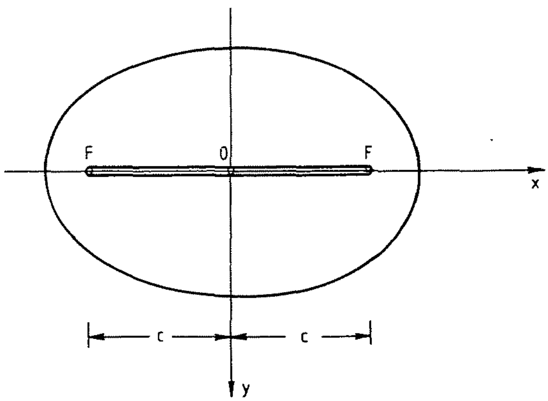

Fig. 3. Cross section bordered by an ellipse split between the focuses ($\xi_1 = 0.805$, $\xi_0 = 0$)

The form of the cross section depends on values ξ_0 and ξ_1.

Case 1: $\xi_0 > 0$. An example of these cross sections is shown in Fig. 2.

Case 2: $\xi_0 = 0$. In this case the cross section is an ellipse being split between the focuses (Fig. 3).

Cross sections bordered by a circular arc and a clover arc

To this group special cross sections belong, the stress function of which written in a polar coordinate system 0 (r, φ) reads as

$$r^2 - C\frac{r_0^{2-n}\, r^n}{n}\cos n\varphi = 0,$$

or

$$\varphi = \pm\frac{1}{n}\operatorname{arc\,cos}\frac{n}{Cr_0^{2n}\, r^{n-2}}.$$

In these formulas n means a positive integer and C means a constant.

The given stress function can be considered as a product of two factors:

$$(r^2 - r_0^2)\left[1 - C\frac{r^{n-2}}{n r_0^{n-2}}\left(1 + \frac{r_0^2}{r^2} + \frac{r_0^4}{r^4} + \ldots + \frac{r_0^{2n}}{r^{2n}}\right)\cos n\varphi\right] = 0.$$

Making these two factors one by one equal to zero, equations of two curves are obtained. One of them is a circle of radius r_0, the other one is a clover curve with n leaves, that in case of $n=1$ degenerates into a circle passing through

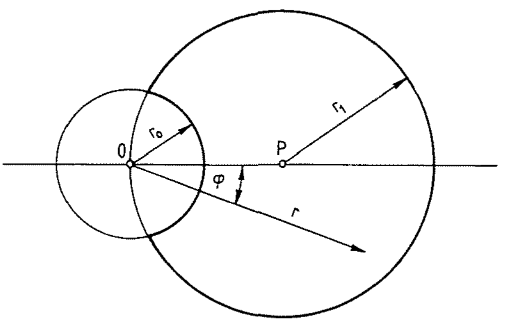

Fig. 4. Cross section bordered by two circular arcs

point 0. The clover curve can be situated in different ways in relation to the circle of radius r_0. If it intersects the circle of radius r_0 then this circle, together with the clover curve, can form the external or internal boundary line of the cross section.

Very different cross sections correspond to the stress functions in question. These were dealt with in detail in an earlier study [7], so here only the case shown in Fig. 4 is mentioned. In this case $n=1$, $C=1$ and the cross section is bordered by two circular arcs: a circular arc of radius r_0 and a clover arc, degenerated into a circular arc of radius r_1, passing through the centre of the circle of radius r_0.

Cross sections bordered by a straight line and a hyperbola

In this group cross sections are to be found, the stress function of which can be expressed by the formula

$$(x-a)(x^2-3y^2+4ax+4a^2-c^2)=0$$

where $a>0$, $c\geqq 0$.

Regarding the form of the cross sections two cases can be distinguished.

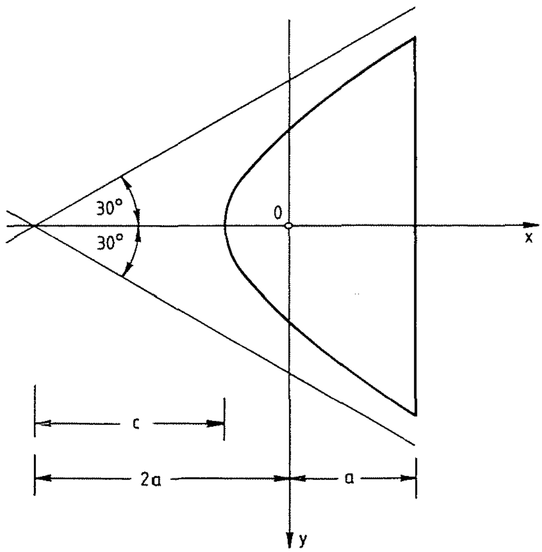

Fig. 5. Cross section bordered by a straight line and a hyperbola arc ($c=1.5a$)

Case 1: $0<c<3a$. In this case (Fig. 5) the cross section is bordered by the straight line $x=a$ and by the hyperbola arc defined by formula

$$x^2-3y^2+4ax+4a^2-c^2=0.$$

Case 2: $c=0$. In this case the hyperbola arc degenerates into a pair of straight lines. This time the cross section is an equilateral triangle, at the same time being a star triangle (Fig. 6). The length of its side is $\sqrt{3}a$, the radius of the circumscribed circle is $R=2a$.

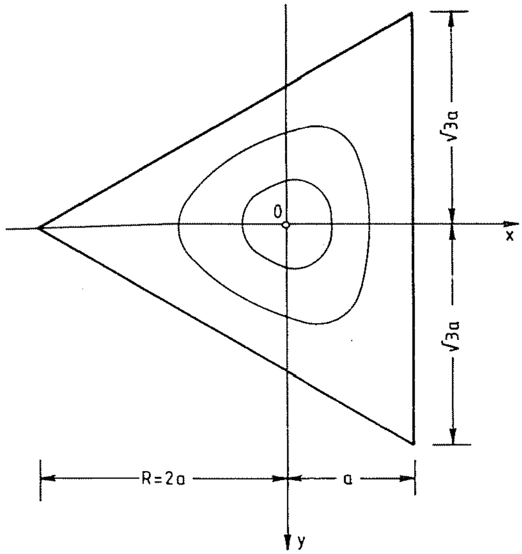

Fig. 6. Cross section bordered by an equilateral triangle (star triangle) ($K_1=2.370a^3$, $K_2=3.630a^3$)

Cross sections bordered by two hyperbolas

Cross sections with stress function of shape

$$\left[x^2-\left(\frac{y}{\sqrt{2}+1}\right)^2+A\right]\left[y^2-\left(\frac{x}{\sqrt{2}-1}\right)^2+B\right]=0$$

belong to this group where A and B are constants. Making the two factors of this equation equal to zero one by one, the equations of the boundary lines of the cross section are obtained. These boundary lines are hyperbolas, the asymptotes of which sustain an angle of 22.5° with axes x, y.

According to the values of constants A and B different cases can be distinguished.

Case 1: $A=-a^2$, $B=-b^2$. In this case the real main radius of one of the hyperbolas bordering the cross section is a, its imaginary main radius is $(\sqrt{2}+1)a$, and the real main radius of the other hyperbola bordering the cross section is b, its imaginary radius is $(\sqrt{2}-1)b$. An example of this type of cross sections is given in Fig. 7.

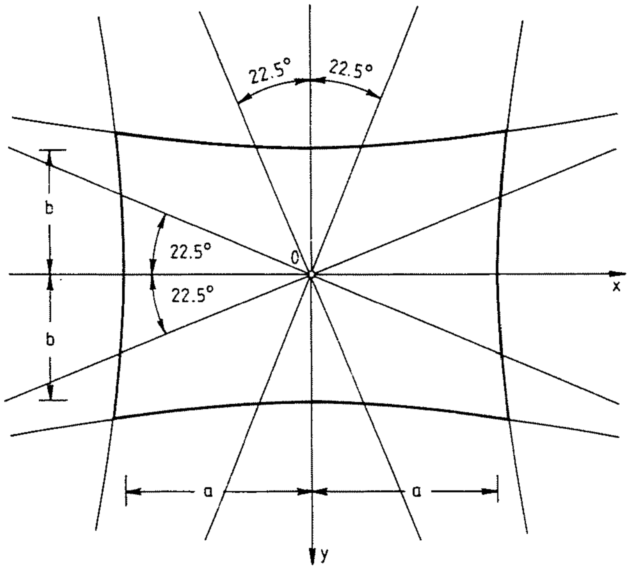

Fig. 7. Concave cross section bordered by two hyperbolas

Case 2: $A=B=-a^2$. In this case the cross section is a star square (Fig. 8). The radius of the circle describing it is $a=0.466R$.

Case 3: $A=-a^2$, $B=0$. In this case the cross section is divided into two half parts connected to each other (Fig. 9). The half cross sections are bordered by a pair of straight lines and by a hyperbola arc. The real main radius of the hyperbola is a and its imaginary main radius is $(\sqrt{2}+1)\ a$.

Case 4: $A=-a^2, B=(\sqrt{2}-1)^2\ a_1^2$. In this case the cross section consists of two half parts having no connection with each other (Fig. 10). The half cross sections are bordered by a convex and a concave hyperbola arc. The real main radius of the concave arc is a, its imaginary main radius is $(\sqrt{2}+1)/a$, and the real main radius of the convex arc is a, its imaginary main radius is $(\sqrt{2}-1)\ a_1$.

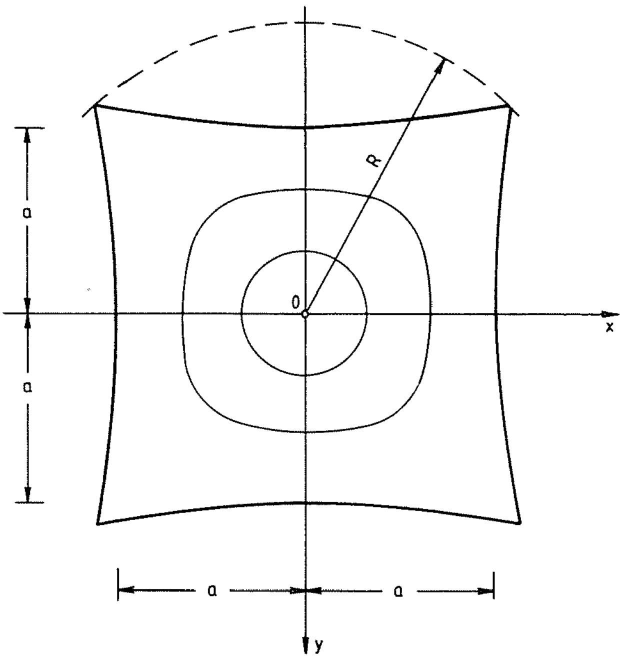

Fig. 8. Cross section bordered by a star square ($K_1=0.299a^2$, $K_2=0.435a^2$)

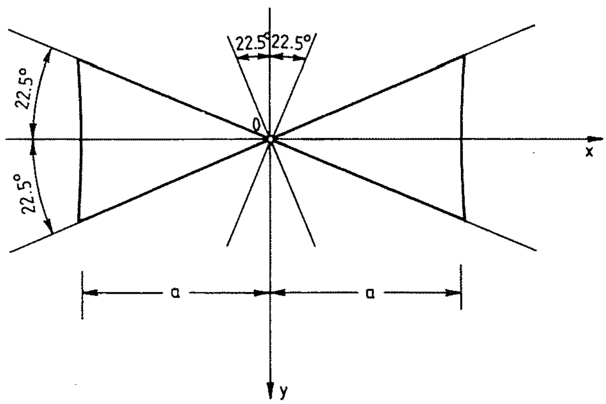

Fig. 9. Half cross sections bordered by a pair of straight lines and a hyperbola arc

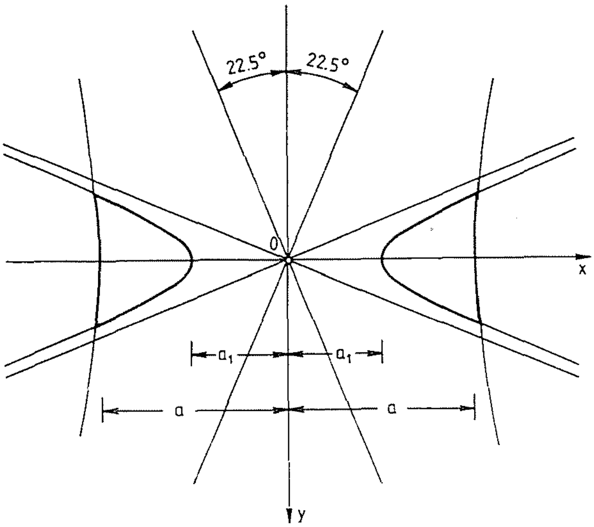

Fig. 10. Cross section bordered by a concave and a convex hyperbola arc

Cross sections bordered by a star polygon

These cross sections are multiple-symmetrical. Their stress function in a polar coordinate system 0 (r, φ) takes the form

$$\frac{r^2}{R^2}+\frac{2}{n}\frac{r^n}{R^n}\cos n\varphi-\frac{n-2}{n}=0,$$

or

$$\varphi=\pm\frac{1}{n}\operatorname{arc\,cos}\left[\frac{n}{2}\frac{R^n}{r^n}\left(\frac{n-2}{n}-\frac{r^2}{R^2}\right)\right].$$

In the above formulas n is the number of the sides of the star polygon ($n=3, 4, 5, \dots$), and R is the radius of the circle described around the star polygon [8].

Among the n sided star polygons here only the 3-, 4-, 5- and 6-sided star polygons are dealt with.

Case 1: $n=3$. The cross section is a star triangle i.e. a regular triangle bordered by three straight lines. The boundary line and two stress lines of the cross section are to be seen in Figure 6.

Case 2: $n=4$. The cross section is a star square. It is bordered by two branches of two hyperbolas. The form of the cross section and its two stress lines are shown in Figure 8.

Case 3: $n=5$. The cross section is a star pentagon. Its form and two of its stress lines are to be seen in Fig. 11.

Case 4: $n=6$. The cross section is a star hexagon. Its form and two of its stress lines are shown in Fig. 12.

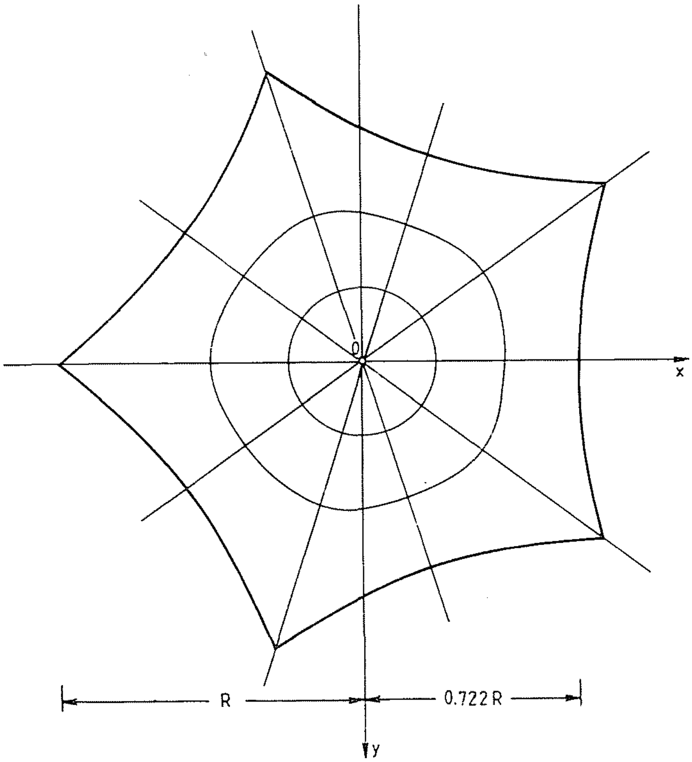

Fig. 11 Cross section bordered by a star pentagon ($K_1=0.358$, $K_2=0.542$)

The differential equation and its boundary conditions are the same for the stress function of pure torsion and for paraboloid shells of revolution subjected to uniformly distributed loads. Thus, the former expressions can also be applied to solve the problem of star shells.

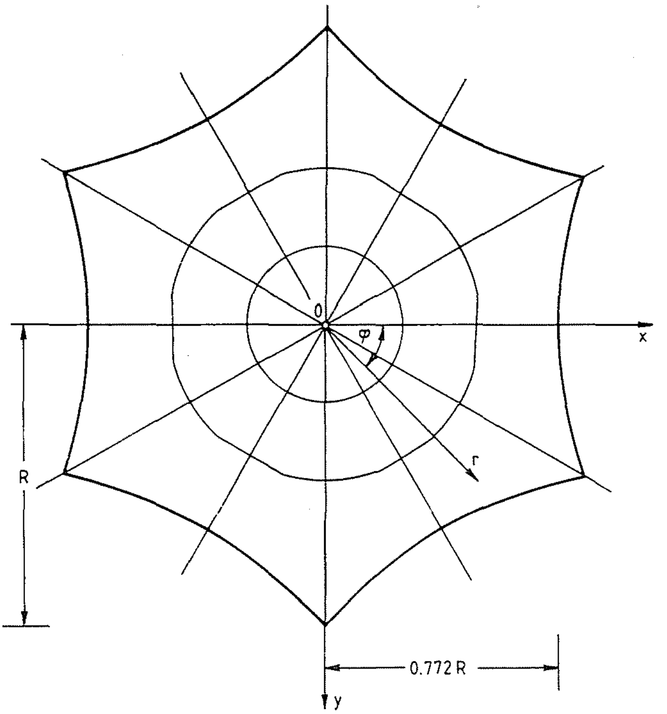

Fig. 12. Cross section bordered by a star hexagon ($K_1 = 0.600$, $K_2 = 0.396$)

Cross sections bordered by two parallel straights and two curve lines

Stress functions of cross sections belonging to this type take the form

$$x^2 - a^2 + C \cos \frac{\pi x}{2a} \cosh \frac{\pi y}{2a} = 0,$$

or

$$y = \pm \frac{2}{\pi} \operatorname{Arc\,cosh} \frac{a^2 - x^2}{C \cos \dfrac{\pi x}{2a}}.$$

In the above formulas a means half of the distance between the two parallel lines and C is a positive number smaller than $4/\pi$.

The form of the cross sections depending on the value of the ratio C/a^2 can be very different.

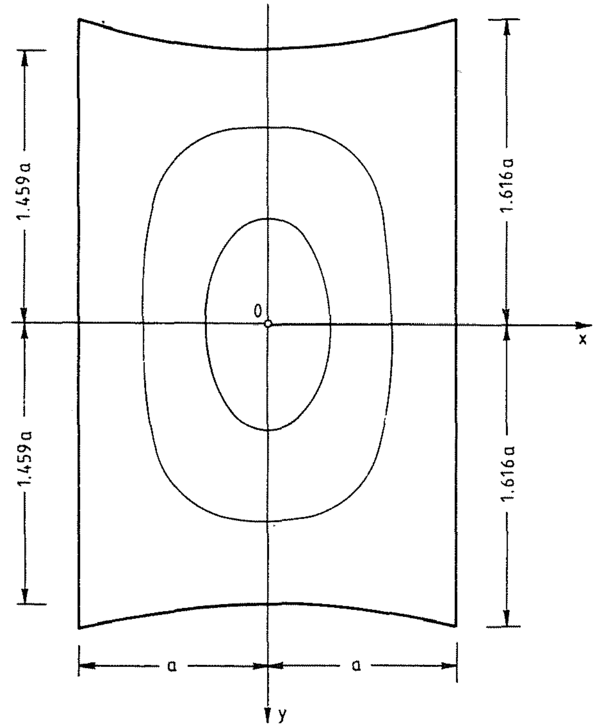
1.459 a
1.459 a
1.616 a
1.616 a
0
x
a
a
y

Fig. 13

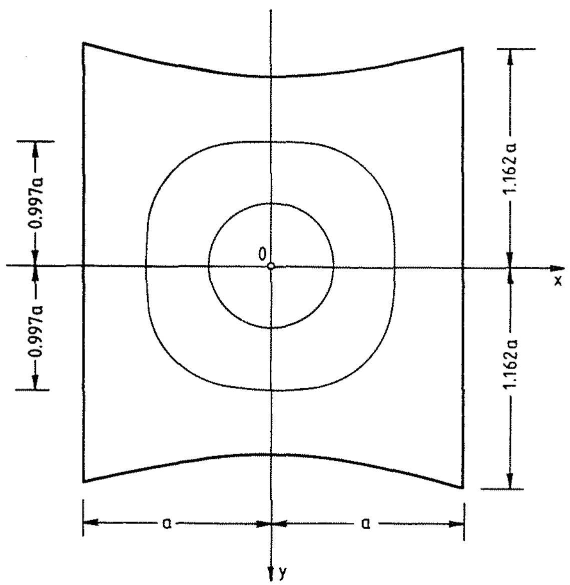
0.997a
0.997a
1.162 a
1.162a
0
x
a
a
y

Fig. 14

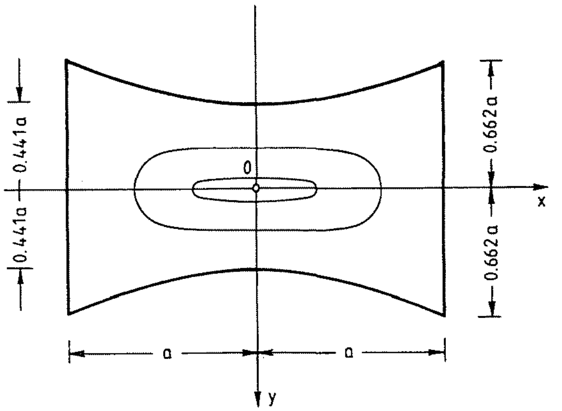

Fig. 15

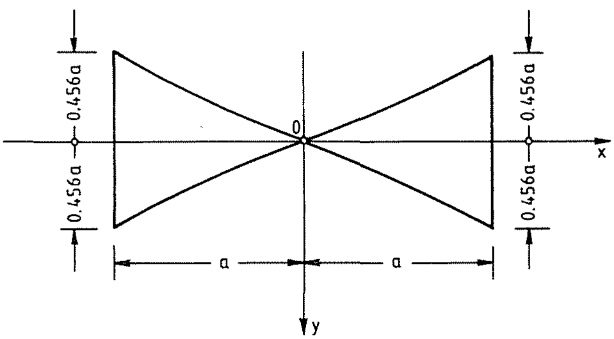

Fig. 16

Case 1: $C=0.2a^2$. The form and two stress lines of the cross section are shown in Fig. 13.

Case 2: $C=0.4a^2$. The form and two stress lines of the cross section are to be seen in Fig. 14.

Case 3: $C=0.8a^2$. The form and two stress lines of the cross section are shown in Fig. 15.

Case 4: $C=1.02a^2$. The cross section is articulated into two half cross sections being connected to each other (Fig. 16).

Case 5: $C \geqq 1.03a^2$. The cross section consists of two half parts having no connection to each other (Fig. 17). The half cross sections are bordered by one straight line and two curved lines.

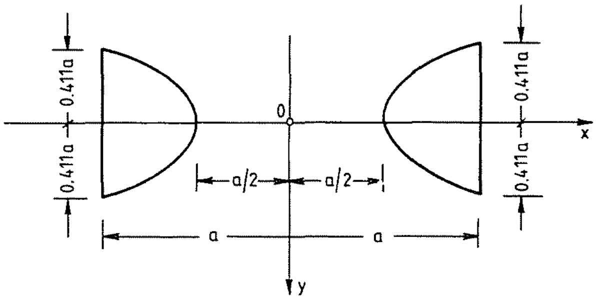

Fig. 17

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