APPLICATION OF THE MODIFIED LAW OF HEAT CONDUCTION AND STATE EQUATION TO DYNAMICAL PROBLEMS OF THERMOELASTICITY

I. FARKAS* and A. SZEKERES

Department of Technical Mechanics,
Technical University, H-1521 Budapest

Received September 20, 1983
Presented by Prof. Dr. Gy. Béda

Summary

In suming up, it can be laid down as a fact that the law of heat conduction and state equation modified for an explicit, linear case can be used in the investigations of dynamical problems of thermoelasticity.

Introduction

The general problem of thermoelasticity can be described by an equation system consisting of the following equations: kinematic equation, constitutive equation, equation of motion, heat conduction equation, first and second law of thermodynamics. After reduction, two equations are obtained with two unknowns such as displacement and temperature. These are also called generalized equation of motion and generalized equation of heat conduction.

In case of special problems like stationary and simplified quasi-static problems these equations fall apart, the connection between the displacement field and temperature field discontinuing; the temperature can then be calculated independently and in the knowledge of temperature, also the displacement field can be obtained. In this way, the heat conduction problem is part of the investigations in the field of thermoelasticity.

The one dimensional dynamic problem of heat conduction without intrinsic heat source is described on the basis of Fourier's law of heat conduction by differential equation

$$aT_{xx} = T_t \tag{1}$$

where the subscript indicates partial derivation.

* Department of Mathematics of the Faculty of Electrical Engineering

An investigation of the differential equation shows that the temperature disturbance propagates at infinite velocity, a phenomenon obviously contradicting experience. To the best of our knowledge, Maxwell had been the first to direct attention to this contradiction in 1867, and many authours have dealt with this problem since [1 thru 10].

Our investigation has a double aim. On one hand conceptual — in the nature no infinite velocity exists; on the other hand practical — to increase the exactness of the solution of some problems. Danyilovskaja was the first who solved the dynamic problem of thermoelasticity in 1950. In her research it was demonstrated that there is a change in some characteristics even before the front of disturbance. The cause of this phenomenon according to Parkus, Vogel, Barenblat is in the structure of the law of heat conduction. The mathematical model of Fourier's law of heat conduction is a parabolic differential equation and its investigation shows that the temperature disturbance propagates by infinite velocity.

To overcome the contradiction, the use of the following hyperbolic equation is recommended in place of parabolic equation (1):

$$aT_{xx} = T_t + \tau T_{tt}. \tag{2}$$

According to this equation, the velocity of propagation of temperature disturbance is obtained as

$$v_T = \sqrt{\frac{a}{\tau}}, \tag{3}$$

a finite value if $\tau \neq 0$.

In earlier investigations, [11], we have elaborated a method to accomodate the law of heat conduction and state equation to the nature of the problem, and tried to explain why in spite of its fundamental error the Fourier law of heat conduction had been one of the most effective model of physics. In our opinion, this is because a modification is significant only in case of rapid dynamical problems which, however, seldom occur in practice.

The nessecity of the modofication has occurred in thermoelasticity. It was not by chance. The elastic wave and its outgrowth: the velocity of propagation of thermodynamic phenomena is at least by two orders of magnitude larger than the velocity of propagation of temperature disturbance. If the fluctuation of temperature is assumed to be harmonic in time, with $2t_0$ period, then in Eq. (2) the ratio of the terms consisting of derivatives with respect to time is $\tau\pi/t_0$. So in the case of internal combustion engine disregarding the term τT_{tt}, gives 8% of error, which depends on τ.

The frequent occurrence of high-speed processes are well-known. In the field of reactor-technics, space research, supersonic flight, modern weapons, magnetohydrodynamic generators and high-speed internal combustion engines, the variation in time is very fast. Because of the nature of the problems, some of the research is conducted in secret. The general researchers are not informed about the concrete problems or the results of these researches.

Discussed below are a new method to modify the law of heat conduction, and state equation, in particular for the explicit, linear case, and, on the other hand, their expectable effect on dynamical problems of thermoelasticity.

The modified law of heat conduction and state equation in an explicit, linear case

With the mechanical interactions neglected, the one dimensional heat conduction problem is described by the first theorem of thermodynamics:

$$\rho u_t = h_x \tag{4}$$

as well as by two constitutive equations: the state equation and the law of heat conduction. Using the conventional formula, these can be written as

$$u = cT, \tag{5}$$

$$h = \kappa T_x . \tag{6}$$

Because of the wave nature of the problem, according to what has been said in [11], equations (5) and (6) cannot be used simultaneously and, with the modifications there and with (4), the phenomenon is described either by equation pair (5) and

$$h = \kappa T_x - \tau h_t , \tag{6'}$$

or by equation pair

$$u = cT + c\tau T_t \tag{5'}$$

and (6). With these equations a second-order hyperbolic partial differential equation is obtained for T (see (2)).

With one of equations (5) or (6) remaining unaltered, let us now investigate what can we do to modify the other equation in such a way that we take a linear combination of all the possible functions (u, u_t, u_x, h, h_t, h_x. T, T_t, T_x).

(a) First, the state equation is left unchanged as in (5), and we try to find the law of heat conduction in formula

$$h = \omega_1 h_t + \omega_2 h_x + \omega_3 T + \omega_4 T_t + \omega_5 T_x, \tag{7}$$

with real unknowns $\omega_1 \dots \omega_5$ (u, u_t and u_x are not included in (7) because these can be expressed from (5) by means of T, T_t, and T_x).

From the system of equations (4), (5), and (7), we obtain the following partial differential equation for T:

$$\omega_5 T_{xx} + \omega_1 \rho c T_{tt} + (\omega_2 \rho c + \omega_4) T_{tx} + \omega_3 T_x - \rho c T_t = 0 . \tag{8}$$

This will be a second-order hyperbolic equation (see (2)), if

$$(\omega_2 \rho c + \omega_4)^2 > 4 \rho c \omega_1 \omega_5 . \tag{9}$$

Sufficient conditions for this are e.g.

$$\operatorname{sign}(\omega_1 \omega_5) = -1 \tag{10}$$

and ω_2, ω_3, ω_4 are arbitrary real numbers. With the introduction of symbols $\omega_1 = -\tau$, $\omega_5 = \kappa$, ($\tau > 0$, $\kappa > 0$) and $a = \dfrac{\kappa}{\rho c}$ we obtain

$$h = \kappa T_x - \tau h_t + \omega_2 h_x + \omega_3 T + \omega_4 T_t . \tag{11}$$

Note that in this case the equation of heat conduction will take the following shape instead of (2):

$$a T_{xx} = T_t + \tau T_{tt} - \left(\omega_2 + \frac{\omega_4}{\rho c} \right) T_{tx} - \frac{\omega_3}{\rho c} T_x . \tag{12}$$

(b) Now we leave the law of heat conduction unchanged as in (6), and we write the state equation with unknowns $a_1 \dots a_7$ as follows:

$$u = a_1 T + a_2 T_t + a_3 T_x + a_4 u_t + a_5 u_x + a_6 h_t + a_7 h_x . \tag{13}$$

Here h is not included because it is in direct proportion with T_x according to (6).

From the partial differential equation system (4), (6), and (13) we obtain a third-order equation for T. If the coefficients of third partial derivatives are assumed to be zero i.e. $a_4 = a_5 = a_6 = a_7 = 0$, then the rest of the second-order equation will take the following shape:

$$\kappa T_{xx} - \rho a_2 T_{tt} - \rho a_3 T_{xt} - \rho a_1 T_t = 0 . \tag{14}$$

This will be hyperbolic if

$$4 \kappa a_2 + \rho a_3^2 > 0 . \tag{15}$$

Sufficient conditions for this are

$$a_2 > 0 \text{ and } a_1,\ a_3 \text{ are arbitrary real numbers.} \tag{16}$$

With the introduction of symbols $a_1 = c$, $a_2 = c\tau$ we obtain

$$u = cT + c\tau T_t + a_3 T_x , \tag{17}$$

and

$$aT_{xx}=T_t+\tau T_{tt}+\frac{a_3}{c}T_{xt}\,. \tag{18}$$

Note that equation (17) is a generalization of (5′) while equation (18) a generalization of (2).

The purpose of this basic research is to outline the possibilities and the direction of studies. In the first step practical applicability is not our aim. The physical interpretation of the modified law of heat conduction (11) and of the state equation (17) is the aim of a later work. This will be done by theoretical and practical methods both in mechanics and in heat technology.

Since the equation system of thermoelasticity has to be written in a generalized shape i.e. in space, for inhomogeneous, anisotropic matter, it is necessary that identities (11) and (17) be produced under such conditions.

In this case, the modified law of heat conduction according to (11) will be

$$h_i=\kappa_{ij}T_{,j}-\tau\dot{h}_i+\omega_j^2 h_{i,j}+\omega_i^3 T+\omega_i^4\dot{T}\,; \qquad i,j=1,2,3\,. \tag{19}$$

The summation convention is true to this case. Since in case of isotropic matter

$$\kappa_{ij}=\kappa\delta_{ij}\,,\quad \omega_j^2=\omega_2 e_j\,,\quad \omega_i^3=\omega_3 e_i\,,\quad \omega_i^4=\omega_4 e_i\,, \tag{20}$$

therefore

$$h_i=\kappa T_{,i}-\tau h_i+\omega_2 e_j h_{i,j}+\omega_3 e_i T+\omega_4 e_i\dot{T}\,. \tag{21}$$

Modified state equation (17) can be written to comply with the above conditions as

$$u=cT+c\tau\dot{T}+a_i^3 T_{,i}\,. \tag{22}$$

Since in case of isotropic matter $a_i^3=a_3 e_i$, therefore

$$u=cT+c\tau\dot{T}+a_3 e_i T_{,i}\,. \tag{23}$$

Application of the system of linear equation to thermoelasticity

According to what has been said introductorily, the problem of thermoelasticity in case of small deformations and reversible processes can be described by the following equation system:

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}) \tag{24}$$

$$\varepsilon_{ij}=a_{ijkl}\sigma_{kl}+\int_0^T \alpha_{ij}dT \tag{25}$$

$$\frac{1}{\rho}\sigma_{ij,j}+q_i=\ddot{u}_i \tag{26}$$

$$\dot{q}=w+(\kappa_{ij}T_{,j})_{,i} \tag{27}$$

$$s=\int_0^T \rho c\frac{dT}{T_1}-\beta_{ij}\varepsilon_{ij} \tag{28}$$

$$\dot{q}=T_1\dot{s} \tag{29}$$

where

$$\varepsilon_{ij},u_i,\sigma_{ij},T_1,q,s$$

are functions of space and time, while material characteristics a_{ijkl}, α_{ij}, β_{ij}, ρ, c, and κ_{ij} functions of space and temperature. Equation (27) can be obtained on the basis of the following considerations (provided the intensity of heat source w is neglected):

$$\dot{q}=h_{i,i};\qquad h_i=\kappa_{ij}T_{,j} \tag{27'}$$

On the basis thereof, using the modified law of heat conduction in (19) and the unchanged state equation in (5), we may write

$$\begin{aligned}\dot{q}\equiv h_{i,i}=(\kappa_{ij}T_{,j})_{,i}-\tau\dot{h}_{i,i}+\omega^2_{j,i}h_{ij}+\omega^2_j h_{i,ji}+\\ +\omega^3_{i,i}T+\omega^3_i T_{,i}+\omega^4_{i,i}\dot{T}+\omega^4_i\dot{T}_{,i}\,.\end{aligned} \tag{30}$$

Taking into consideration that

$$\ddot{q}=\dot{h}_{i,i};\qquad \dot{q}_{,j}=h_{i,ij}, \tag{31}$$

we obtain after reduction equation

$$\begin{aligned}\tau\ddot{q}+\dot{q}-(\kappa_{ij}T_{,j})_{,i}=\omega^4_{i,i}\dot{T}+\omega^4_i\dot{T}_{,i}+\omega^3_i T_{,i}+\omega^3_{i,i}T+\\ +\omega^2_j\dot{q}_{,j}+\omega^2_{j,i}h_{i,j}\,.\end{aligned} \tag{32}$$

Hence, in accordance with the modified law of heat conduction, the equation system of thermoelasticity consists of equations (24 thru 26, 32, 28, 29). Thus the modification had been introduced by means of a system formally most similar to equation system (24 thru 29), however, heat flux intensity h_i has appeared as a new unknown and therefore we would need a new equation as a supplement, e.g. (27'/1). To avoid this, the following ways are offering themselves:

Use h_i as an unknown function in place of heat quantity q in the equations. In this case, the system of thermoelasticity will be given by (11) and (33) in addition to (24 thru 26, 28):

$$h_{i,i}=T_1\dot{s}\,. \tag{33}$$

Another method is to restrict the investigation merely to homogeneous, isotropic matter in accordance with the majority of problems encountered in practice. In this case, system (24 thru 26, 32′, 28, 29) will apply where

$$\tau\ddot{q}+\dot{q}-\kappa T_{,ii}=[\omega_4 \dot{T}_{,i}+\omega_3 T_{,i}+\omega_2 \dot{q}_{,j}]\, e_i\,. \tag{32'}$$

Consider now the equation system of thermoelasticity with the modified state equation. In this case, the first theorem of thermodynamics can be written in the following shape instead of (28):

$$\rho\dot{u}=\dot{q}+\sigma_{ij}\dot{\varepsilon}_{ij}\,. \tag{28/a}$$

After completion of equations (24 thru 27, 29) of thermoelasticity with this and with state equation (22), we obtain a system consisting of seven equations. Here intrinsic energy appears as a new unknown.

Symbols

u — intrinsic energy
h_i — heat flux intensity
ρ — density
c — specific heat
a — coefficient of thermometric conductivity
t — time
x — space
T — temperature difference
T_0 — temperature in normal condition
T_1 — instantaneous temperature
τ — relaxation time
v_T — velocity of propagation of temperature disturbance
κ_{ij} — heat conduction matrix
κ — coefficient of heat conduction
$a_1 \ldots a_7$ — constants
$\omega_1 \ldots \omega_5$ — constants
$\omega_j^2, \omega_j^3, \omega_j^4$ — vector comprising constants
$(\ )_{,j}=\dfrac{\partial(\ )}{\partial x_j}$
δ_{ij} — Kronecker delta
$e_i^*=[1, 1, 1]$
a_i^3 — vector comprising constants
ε_{ij} — strain

u_i — displacement
a_{ijkl} — elasticity matrix
σ_{kl} — stress
α_{ij} — heat conduction matrix
g_i — field strength intensity
q — heat quantity
w — heat source intensity
s — entropy
$\beta_{ij} = -(3\lambda + 2\mu)\,\alpha_{ij}$

λ, μ — Lamé's constants

References

1. BRAUN, M.: Zur Ausbreitung von Unstetigkeitsflächen in thermoelastischen Stoffen. — Doctor's Thesis, Stuttgart 1974.
2. BEEVERS, C. E.—ENGELBRECHT, J.: Theory of Thermoviscoelasticity. Publ. of the Academy of Sci. Sowiet Union, Vol. 24. 1975.
3. PURI, P.: Plane Waves in Generalized Thermoelasticity. — Int. Journal of Engng. Sci. 1973. Vol. 11.
4. MEIXNER, J.: On the Linear Theory of Heat Conduction. — Arch. Rat. Mech. Anal. *39*, 108 (1970).
5. GURTIN, M. E.—PIPKIN, A. C.: A General Theory of Heat Conduction with Finite Wave Speeds. — Arch. Rat. Mech. Anal. *31*, No. 2. (1968).
6. MAURER, M. J.: Relaxation Model for Heat Conduction in Metals. — Journal of Applied Physics, *40*, No. 2. (1969).
7. LORD, H. W.—SHULMAN, Y.: A Generalized Dynamical Theory of Thermoelasticity. — J. Mech. Phys. Solids *15*, 299 (1967).
8. GYARMATI, I.: On the Wave Approach of Thermodynamics and some Problems of Non-Linear Theories. — J. Non-Equilib. Thermodyn. *2*, 233 (1977).
9. FARKAS, H.: Generalization of the Fourier Law. — Per. Pol. Mech. Eng. *24*, 291 (1980).
10. Лыков, А. В.: Теория теплопроводности: Изд. Высшая Школа, Москва, 1967.
11. SZEKERES, A.: Equation System of Thermoelasticity Using the Modified Law of Thermal Conductivity. — Per. Pol. Mech. Eng. *24*, 254 (1980).
12. PETROVSZKIJ, J. G.: Dicussions on Partial Differential Equations. — Akadémiai Kiadó, Budapest, 1955.

Dr. Iréne FARKAS
Dr. András SZEKERES
H-1521 Budapest