DIFFERENTIAL EQUATIONS FOR THE TRANSFORMATION KERNEL OF SPECTRAL DENSITY FUNCTIONS IN THE CASE OF RANDOM TRANSFORMATIONS OF TIME (TO PUT FOR VEHICLES WITH NON-CONSTANT SPEED)

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Summary

Distortions of the spectrum of stochastic road profiles are investigated in such situations, when the distance — as integral of a random velocity process — is also random. Explicit results are obtained for the simplest Markovian model of urban traffic; the kernel function of the integral operator defined by this transformation of spectral densities is found to be a rational function of frequencies.

Introduction

Vibrations caused by unevenness of the road have an essential influence on life time of vehicles. In a simplified treatement the problem is formulated in terms of the process $\eta=\eta(s)$ of road profile, i.e. η denotes the level of the road as a function of distance s. Analysis of vibrations of vehicles designed for public traffic is usually based on the assumption that η is a stationary stochastic process, let $f=f(\mu)$ denote the spectral density of η as a function of the circular frequency μ. In the case of vehicles travelling with a constant velocity v, the calculation of stresses should be based on the effective spectral density $\tilde{f}(\mu) = \frac{1}{v} f\left(\frac{\mu}{v}\right)$, see [1, 2]. In the case of urban traffic, however, velocity of vehicles should be considered as a stationary process $\xi(t)$, and the effective energy spectrum turns out to be the spectral density of the composed process $\tilde{\eta}(t) = \eta(\zeta(t))$, where $\zeta(t) = \int_0^t \xi(x)dx$. The spectral density $\tilde{f}$ of $\tilde{\eta}$ can be calculated by means of formulae (3), (4) and (5), see [3, 4]. Since f can experimentally be determined, the main problem is to find $\tilde{f}$ as an integral transform $\tilde{f} = \mathbf{K}f$ of f. In this paper we introduce and investigate a Markovian model of $\xi(t)$ with two possible values of the velocity. Our main result is the formula

$$\tilde{f}(\mu)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{2\lambda v^2\rho^2 z^2 f(z)dz}{((\mu-vz)^2-v^2\rho^2z^2)^2+4\lambda^2(\mu-vz)^2}. \tag{1}$$

see Theorem 1, 2, where v is the mean velocity of our vehicle, and ρ characterizes the relative fluctuations of the velocity process. Of course, (1) can be applied only if the undelying model of the velocity process $\xi(t)$ fits well to the given problem. We are going to consider such a model that the vehicle moves with velocity v_1 or v_2, and the average length of the corresponding consecutive intervals is $\frac{1}{\lambda}$ for both velocities. The assumption that the velocity takes on only two values may be realistic in the following two situations. In long-distance transport we have e.g. $v_1=80$ km/h on the highway, while $v_2=60$ km/h inside villages. In public transport in cities the typical velocity is e.g. $v_1=40$ km/h, but we have $v_2=0$ during the stay of the bus at stops. The model is given by a sequence of random times $0<\tau_1<\tau_2<\ldots<\tau_n<\ldots$ when the vehicle changes its velocity from v_1 to v_2 or from v_2 to v_1; λ should experimentally be determined. In this model periods of breaking and accelerating the vehicle are neglected. If $v_2=0$ then the associated intervals correspond to stops. This case applies to urban traffic, provided that the vehicle has to wait in stops in the average as long as it travels between consecutive stops (see Figs 1, 2).

In this case the effective spectrum reads as follows:

$$\tilde{f}(\mu)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\frac{1}{v_1}f\left(\frac{x}{v_1}\right)\frac{\lambda}{2}x^2dx}{\mu^2(\mu-x)^2+\lambda^2(2\mu-x)^2}. \tag{2}$$

Mathematical formulation and discussion of the model suppose that η and ξ are completely independent stationary processes, and let $\zeta(t)=\int_0^t\xi(x)dx$. Then the composed process $\tilde{\eta}(t)=\eta(\zeta(t))$ is again a stationary process, see [3]. Let f and $\tilde{f}$ denote the spectral density of η and of $\tilde{\eta}$, respectively, then $\tilde{f}\sim\mathbf{K}f$ with an integral operator $\mathbf{K}$, i.e.

$$\tilde{f}(\mu)=\int_{-\infty}^{\infty}k(\mu,z)f(z)dz, \tag{3}$$

where

$$k(\mu,z)=\mathrm{Re}\frac{1}{\pi}\int_0^{\infty}\mathbf{E}(\exp iz\zeta(t))\exp(-i\mu t)dt, \tag{4}$$

$$\zeta(t) = \int_0^t \xi(x)dx, \tag{5}$$

where $\mathbf{E}$ denotes expectation, while $i^2 = -1$, see [3].

Let us remark that (4) holds under some natural integrability conditions on the correlation function of η, see Theorem 3 in [4], $\mathbf{K}$ is defined on $\mathbf{L}_1(-\infty, \infty)$.

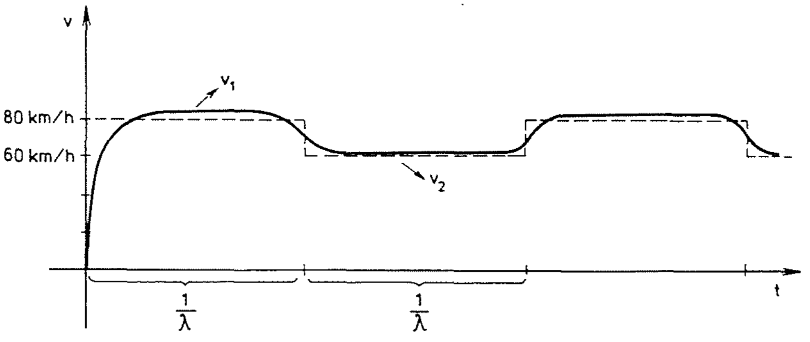

Fig. 1

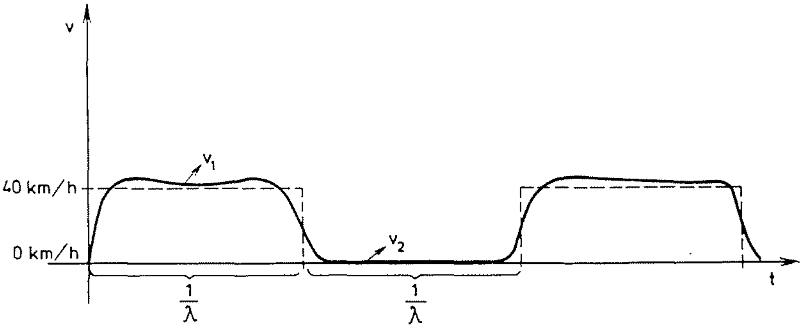

Fig. 2

The kernel k of our transformation will be calculated in the following case. Let $\delta_1, \delta_2, \ldots, \delta_n, \ldots$ be a sequence of independent exponentially distributed random variables of common parameter $\lambda > 0$, then $\tau_1 = \delta_1$, $\tau_2 = \delta_1 + \delta_2, \ldots, \tau_n = \delta_1 + \ldots + \delta_n, \ldots$ are points of a stationary Poisson process of intensity λ. Now we put

$$\xi(t) = \xi(0), \quad \text{if} \quad t \in [\tau_{2n}, \tau_{2n+1}),$$

$$\xi(t) = \overline{\xi(0)}, \quad \text{if} \quad t \in [\tau_{2n+1}, \tau_{2n+2}),$$

$$\overline{\xi(0)} = \begin{cases} v_1, & \text{if} \quad \xi(0) = v_2 \\ v_2, & \text{if} \quad \xi(0) = v_1, \end{cases}$$

where $\tau_0=0$, $n=1, 2, \ldots$, $0 \le v_2 < v_1$ and

$$\mathbf{P}(\xi(0)=v_1)=\mathbf{P}(\xi(0)=v_2)=\frac{1}{2}. \tag{6}$$

It is easy to see that ξ is a stationary Markov process with transition probabilities

$$\begin{aligned} \mathbf{P}(\xi(t+s)=v_1 \mid \xi(t)=v_1) &= (\mathrm{ch}(\lambda s))\exp(-\lambda s), \\ \mathbf{P}(\xi(t+s)=v_2 \mid \xi(t)=v_2) &= (\mathrm{ch}(\lambda s))\exp(-\lambda s) \\ \mathbf{P}(\xi(t+s)=v_1 \mid \xi(t)=v_2) &= (\mathrm{sh}(\lambda s))\exp(-\lambda s), \\ \mathbf{P}(\xi(t+s)=v_2 \mid \xi(t)=v_1) &= (\mathrm{sh}(\lambda s))\exp(-\lambda s). \end{aligned} \tag{7}$$

Since

$$\mathbf{E}(\exp iz\zeta(t))=\mathbf{E}(\mathbf{E}(\exp iz\zeta(t) \mid \xi(0)))=$$

$$=\frac{1}{2}\mathbf{E}(\exp iz\zeta(t) \mid \xi(0)=v_1)+\frac{1}{2}\mathbf{E}(\exp iz\zeta(t) \mid \xi(0)=v_2),$$

introducing

$$\begin{aligned} \omega_1(t, z) &= \mathbf{E}(\exp iz \int_0^t \xi(x)dx \mid \xi(0)=v_1), \\ \omega_2(t, z) &= \mathbf{E}(\exp iz \int_0^t \xi(x)dx \mid \xi(0)=v_2), \end{aligned} \tag{8}$$

we obtain that

$$k(\mu, z)=\mathrm{Re}\frac{1}{2\pi}\int_0^\infty (\omega_1(t, z)+\omega_2(t, z))\exp(-i\mu t)dt. \tag{9}$$

In view of the following theorem, calculation of the kernel function (9) reduces to solution of the following system of linear differential equations.

Theorem 1. For ω_1 and ω_2 defined by (8) we have

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} -\lambda - izv_1 & \lambda \\ \lambda & -\lambda - izv_2 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \tag{10}$$

with initial condition $\omega_1(0, z)=\omega_2(0, z)=1$.

Proof: Introduce

$$\alpha(t, u, v)=\mathbf{E}((\exp iz \int_0^t \xi(x)dx)\delta(v, \xi(t)) \mid \xi(0)=u),$$

$$\delta(v, \xi(t))=\begin{cases} 1, & \text{if } \xi(t)=v \\ 0, & \text{if } \xi(t)\neq v, \end{cases} \tag{11}$$

$$\beta(t, u)=\mathbf{E}(\exp iz \int_0^t \xi(x)dx \mid \xi(0)=u),$$

since ξ is a Markov process, an easy calculation shows that

$$\beta(t+s, u) = \sum_y \alpha(s, u, y)\beta(t, y). \tag{12}$$

We shall see that α and β are differentiable functions of t. It is plain that

$$\omega_1(t, z) = \beta(t, v_1), \quad \omega_2(t, z) = \beta(t, v_2) \tag{13}$$

thus differentiating both sides of (12) with respect to s we obtain that

$$\frac{d}{ds}\beta(t+s, u)\bigg|_{s=0} = \sum_y \frac{d}{ds}\alpha(s, u, y)\bigg|_{s=0} \beta(t, y). \tag{14}$$

However

$$\frac{d}{ds}\beta(t+s, u)\bigg|_{s=0} = \frac{d}{dt}\beta(t, u) \tag{15}$$

thus in view of (13), (14) and (15) it is sufficient to determine the derivative of α at $s=0$. We obtain

$$\frac{d}{ds}\alpha(s, u, y)\bigg|_{s=0} = \lim_{s\to 0}\frac{1}{s}(\alpha(s, u, y) - \alpha(0, u, y)) =$$

$$= \lim_{s\to 0}\frac{1}{s}\mathbf{E}((\exp iz\zeta(s))\delta(\xi(s), y) - \delta(\xi(0), y) \mid \xi(0) = u) =$$

$$= \lim_{s\to 0}\frac{1}{s}\mathbf{E}((\exp iz\zeta(s))\delta(\xi(s), y) - \delta(\xi(s), y) \mid \xi(0) = u) +$$

$$+ \lim_{s\to 0}\frac{1}{s}\mathbf{E}(\delta(\xi(s), y) - \delta(\xi(0), y) \mid \xi(0) = u) = A_1 + A_2 \tag{16}$$

$$A_1 = \lim_{s\to 0}\mathbf{E}(\delta(\xi(s), y)\,(\exp iz\zeta(s) - \exp iz0)\frac{1}{s} \mid \xi(0) = u) =$$

$$= \mathbf{E}(\delta(\xi(s), y)\,(iz\xi(s)\exp iz\int_0^s \xi(x)dx)_{s=0} \mid \xi(0) = u) =$$

$$= \mathbf{E}(\delta(\xi(0), y) iz\xi(0) \mid \xi(0) = u) =$$

$$= \delta(u, y) izu = \begin{cases} izy, & \text{if } u = y \\ 0, & \text{if } u \neq y. \end{cases} \tag{17}$$

Since

$$\mathbf{E}(\delta(\xi(s), y) \mid \xi(0) = u) = \mathbf{P}(\xi(s) = y \mid \xi(0) = u) =$$

$$= \begin{cases} (\operatorname{ch}\lambda s)\exp(-\lambda s), & \text{if } u = y \\ (\operatorname{sh}\lambda s)\exp(-\lambda s), & \text{if } u \neq y, \end{cases}$$

and

$$\mathbf{E}(\delta(\xi(0), y) \mid \xi(0) = u) = \begin{cases} 1, & \text{if } u = y \\ 0, & \text{if } u \neq y, \end{cases}$$

consequently

$$A_2 = \begin{cases} \frac{d}{ds}(\text{ch}(\lambda s)\exp(-\lambda s))_{s=0}, & \text{if} \quad u=y \\ \frac{d}{ds}(\text{sh}(\lambda s)\exp(-\lambda s))_{s=0}, & \text{if} \quad u \neq y, \end{cases}$$

whence

$$A_2 = \begin{cases} -\lambda, & \text{if} \quad u=y \\ \lambda, & \text{if} \quad u \neq y. \end{cases} \tag{18}$$

Substituting (18) and (17) into (16) we obtain

$$\frac{d}{ds}\alpha(s, u, y)_{s=0} = \begin{cases} -\lambda + izy, & \text{if} \quad u=y \\ \lambda, & \text{if} \quad u \neq y. \end{cases} \tag{19}$$

Comparing (19) and (14) it can be see that α and β are really differentiable functions, furthermore, by means of (15) it can be concluded that

$$\begin{aligned} \frac{d}{dt}\beta(t, v_1) &= (-\lambda + izv_1)\beta(t, v_1) + \lambda\beta(t, v_2), \\ \frac{d}{dt}\beta(t, v_2) &= (-\lambda + izv_2)\beta(t, v_2) + \lambda\beta(t, v_1). \end{aligned} \tag{20}$$

Finally, as (11) implies

$$\beta(0, v_1) = 1 \qquad \text{and} \qquad \beta(0, v_2) = 1, \tag{21}$$

the statement of the theorem follows in view of notations introduced in (13).

Theorem 2. For the model defined above we have (1) with $v = \frac{1}{2}(v_1 + v_2)$ and $\rho = \frac{v_1 - v_2}{v_1 + v_2}$.

Proof: Theorem 1 implies

$$\omega_1(t, z) = \frac{\lambda + iv\rho z + \sqrt{\lambda^2 - v^2\rho^2 z^2}}{2\sqrt{\lambda^2 - v^2\rho^2 z^2}} \exp \kappa_1 t + \frac{-\lambda - iv\rho z + \sqrt{\lambda^2 - v^2\rho^2 z^2}}{2\sqrt{\lambda^2 - v^2\rho^2 z^2}} \exp \kappa_2 t,$$

$$\omega_2(t, z) = \frac{\lambda + iv\rho z + \sqrt{\lambda^2 - v^2\rho^2 z^2}}{2\lambda\sqrt{\lambda^2 - v^2\rho^2 z^2}} (-iv\rho z + \sqrt{\lambda^2 - v^2\rho^2 z^2}) \exp \kappa_1 t + \frac{-\lambda - iv\rho z + \sqrt{\lambda^2 - v^2\rho^2 z^2}}{2\lambda\sqrt{\lambda^2 - v^2\rho^2 z^2}} (-iv\rho z - \sqrt{\lambda^2 - v^2\rho^2 z^2}) \exp \kappa_2 t,$$

where

$$\kappa_1 = -\lambda + ivz + \sqrt{\lambda^2 - v^2\rho^2 z^2},$$
$$\kappa_2 = -\lambda + ivz - \sqrt{\lambda^2 - v^2\rho^2 z^2}.$$

Thus an easy calculation shows that

$$k(\mu, z) = \frac{1}{2\pi} \operatorname{Re} \int_0^\infty (\omega_1(t, z) + \omega_2(t, z)) \exp(-i\mu t) dt =$$

$$= \frac{1}{2\pi} \operatorname{Re} \left(\frac{\lambda + \sqrt{\lambda^2 - v^2\rho^2 z^2}}{\sqrt{\lambda^2 - v^2\rho^2 z^2}} \cdot \frac{-1}{\kappa_1 - i\mu} + \frac{-\lambda + \sqrt{\lambda^2 - v^2\rho^2 z^2}}{\sqrt{\lambda^2 - v^2\rho^2 z^2}} \cdot \frac{-1}{\kappa_2 - i\mu} \right) =$$

$$= \frac{1}{2\pi} \operatorname{Re} \frac{-4\lambda - 2i(\mu - vz)}{(\mu - vz)^2 - v^2\rho^2 z^2 - i2\lambda(\mu - vz)} =$$

$$= \frac{1}{\pi} \frac{2\lambda v^2 \rho^2 z^2}{((\mu - vz)^2 - v^2\rho^2 z^2)^2 + 4\lambda^2(\mu - vz)^2}$$

whence (1) follows by (3); while (2) is obtained as a particular case of (1).

References

1. ZASCHEL, J. M.: Die Analyse und Beschreibung von Betriebsbeanspruchungen zur Lebensdauerbeurteilung. Fatigue Life Under Random Load, Conference 1978, Università degli Studii di Trieste
2. ILOSVAI, I.—KERESZTES, A., MICHELBERGER, P., PETER, T.: Mathemathical vibration analysis of buses operating in towns. Per. Polytechn. Transp. Eng., *7*, 139 (1979)
3. FARKAS, M., FRITZ, J., MICHELBERGER, P.: On the effect of stochastic road profiles on vehicles travelling at varying speed. Acta Techn. Acad. Sci. Hungar., *91*, 303 (1980)
4. FARKAS, M., BELLAY, Á.: On the effect of nonstationary excitations of vehicles (in Hungarian) Technical Report, Technical University Budapest, 1979.

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