COMPARISON OF SOME PLASTICITY THEORIES BY MEANS OF THE FINITE ELEMENT METHOD

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Summary

This paper deals with six plasticity theories by means of finite element method. These are the time-independent theory of Prandtl—Reuss, Hencky—Nadai, Ilyushin and Valanis and the elasto-viscoplastic theory of Perzyna and Krempl. The stress-strain relation in the plasticity theories investigated is written uniformly in incremental form.

The time-independent theories are compared by computing the deformation of a thin-walled tube subjected to combined tension-torsion.

The two elasto-viscoplastic theories are applied to a thick-walled cylinder subjected to a pressurization rate of 1 Mpa/s. The hoop stress distribution is presented.

Introduction

Recently, calculations based on plasticity theory have found increasing use in the structural analysis calculations. Also, the increasing requirements imposed upon operation of the structural units of machinery demand that models more accurately describing the behaviour of material be used. Because of the rather sophisticated models, plasticity calculations can usually be made only by numerical methods.

Among the different numerical methods, e.g. the finite element method offers a rather effective approach. This method permits actual practical calculations to be made and different plasticity theories to be compared. On the basis of numerical investigations and comparison with the measurement results, the applicability limits and deficiencies of the different theories can be determined. In this study, six plasticity theories are compared by means of finite element method. Among the theories investigated, the first four theories — Prandtl—Reuss theory of plastic flow, Hencky—Nádai theory of deformation, Ilyushin theory of geometry and the Valanis theory — discuss time-independent elasto-plastic strains while the last two theories include the elasto-viscoplastic behaviour of materials elaborated by Perzyna and Krempl.

A computer program [6] developed earlier is used in finite element calculations.

Basic equations

To comply with the finite element calculation, the stress-strain relation in the plasticity theories investigated is written in incremental form. This can be expressed for the six theories uniformly in the following way:

$$\Delta\sigma_{ij} = C_{ijkl}\Delta\varepsilon_{kl} - \Delta P_{ij} \tag{1}$$

For the four time-independent cases, $\Delta P_{ij} = 0$ and tensor C_{ijkl} can be written as

$$C_{ijkl} = aT_{ijkl} + bL_{ijkl} + cM_{ijkl} \tag{2}$$

where

$$T_{ijkl} = I_{ijkl} - \frac{1}{3}L_{ijkl}$$

$$M_{ijkl} = s_{ij}s_{kl}$$

The scalar parameters *a*, *b*, *c* for the different theories are tabulated in Table 1 below:

	a	b	c
Prandtl—Reuss	$2G$	K	$-\dfrac{9G^2}{\sigma^2(H+3G)}$
Hencky—Nádai	$\dfrac{2G}{1+3G/H_s}$	K	$-\dfrac{9G^2(H_s-H)}{\sigma^2(H+3G)(H_s+3G)}$
Ilyushin	N	K	$-\dfrac{N-P}{s_{ij}s_{ij}}$
Valanis [2]	$2G$	K	$-\dfrac{\alpha_1}{\dfrac{1+\beta\zeta}{\beta d\zeta}s_{ij}d\varepsilon_{ij}}$

A detailed discussion of the first three theories is given by Zyczkowski [1]. The Mises yield condition has been used for the Prandtl—Reuss theory in Table I. The use of a more generalized plasticity condition is also possible:

$$F = \left(\frac{3}{2}N_{ijkl}\bar{s}_{ij}\bar{s}_{kl}\right)^{1/2} - \sigma_y(\varepsilon_p) = 0 \tag{3}$$

where

$$N_{ijkl}=I_{ijkl}+A(\varepsilon_p)\cdot\varepsilon_{ij}^p\varepsilon_{kl}^p$$

The yield function in (3) can describe anisotropic hardening [7]. In this case, M_{ijkl} and c take the following form:

$$M_{ijkl}=d_{ij}d_{kl}$$

$$C=\frac{1}{d_{pq}a_{pq}+\frac{2}{3}H(1-\omega)a_{pq}a_{pq}-g_{pq}\cdot a_{pq}+h\left(\frac{2}{3}a_{pq}\cdot a_{pq}\right)^{1/2}}$$

In case of the Perzyna and Krempl theory equation (1) gives the stress increment in time interval Δt. Quantities C_{ijkl} and ΔP_{ij} will then change, as follows:

For the Perzyna theory [3]:

$$C_{ijkl}=(D_{ijkl}^{-1}+\Theta\Delta tH_{ijkl}^{\sigma})^{-1} \tag{4a}$$

$$\Delta P_{ij}=\Delta tC_{ijkl}\dot{\varepsilon}_{kl}^{vp} \tag{4b}$$

where

$$\dot{\varepsilon}_{kl}^{vp}=\gamma\langle\Phi(F)\rangle\frac{\partial F}{\partial\sigma_{kl}}$$

$$H_{ijkl}^{\sigma}=\frac{\partial\dot{\varepsilon}_{ij}^{vp}}{\partial\sigma_{kl}}$$

For the Krempl theory [4]:

$$C_{ijkl}=(D_{ijkl}^{-1}+\Theta\Delta tH_{ijmn}^{\sigma})^{-1}(I_{mnkl}-\Theta\Delta tH_{mnkl}^{\varepsilon}) \tag{5a}$$

$$\Delta P_{ij}=(D_{ijmn}^{-1}+\Theta\Delta tH_{ijmn}^{\sigma})^{-1}\Delta t\dot{\varepsilon}_{mn}^{in}$$

where

$$\dot{\varepsilon}_{mn}^{in}=D_{mnrs}^{-1}\frac{\sigma_{rs}-G_{rs}}{k(\Gamma)}$$

$$G_{rs}=\frac{g(\varphi)}{\varphi E}D_{rsmn}\varepsilon_{mn}$$

$$\Gamma=[(\sigma_{ij}-G_{ij})(\sigma_{ij}-G_{ij})]^{1/2}$$

$$H_{ijmn}^{\sigma}=\frac{\partial\dot{\varepsilon}_{ij}^{in}}{\partial\sigma_{mn}}$$

$$H_{ijmn}^{\varepsilon}=\frac{\partial\dot{\varepsilon}_{ij}^{in}}{\partial\varepsilon_{mn}}$$

Θ in equations (4) and (5) means the weight factor in time-step schemes based on the difference method which is used in integration by time:

$$\Delta\varepsilon_{ij}^{vp}=\Delta t[(1-\Theta)^{t}\dot{\varepsilon}_{ij}^{vp}+\Theta^{t+\Delta t}\dot{\varepsilon}_{ij}^{vp}] \quad (6)$$

The following equation is used in finite element calculations:

$$(\mathbf{K}_L^t+\mathbf{K}_{NL}^t)\Delta\mathbf{u}=\mathbf{R}^{t+\Delta t}-\mathbf{F}^t+\mathbf{V}^t \quad (7)$$

Where $\mathbf{K}_L^t$, $\mathbf{K}_{NL}^t$ — linear and non-linear tangential stiffness matrix
$\mathbf{u}$ – nodal displacement increment vector
$\mathbf{R}^{t+\Delta t}$ — external load vector in the time $t+\Delta t$
$\mathbf{F}^t$ — internal load vector
$\mathbf{V}^t$ — pseudo-load vector.

A solution to the above equation is given by the Newton—Raphson or by the modified Newton—Raphson method. The solution to two simple problems is presented to illustrate the application of the finite element program elaborated on the basis of the theories discussed in detail above.

Examples

Combined tension and torsion of thin-walled tube

The first problem includes a thin-walled tube subjected to non-proportional load. The combined tensile-torsional load path is given in Fig. 1. Plane stress state has been assumed in finite element calculations, the finite

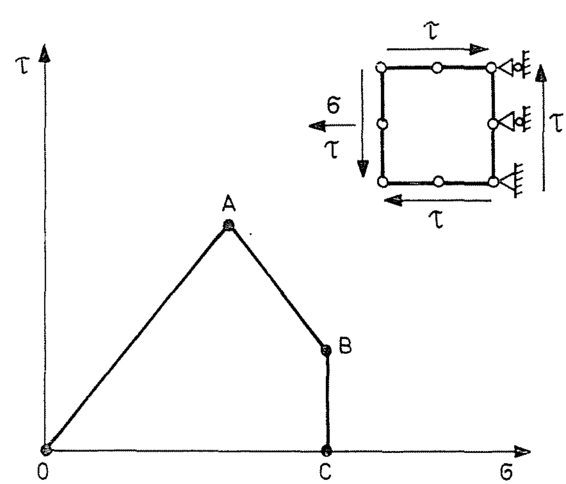

Fig. 1. Loading path and finite element model for analysis of the thin-walled tube

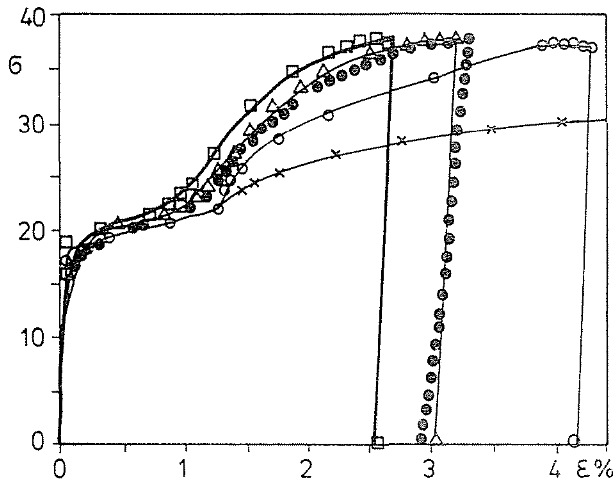

Fig. 2.a Axial stress-strain curves

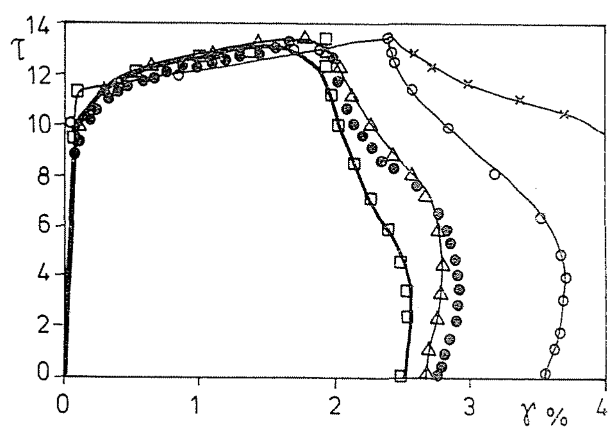

Fig. 2.b Torsional stress-strain curves

——— experiment, Liu [5]

□ anisotropic hardening [7]

○ ○ ○ kinematic hardening

× × × Hencky—Nádai theory

● ● ● Valanis theory

△ △ △ Ilyushin theory

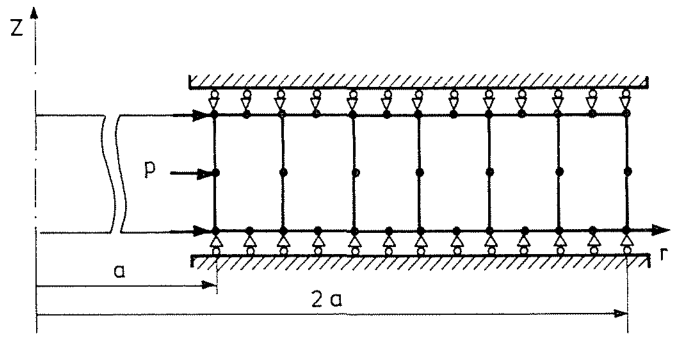

Fig. 3. Finite element mesh for a thick-walled cylinder

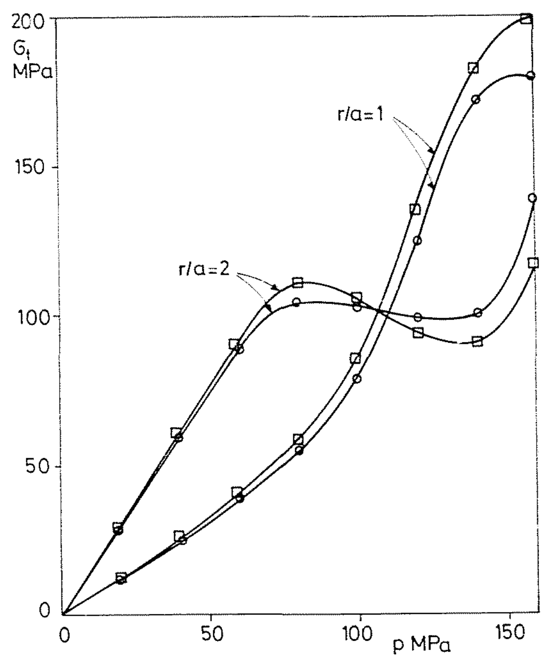

Fig. 4. Hoop stress versus internal pressure at $r/a=1$ and $r/a=2$ ratios for a pressurization rate of 1 MPa/s

○ ○ ○Perzyna theory

□ □ □Krempl theory

element model being shown in the upper right part of the Figure. Measurements for this load path were made by Liu [5]. The results of measurements and those obtained on the basis of the different theories are shown in Fig. 2, Fig. 2.a and Fig. 2.b show the axial stress-strain curves and shear stress-strain curves, respectively.

Solid lines indicate the results of measurements. It can be seen that best results are supplied by the anisotropic hardening model. Acceptable results are obtained on the basis of the Valanis theory, and Ilyushin theory too, however, the Hencky—Nádai theory and the kinematic hardening model yield rather divergent results as compared with the measurements.

Elasto-viscoplastic strain of thick-walled tube

The second problem is a comparison of the Perzyna theory and Krempl theory. Here the elasto-viscoplastic strain of a thick-walled tube is investigated, the load being applied to the tube for a pressurization rate $\dot{p}=1$ MPa/s. The axissymmetric finite element model is shown in Fig. 3.

The results of calculations are given in Fig. 4. In the Figure, the change of tangential stress as a function of pressure can be seen for both the internal and external surface of the tube.

Nomenclature

σ_{ij} — second Piola—Kirchhoff stress tensor

ε_{ij} — Green—Lagrange strain tensor

s_{ij} — deviatoric stress tensor

e_{ij} — deviatoric strain tensor

ε_{ij}^{p} — plastic strain tensor

$\dot{\varepsilon}_{ij}^{vp}$, $\dot{\varepsilon}_{ij}^{in}$ — viscoplastic strain rate tensor

E — elastic modulus

ν — Poisson's ratio

K — bulk modulus

$G=\dfrac{E}{2(1+\nu)}$

δ_{ij} — Kronecker delta

$I_{ijkl}=\dfrac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$

$L_{ijkl}=\delta_{ij}\delta_{kl}$

$D_{ijkl}=2GT_{ijkl}+K\cdot L_{ijkl}$

$\sigma\equiv\sqrt{\frac{3}{2}s_{ij}s_{ij}}$ — equivalent stress

$\varepsilon_p\equiv\sqrt{\frac{2}{3}\varepsilon^p_{ij}\varepsilon^p_{ij}}$ — equivalent plastic strain

$\sigma_y(\varepsilon_p)$ — hardening function

H — plastic modulus

H_s — plastic secant modulus

$\alpha_1, \beta_1, \beta_2$ — endochron material parameters

$(\beta d\zeta)^2=\beta_1 d\varepsilon_{ii}d\varepsilon_{jj}+\beta_2 d\varepsilon_{ij}d\varepsilon_{ij}$

$a_{ij}=\frac{\partial F}{\partial\sigma_{ij}}$

F — yield function

$g_{ij}=\frac{\partial F}{\partial\varepsilon^p_{ij}}$

$h=\frac{\partial F}{\partial\varepsilon_p}$

ω — mixed hardening parameter

$d_{ij}=D_{ijkl}a_{kl}$

$A(\varepsilon_p)$ — anisotrop hardening function

γ — fluidity parameter

$\langle\Phi(F)\rangle=\begin{cases}\Phi(F) & F>0\\ 0 & F\leqq 0\end{cases}$

$g(\varphi)$ — viscosity function

$k(\Gamma)$ — hardening function

α_{ij} — translation tensor

$\bar{\sigma}_{ij}=\sigma_{ij}-\alpha_{ij}$

$N=f(e_{ij}e_{ij})$

$P=\dot{f}$

References

1. ZYCZKOWSKI, M.: Combined loading in the Theory of Plasticity, PWN-Polisch Scientific Publishers, Warsaw, 1981.
2. VALANIS, K. C.: Archiwum Mechaniki Stossowanej, *23*, 517 (1971)
3. PERZYNA, P.: Fundamental problems in visco-plasticity in: Recent Advances in Applied Mechanics, Academic Press, New York, 1966.
4. KREMPL, E.: Acta Mechanics, *36*, 263 (1980)

5. LIU, K. C.: Inelastic Behavior of Pressure Vessel and Piping Components, PVP-PB-028, 1—12, 1978.
6. SZABÓ, L.: Elasto-plastic and creep problems by finite element method, Research Report, VEIKI, Budapest, 1983.
7. SZABÓ, L.: Int. J. Num. Meth. Eng., 1983. (in press)

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