TWO FORMULI OF THE SHEAR CENTER

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Summary

In literature there are essentially two different formuli for the coordinates of the shear center. One of them — which can be derived analitically from the solution of the Saint-Venant problems — contains two terms: one of these contains the torsion stress function, the other contains the warping function at twisting. The other formula was derived by Trefftz on the base of energetic considerations, and differs from the previous one in that the term containing the stress function is missing. Why do these formuli differ when both of them were derived from the analitical solution of the Saint-Venant problem?

In the Saint-Venant problems the Saint-Venant principle has been applied, i.e. the distributed forces on the end-section $z=l$, have been replaced by a concentrate force and couple statically equivalent with it. In analytically deriving the formula of the shear center, only this statical equivalence is necessary. But Trefftz's conception implies the consideration, that the equivalence also holds for the energy i.e. the work done by the two, statically equivalent force systems is unchanged in the course or deformation.

The paper deals with this problem, studies the rightfulness and conditions of using the energy and work theorems in the Saint-Venant problems. It verifies, that considering these conditions, Betti's theorem (the way suggested by Trefftz) gives the same formula for the shear center as the analytical solution. In addition it shows how to determine the coordinates of the shear center, when the origin of the system of coordinates in solving the boundary problem for the warping function is an arbitrary point of the cross section.

Introduction

In literature essentially two formuli exist for the coordinates of the shear center; one of them e.g. for single-connected regions [1]:

$$\begin{aligned} x_T &= \frac{\nu}{1+\nu}\frac{1}{I_x}\int_A \Phi x\,dA - \frac{1}{I_x}\int_A \varphi y\,dA\,; \\ y_T &= \frac{\nu}{1+\nu}\frac{1}{I_y}\int_A \Phi y\,dA + \frac{1}{I_y}\int_A \varphi x\,dA\,; \end{aligned} \tag{1}$$

while the other one was derived by Trefftz [2] on the basis of energetic considerations for both single- and multi-connected regions:

$$x_T = -\frac{1}{I_x}\int_A \varphi y\, dA; \qquad y_T = \frac{1}{I_y}\int_A \varphi x\, dA. \tag{2}$$

In both formuli x and y are the principal central axes of inertia of the cross section, z is the axis of the bar (Fig. 1); $\Phi(x, y)$ — the torsion stress function; $\varphi(x, y)$ — the warping function at twisting the bar.

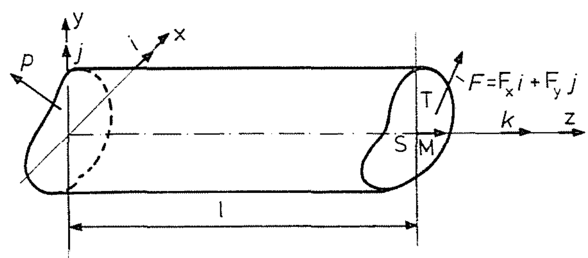

Fig. 1

Both solutions were derived for the Saint-Venant problems, the fundamental conditions of which are the following:

1. The prismatic bar is loaded only at its ends: at end $z=l$ by the active forces, at end $z=0$ by the reactive forces maintaining equilibrium;

2. The distributed active force at $z=l$ can be replaced by the statically equivalent concentrate force and couple at the centroid of the cross section.

3. The end-conditions at $z=0$ must not restrict the warping of the cross section.

The shear center (T) had been defined by both authors as the point of the cross section along which it does not twist the bar if concentrate force $\underline{F}$ acts.

The basis of derivation of formula (1) is the following. The moment about axis z of the concentrate $\underline{F}$ acting at the end-section $z=l$ through its shear center, is equal to the moment of shear stress in a cross section about the same axis. This ensures statical equivalence of the internal forces and force $\underline{F}$. (The other conditions of statical equivalence are realized in the solution.)

The basic conception of Trefftz's derivation is as follows. If at the impact of force $\underline{F}$ the end-section $z=l$ does not twist about axis z, then couple $M\underline{k}$ (causing the twist of the bar) acting at the same time, does not do any work through the displacements involved by force $\underline{F}$, and from this Trefftz concluded that the subsequent strain energy

$$U_{12} \equiv \int_V \boldsymbol{\sigma}^{(1)} \cdot\cdot\, \boldsymbol{\varepsilon}^{(2)}\, dV = \frac{l}{G} \int_A \underline{\tau}_z^{(1)} \cdot \underline{\tau}_z^{(2)}\, dA$$

is equal to zero. Here $\underline{\tau}_z^{(1)}$ and $\underline{\tau}_z^{(2)}$ are the shear stresses in a cross section, involved by force $\underline{F}$ and couple $M\underline{k}$, respectively.

In connection with Trefftz's conception some questions arise. Is it right, when using the energy theorems to replace the actual distributed forces acting at the end-section $z=l$ with the statically equivalent concentrate force and couple, i.e. is the work done by the distributed forces at $z=l$ equal to the work done by the statically equivalent concentrate force and couple along the displacements, compatible with the solution of Saint-Venant's problem?

If it is right, we can say that the distributed forces and concentrate force and couple are equivalent not only statically, but energetically, too. Trefftz's conception is right if this energy equivalence holds.

A further question: is the work done by reactive forces at the end-section $z=0$ equal to zero? If so the work done by forces at the end-section $z=l$ must be equal to the strain energy, determined by stresses and strains. The formula derived by Trefftz is right, if the answers to these questions are positive.

The paper deals with clearing up these questions.

1. It will be verified, that the work done by the system of distributed forces at the end $z=l$, parallel to the section, is equal to the work done by the statically equivalent concentrate force and couple through the displacements compatible with the solution of Saint-Venant's problem.

2. It will be verified, that in the work done by external forces the work done by reactive forces at $z=0$, must be considered too, as in this way only it will be equal to the strain energy determined by stresses and strains.

3. Computing in accordance with the above, Betti's theorem (concept of Trefftz) gives a formula for the shear center corresponding to (1) for multi-connected regions and for axes x, y that are arbitrary axes at the centroid of the cross section.

4. It will be shown, how the coordinates of the shear center can be computed when the origin of the system of coordinates in solving the boundary problem for the warping function, is an arbitrary point of the cross section.

In the former and the following the first system of external forces consists of the concentrate force $\underline{F}$ acting at the end-section $z=l$ through the shear center T and of $p^{(1)}$ distributed force on end-section $z=0$ maintaining equilibrum. If the stress field is $\boldsymbol{\sigma}^{(1)}$, then

$$\underline{p}^{(1)} = \boldsymbol{\sigma}^{(1)}|_{z=0} \cdot (-\underline{k})\,. \tag{3}$$

The 2nd system of external forces consists of the couple $M\underline{k}$ at the end-section $z=l$ and of $\underline{p}^{(2)}$ distributed force on the end-section $z=0$, maintaining equilibrium. If the stress field is $\boldsymbol{\sigma}^{(2)}$, then

$$\underline{p}^{(2)}=\boldsymbol{\sigma}^{(2)}|_{z=0}\cdot(-\underline{k}). \tag{4}$$

Here and in the following, $\cdot$, notes the scalar product. The dyadic product has no sign (e.g. ab).

Verification of the energetical equivalence, Clapeyron's theorem

The stress and displacement fields, the work done by external forces and the strain energy in case of the 1st system of forces

The stress field [1]* is:

$$\begin{aligned}&\sigma_x^{(1)}=\sigma_y^{(1)}=\tau_{xy}^{(1)}=0\,;\\ &\sigma_z^{(1)}=\frac{z-l}{D}\underline{R}\cdot\mathbf{I}_S\cdot\underline{F}\\ &\underline{\tau}_z^{(1)}=\nabla\chi(x,y)\times\underline{k}-\underline{H}\,,\end{aligned}\tag{5}$$

where $\underline{R}=x\underline{i}+y\underline{j}$ the position vector started from the centre of gravity;

$\mathbf{I}_S$ — tensor of inertia of the cross section at its centroid;

$$D\equiv I_xI_y-I_{xy}^2\,;$$

$$\frac{1}{D}\underline{R}\cdot\mathbf{I}_S\cdot\underline{F}\equiv\frac{xI_x-yI_{xy}}{D}F_x+\frac{yI_y-xI_{xy}}{D}F_y\,;$$

$\chi(x,y)$ — bending stress function, with which the equation of the equilibrum

$$\nabla\cdot\underline{\tau}_z^{(1)}+\frac{\partial\sigma_z^{(1)}}{\partial z}=0\tag{6}$$

is identically satisfied;

$\underline{H}$ — a vector, for which the identity

$$\nabla\cdot\underline{H}=\frac{\partial\sigma_z^{(1)}}{\partial z}=\frac{1}{D}\underline{R}\cdot\mathbf{I}_S\cdot\underline{F}\tag{7}$$

* In the papers mentioned formerly the formulae are in a scalar form. It is obvious, that these vectorial expressions are identical.

is valid, so a possible form of $\underline{H}$ is:

$$\underline{H}=\frac{1}{3D}\underline{RR}\cdot\mathbf{I}_S\cdot\underline{F}.$$

The stress function $\chi(x, y)$ is the solution of the boundary problem:

$$\begin{aligned}\Delta\chi&=-\frac{2\nu-1}{1+\nu}\frac{1}{3D}\underline{R}\cdot(\underline{k}\times\mathbf{I}_S\cdot\underline{F})\\ \left.\frac{\partial\chi}{\partial s}\right|_{g_i}&=\left.\underline{H}\cdot\underline{n}\right|_{g_i}\qquad(i=1,2,\ldots,n)\end{aligned}\tag{9}$$

in the case of an n-connected region; n — the exterior unit-normal vector.

The displacement field being compatible with this solution [3], with the identity

$$\underline{u}^{(1)}=\underline{\mathring{u}}+w\underline{k},$$

where $\underline{\mathring{u}}$ is the displacement in the plane (xy):

$$\begin{aligned}\underline{\mathring{u}}&=\frac{\nu}{E}\frac{l-z}{D}\left(\underline{RR}-\frac{R^2}{2}\mathbf{E}\right)\cdot\mathbf{I}_S\cdot\underline{F}+\frac{1}{ED}\left(\frac{lz^2}{2}-\frac{z^3}{3}+\alpha_0 z\right)\mathbf{I}_S\cdot\underline{F}+\underline{\beta}_0\\ w&=\frac{1}{ED}\left(\frac{z^2}{2}-lz-\alpha_0\right)\underline{R}\cdot\mathbf{I}_S\cdot\underline{F}+w_1(x,y),\end{aligned}\tag{10}$$

where E — elasticity modulus;
ν — Poisson's ratio;
$\mathbf{E}$ — two-dimensional unit-tensor;
$\underline{\beta}_0, d_0$ — constants, $\underline{\beta}_0\cdot\underline{k}=0$.

Prescribing the displacement of the centroid of the end-section $z=0$ to be equal to zero, $\underline{\beta}_0=\underline{0}$. The displacement of this point in direction z is given by w_1. The boundary problem for w_1:

$$\begin{aligned}\Delta w_1(x,y)&=-\frac{2}{D}\underline{R}\cdot\mathbf{I}_S\cdot\underline{F}\\ \left.\frac{\partial w_1}{\partial s}\right|_{g_i}&=\frac{\nu}{ED}\underline{n}\cdot\left(\underline{RR}-\frac{R^2}{2}\mathbf{E}\right)\cdot\mathbf{I}_S\cdot\underline{F},\end{aligned}\tag{11}$$

with the solution of which the shear stress can be expressed:

$$\underline{\tau}_z^{(1)}=G\underline{\gamma}_z^{(1)}=G\left[\nabla w_1-\frac{\nu}{ED}\left(\underline{RR}-\frac{R^2}{2}\mathbf{E}\right)\cdot\mathbf{I}_S\cdot\underline{F}\right],\tag{12}$$

where G is the shear modulus.

The traction on the end-section $z=0$ according to (3) is

$$\underline{p}^{(1)} = -(\underline{\tau}_z^1 + \sigma_z^{(1)}\underline{k})_{z=0}\,. \tag{13}$$

The work done by external forces

$$W_{11} \equiv \frac{1}{2}\underline{F}\cdot\underline{u}^{(1)}\Big|_{\substack{\underline{R}=\underline{R}_T\\ z=l}} + \frac{1}{2}\int_A \underline{p}^{(1)}\cdot\underline{u}^{(1)}|_{z=0}\,dA\,. \tag{14}$$

On the basis of (11) (since $\underline{F}\cdot\underline{k}=0$)

$$\mathring{\underline{u}}|_{z=l} = \frac{1}{ED}\left(\frac{l^3}{3}+\alpha_0 l\right)\mathbf{I}_S\cdot\underline{F} = \text{const}, \tag{15}$$

from which it can be seen, that the end-section $z=l$ is displaced in the plane (xy) as a rigid body. And this means, that the work done by any distributed force at the end-section $z=l$ — if it is parallel to the section and statically equivalent to a concentrate force at point T — is equal to the work done by this force through the displacement field compatible with the solution of the Saint-Venant problem; i.e. they are energetically equivalent, too.

The displacement of the end-section $z=0$ is

$$\underline{u}^{(1)}|_{z=0} = \frac{\nu l}{ED}\left(\underline{\underline{RR}} - \frac{R^2}{2}\mathbf{E}\right)\cdot\mathbf{I}_S\cdot\underline{F} + \left[-\frac{\alpha_0}{ED}\underline{R}\cdot\mathbf{I}_S\cdot\underline{F} + w_1(x,y)\right]\underline{k}\,. \tag{16}$$

Substituting (13), (15) and (16) into (14) and taking into account (5):

$$W_{11} = \frac{1}{2ED}\left(\frac{l^3}{3}+\alpha_0 l\right)\underline{F}\cdot\mathbf{I}_S\cdot\underline{F} - \frac{\nu l}{2ED}\int_A \underline{\tau}_z^{(1)}\cdot\left(\underline{\underline{RR}} - \frac{R^2}{2}\mathbf{E}\right)dA\cdot\mathbf{I}_S\cdot\underline{F} -$$

$$- \frac{\alpha_0}{2ED}\underline{F}\cdot\mathbf{I}_S\cdot\int_A \underline{\underline{RR}}\,dA\cdot\mathbf{I}_S\cdot\underline{F} + \frac{l}{2D}\int_A w_1\underline{R}\,dA\cdot\mathbf{I}_S\cdot\underline{F}\,.$$

Since

$$\frac{1}{D}\int_A \underline{\underline{RR}}\,dA \equiv \mathbf{I}_S^{-1} \qquad \text{and} \qquad \mathbf{I}_S^{-1}\cdot\mathbf{I}_S = \mathbf{E} \tag{17}$$

the terms containing α_0 drop out, and the expression of work is:

$$W_{11} \equiv \frac{l^3}{6ED} \underline{F} \cdot \mathbf{I}_S \cdot \underline{F} + \frac{l}{2D} \int_A w_1 \underline{R}\, dA \cdot \mathbf{I}_S \cdot \underline{F} -$$

$$- \frac{\nu l}{2ED} \int_A \underline{\tau}_z^{(1)} \cdot \left(\underline{\underline{RR}} - \frac{R^2}{2} \mathbf{E} \right) dA \cdot \mathbf{I}_S \cdot \underline{F} . \tag{18}$$

The strain energy

$$U_{11} \equiv \frac{1}{2E} \int_V \sigma_z^{(1)} \sigma_z^{(1)}\, dV + \frac{1}{2G} \int_V \underline{\tau}_z^{(1)} \cdot \underline{\tau}_z^{(1)}\, dV.$$

The first integral, considering (5)

$$\frac{1}{2E} \int_A \int_0^l \frac{(z-l)^2}{D^2} dz\, \underline{F} \cdot \mathbf{I}_S \cdot \underline{\underline{RR}} \cdot \mathbf{I}_S \cdot \underline{F}\, dA = \frac{l^3}{6ED} \underline{F} \cdot \mathbf{I}_S \underline{F} ,$$

i.e.

$$\frac{1}{2E} \int_V \sigma_z^{(1)} \sigma_z^{(1)}\, dV = \frac{l^3}{6ED} \underline{F} \cdot \mathbf{I}_S \cdot \underline{F} .$$

Putting into the second integral expression (12) instead of one $\underline{\tau}_z^{(1)}$, and taking into account, that $\underline{\tau}_z^{(1)}$ does not depend on z:

$$\frac{1}{2G} \int_V \underline{\tau}_z^{(1)} \cdot \underline{\tau}_z^{(1)}\, dV = \frac{l}{2} \int_A \underline{\tau}_z^{(1)} \cdot \nabla w_1\, dA -$$

$$- \frac{\nu l}{2ED} \int_A \underline{\tau}_z^{(1)} \cdot \left(\underline{\underline{RR}} - \frac{R^2}{2} \mathbf{E} \right) dA \cdot \mathbf{I}_S \cdot \underline{F} .$$

The first right-hand integral can be transformed as:

$$\frac{l}{2} \int_A \underline{\tau}_z^{(1)} \cdot \nabla w_1\, dA = \frac{l}{2} \int_A \nabla \cdot (w_1 \underline{\tau}_z^{(1)})\, dA - \frac{l}{2} \int_A w_1 \nabla \cdot \underline{\tau}_z^{(1)}\, dA =$$

$$= \frac{l}{2} \sum_{i=1}^{n} \oint_{g_i} \underline{n} \cdot \underline{\tau}_z^{(1)} w_1\, ds - \frac{l}{2} \int_A w_1 \nabla \cdot \underline{\tau}_z^{(1)}\, dA .$$

According to (6) and (7)

$$\nabla \cdot \underline{\tau}_z^{(1)} = -\frac{\partial \sigma_z^{(1)}}{\partial z} = -\frac{1}{D} \underline{R} \cdot \mathbf{I}_S \cdot \underline{F},$$

and on the boundary curves

$$\underline{\tau}_z^{(1)} \cdot \underline{n} \,|_{g_i} = 0,$$

so

$$U_{11} \equiv \frac{l^3}{6ED} \underline{F} \cdot \mathbf{I}_S \cdot \underline{F} + \frac{l}{2D} \int_A w_1 \underline{R} \, dA \cdot \mathbf{I}_S \cdot \underline{F} -$$

$$- \frac{\nu l}{2ED} \int_A \underline{\tau}_z^{(1)} \cdot \left(\underline{RR} - \frac{R^2}{2} \mathbf{E} \right) dA \cdot \mathbf{I}_S \cdot \underline{F}.$$

Comparing this expression with (18), it can be established that Clapeyron's theorem — $U_{11} = W_{11}$ — is realized if we take into account the work done by reactive forces at the end-section $z=0$ in the course of deformation.

The stress and displacement fields, the work done by external forces and the strain energy in case of the 2nd system of forces

The stress field is

$$\begin{gathered} \sigma_x^{(2)} = \sigma_y^{(2)} = \sigma_z^{(2)} = \tau_{xy}^{(2)} = 0 \\ \underline{\tau}_z^{(2)} = G\vartheta \, \nabla \Phi \times \underline{k}, \end{gathered} \tag{19}$$

where $\vartheta \equiv d\Psi/dz = \text{constant}$ — the twist of the bar; $\Phi(x, y)$ the torsion stress function, the solution of the boundary problem:

$$\Delta \Phi = -2; \qquad \Phi|_{g_i} = C_i = \text{const} \qquad (i = 1, \ldots, n) \tag{20}$$

in the case of an n-connected region. One of the constants, C_i, may be arbitrary, for example

$$C_1 = 0,$$

where g_1 is the curve surrounding all the others. From the statical equivalence of couple $M\underline{k}$ and stress $\underline{\tau}_z^{(2)}$:

$$M = 2G\vartheta \left[\int_A \Phi \, dA + \sum_{i=2}^{n} C_i A_i \right] \tag{21}$$

where A_i is the area of the region surrounded by curve g_i.

The compatible displacement field:

$$\underline{u}^{(2)} = \vartheta z \underline{k} \times \underline{R} + \vartheta \varphi(x, y)\, \underline{k}\,, \tag{22}$$

where $\varphi(x, y)$ is the warping function, and the solution of the boundary problem:

$$\Delta\varphi = 0; \qquad \left.\frac{\partial\varphi}{\partial n}\right|_{g_i} = \underline{R} \cdot \underline{e}\,|_{g_i} \qquad (i = 1, \ldots, n)\,. \tag{23}$$

Here $\underline{e} = \underline{k} \times \underline{n}$, the tangential unit-vector. The stress can be expressed by the warping function:

$$\underline{\tau}_z^{(2)} = G\underline{\gamma}_z^{(2)} = G\vartheta\,[\nabla\varphi - \underline{R} \times \underline{k}]\,. \tag{24}$$

The work done by the external forces

$$W_{22} \equiv \frac{1}{2}\,\underline{M} \cdot \underline{\omega}^{(2)}\,|_{z=l} + \frac{1}{2}\int_A \underline{p}^{(2)} \cdot \underline{u}^{(2)}\,|_{z=0}\, dA\,,$$

where

$$\underline{\omega}^{(2)} = \frac{1}{2}\,\nabla \times \underline{u}^{(2)}; \qquad \underline{M} = M\underline{k}\,.$$

It is to be seen that $\underline{\omega}^{(2)} = \vartheta z \underline{k}$, so

$$\underline{\omega}^{(2)}\,|_{z=l} = \vartheta l \underline{k} = \text{const},$$

and the work done by any distributed force p_l at $z = l$ — if it is parallel to the section and statically equivalent to couple $M\underline{k}$ — is equal to the work done by couple $M\underline{k}$ (at any point of the cross section) through the displacements (22):

$$\frac{1}{2}\int_A \underline{p}_l \cdot (\vartheta l \underline{k} \times \underline{R} + \vartheta\varphi\underline{k})\, dA = \frac{\vartheta l}{2}\,\underline{k} \cdot \int_A \underline{R} \times \underline{p}_l\, dA \equiv \frac{1}{2}\,\underline{M} \cdot \underline{\omega}^{(2)}\,|_{z=l}\,,$$

i.e. the distributed force $\underline{p}_l$ is equivalent to couple $M\underline{k}$ not only statically but also energetically.

The traction on the end-section $z = 0$ (see(5)), and $\underline{u}^{(2)}\,|_{z=0}$

$$\underline{p}^{(2)} = -\underline{\tau}_z^{(2)}\,|_{z=0}\,; \qquad \underline{u}^{(2)}\,|_{z=0} = \vartheta\varphi\underline{k}\,.$$

Thus the second term in W_{22} equals zero, and

$$W_{22} \equiv \frac{\vartheta l}{2}\, M\,. \tag{25}$$

The strain energy

$$U_{22} \equiv \frac{1}{2} \int_V \underline{\underline{\tau}}_z^{(2)} \cdot \underline{\underline{\tau}}_z^{(2)} \, dV.$$

Taking into account (24) and that the integrand does not depend on z:

$$U_{22} \equiv \frac{\vartheta l}{2} \int_A \underline{\underline{\tau}}_z^{(2)} \cdot (\nabla \varphi + \underline{k} \times \underline{R}) \, dA =$$

$$= \frac{\vartheta l}{2} \underline{k} \cdot \int_A \underline{R} \times \underline{\underline{\tau}}_z^{(2)} \, dA + \frac{\vartheta l}{2} \int_A \nabla \cdot (\underline{\underline{\tau}}_z^{(2)} \varphi) \, dA - \frac{\vartheta l}{2} \int_A \varphi \nabla \cdot \underline{\underline{\tau}}_z^{(2)} \, dA .$$

According to the statical equivalent of internal forces $\underline{\underline{\tau}}_z^{(2)}$ and external forces:

$$\underline{k} \cdot \int_A \underline{R} \times \underline{\underline{\tau}}_z^{(2)} \, dA = M ,$$

and to the equation of equilibrium (6) $\nabla \cdot \underline{\underline{\tau}}_z^{(2)} = 0$, after transforming the second integral with the theorem of Gauss—Ostrogradskij, we receive

$$\int_A \nabla \cdot (\underline{\underline{\tau}}_z^{(2)} \varphi) \, dA = \sum_{i=1}^{n} \oint_{g_i} \underline{n} \cdot \underline{\underline{\tau}}_z^{(2)} \varphi \, dA = 0,$$

since $\underline{\underline{\tau}}_z^{(2)} \cdot \underline{n} \,|_{g_i} = 0$.

Thus Clapeyron's theorem, $U_{22} = W_{22}$, holds for this case, too.

The coordinates of the shear center

The realization of Clapeyron's theorem for both the 1st and the 2nd system of forces indicates; that we computed the work of external forces in the right way: Computing in the same way the work in the course of both system of forces acting simultaneously, Betti's theorem

$$W_{12} = W_{21}$$

gives the right expression for the coordinates of the shear center.

The work W_{12}:

$$W_{12} \equiv \underline{F} \cdot \underline{u}^{(2)} \Big|_{\substack{\underline{R} = \underline{R}_T \\ z = l}} + \int_A \underline{p}^{(1)} \cdot \underline{u}^{(2)} \,|_{z=0} \, dA .$$

According to (22) and (1)

$$\underline{u}^{(2)}\Big|_{\substack{\underline{R}=\underline{R}_T\\ z=l}} = \vartheta l\underline{k}\times\underline{R}_T+\vartheta\varphi(x,y)\,\underline{k}\,;$$

$$\underline{u}^{(2)}\,|_{z=0}=\vartheta\varphi(x,y)\,\underline{k}\,,$$

$$\underline{p}^{(1)}=-(\underline{\tau}_z^{(1)}+\sigma_z^{(1)}\underline{k})_{z=0}\,,$$

thus, taking into account (5) and that $\underline{F}\cdot\underline{k}=0$:

$$W_{12}\doteq\vartheta l(\underline{k}\times\underline{R}_T)\cdot\underline{F}+\frac{\vartheta l}{D}\int_A\varphi\underline{R}\,dA\cdot\mathbf{I}_S\cdot\underline{F}\,. \tag{26}$$

The work W_{21}:

$$W_{21}\equiv M\omega_z^{(1)}\Big|_{\substack{\underline{R}=\underline{0}\\ z=l}}+\int_A\underline{p}^{(2)}\cdot\underline{u}^{(1)}\,|_{z=0}\,dA\,.$$

where the rotation around axis z:

$$\omega_z^{(1)}=\frac{1}{2}\underline{k}\cdot(\nabla\times\underline{u}^{(1)})\,.$$

Taking into account (10) for $\underline{u}^{(1)}$:

$$\omega_z^{(1)}=\frac{1}{2}\,\frac{\nu}{E}\,\frac{l-z}{D}(\underline{k}\times\nabla)\cdot\left(\underline{RR}-\frac{R^2}{2}\mathbf{E}\right)\cdot\mathbf{I}_S\cdot\underline{F}\,,$$

from which

$$\omega_z^{(1)}\,|_{z=l}=0$$

in all points of the cross section. Thus the twisting couple does not do any work in the displacements involved by force $\underline{F}$. (Trefftz stated this too, but he identified it with work W_{21}.)

Considering (11), (4) and (12)

$$W_{21}\equiv-\frac{\nu l}{ED}\int_A\underline{\tau}_z^{(2)}\cdot\left(\underline{RR}-\frac{R^2}{2}\mathbf{E}\right)\cdot\mathbf{I}_S\cdot\underline{F}\,dA\,. \tag{27}$$

Substituting $\underline{\tau}_z^{(2)}$ according to (19)

$$W_{21}\equiv\frac{\nu l}{ED}G\vartheta\int_A(\underline{k}\times\nabla\Phi)\cdot\left(\underline{RR}-\frac{R^2}{2}\mathbf{E}\right)dA\cdot\mathbf{I}_S\cdot\underline{F}\,.$$

The transformation of this integral is:

$$\underline{k}\cdot\int_A \nabla\Phi\times\left(\underline{RR}-\frac{R^2}{2}\mathbf{E}\right)dA=$$

$$=\underline{k}\cdot\int_A \nabla\times\left[\Phi\left(\underline{RR}-\frac{R^2}{2}\mathbf{E}\right)\right]dA-\underline{k}\cdot\int_A \Phi\nabla\times$$

$$\times\left(\underline{RR}-\frac{R^2}{2}\mathbf{E}\right)dA=$$

$$=\sum_{i=1}^{n}\underline{k}\cdot\oint_{g_i}\Phi\underline{n}\times\left(\underline{RR}-\frac{R^2}{2}\mathbf{E}\right)ds-2\int_A \Phi\underline{R}\,dA\times\underline{k},$$

since it is to be seen, that

$$\underline{k}\cdot(\nabla\times\underline{RR})-\left(\underline{k}\times\nabla\frac{R^2}{2}\right)\cdot\mathbf{E}=\underline{R}\times\underline{k}-\underline{k}\times\underline{R}=2\underline{R}\times\underline{k}.$$

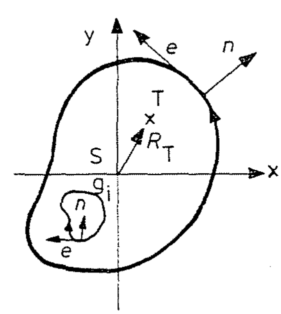

Fig. 2

On the curves g_i $\Phi|_{g_i}=C_i=$const, and (see Fig. 2)

$$\underline{k}\cdot\oint_{g_i}\underline{n}\times\left(\underline{RR}-\frac{R^2}{2}\mathbf{E}\right)ds=-\underline{k}\cdot\int_{A_1}\nabla\times\left(\underline{RR}-\frac{R^2}{2}\mathbf{E}\right)dA=$$

$$=-2\int_{A_i}\underline{R}\,dA\times\underline{k}=-2\underline{S}_i\times\underline{k},$$

where $\underline{S}_i$ is the statical moment of area A_i regarding the centroid of the cross section. Finally

$$W_{21} \equiv -\frac{\nu I G}{ED} 2\vartheta \left[\int_A \underline{R}\, dA + \sum_{i=2}^{n} C_i \underline{S}_i \right] \cdot (\underline{k} \times \mathbf{I}_S \cdot \underline{F}) . \tag{28}$$

Writing (26) and (28) into equation $W_{12} = W_{21}$, we obtain a formula for the coordinates of the shear center:

$$(\underline{k} \times \underline{R}_T) \cdot \underline{F} = -\frac{\nu}{1+\nu} \frac{1}{D} \left(\int_A \Phi \underline{R}\, dA + \sum_{i=2}^{n} C_i \underline{S}_i \right) \cdot (\underline{k} \times \mathbf{I}_S) \cdot \underline{F} -$$

$$-\frac{1}{D} \int_A \varphi \underline{R}\, dA \cdot \mathbf{I}_S \cdot \underline{F} ,$$

i.e.

$$\underline{k} \times \underline{R}_T = -\frac{\nu}{1+\nu} \frac{1}{D} \left(\int_A \Phi \underline{R}\, dA + \sum_{i=2}^{n} C_i \underline{S}_i \right) \cdot (\underline{k} \times \mathbf{I}_S) - \frac{1}{D} \int_A \varphi \underline{R}\, dA \cdot \mathbf{I}_S ,$$

since $E = 2G(1+\nu)$; furthermore $\underline{k} \times \underline{R}_T \times \underline{k} = \underline{R}_T$; $\frac{1}{D} \underline{k} \times \mathbf{I}_S \times \underline{k} = \mathbf{I}_S^{-1}$ and $\frac{1}{D} \mathbf{I}_S \times \underline{k} = \underline{k} \times \mathbf{I}_S^{-1}$:

$$\underline{R}_T = \left[\frac{\nu}{1+\nu} \left(\int_A \Phi \underline{R}\, dA + \sum_{i=2}^{n} C_i \underline{S}_i \right) - \int_A \varphi \underline{R}\, dA \times \underline{k} \right] \cdot \mathbf{I}_S^{-1} . \tag{29}$$

In the case of a single-connected region and $I_{xy} = 0$, the formula (29) coincides with (1), which was found on the basis of statical equivalence only.

The role of the origin of the system of coordinates in the torsion problem and its effect on the formula of shear center

The displacement field at twisting depends of the origin of the system of coordinates. In the case of $0 \equiv S$:

$$\underline{u} = \vartheta z \underline{k} \times \underline{R} + \vartheta \varphi(x, y)\, \underline{k} ,$$

and in the case of $0 \equiv A$ — an arbitrary point of the cross section (Fig. 3)

$$\underline{u}_A = \vartheta z \underline{k} \times \underline{r} + \vartheta \varphi_A(x, y)\, \underline{k} .$$

In both cases the cross sections rotate about the origin. However, the stress field does not depend on the selection of the origin, thus — according to the signs of Fig. 3:

$$\underline{\tau}_z^{(2)} = G\vartheta\,(\nabla\varphi - \underline{R}\times\underline{k}) = G\vartheta\,(\nabla\varphi_A - \underline{r}\times\underline{k}),$$

from which

$$\nabla(\varphi-\varphi_A) = (\underline{R}-\underline{r})\times\underline{k} = \underline{R}_A\times\underline{k}.$$

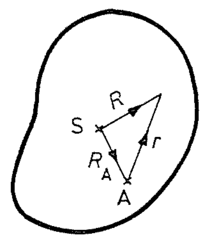

Fig. 3

The deviation of the two warping functions is

$$\varphi-\varphi_A = \underline{R}\cdot(\underline{R}_A\times\underline{k}) + C$$

(since $\nabla\underline{R} = \mathbf{E}$). Writing this into expression (26) of W_{12}

$$W_{12} \equiv \vartheta\varphi(\underline{k}\times\underline{R}_T)\cdot\underline{F} + \frac{\vartheta l}{D}\int_A \varphi_A\underline{R}\,dA\cdot\mathbf{I}_S\cdot\underline{F} +$$

$$+\frac{\vartheta l}{D}(\underline{R}_A\times\underline{k})\cdot\int_A \underline{RR}\,\mathrm{dA}\cdot\mathbf{I}_S\cdot\underline{F} + \frac{\vartheta l}{D}C\int_A \underline{R}\,dA\cdot\mathbf{I}_S\cdot\underline{F}.$$

Taking into account (17) and that $\int_A \underline{R}\,dA = \underline{0}$, we obtain

$$W_{12} \equiv \vartheta l\,[\underline{k}\times(\underline{R}_T-\underline{R}_A)]\cdot\underline{F} + \frac{\vartheta l}{D}\int_A \varphi_A\underline{R}\,dA\cdot\mathbf{I}_S\cdot\underline{F}.$$

In this formula the first term is the work done by force $\underline{F}$ in the course of rotation ϑl of end-section $z=l$ around point A. Thus formula (29) of the shear center changes as follows

$$\underline{R}_T = \underline{R}_A + \left[\frac{\nu}{1+\nu}\left(\int_A \Phi\underline{R}\,dA + \sum_{i=2}^{n} C_i\underline{S}_i\right) - \int_A \varphi_A\underline{R}\,dA\times\underline{k}\right]\cdot\mathbf{I}_S^{-1}.$$

where the φ_A is the solution of the boundary problem:

$$\Delta\varphi_A=0, \qquad \left.\frac{\partial\varphi_A}{\partial n}\right|_{g_i}=\underline{r}\cdot\underline{e}\,|_{g_i} \qquad (i=1,\ldots,n).$$

From this formula of the shear center we can conclude, that if formula (29) contains warping function φ_A—belonging to an arbitrary point A, — it gives the place of the shear center relative to point A, supposing, that $\underline{R}$ begins at the centroid of the cross section in this case, too.

Conclusion

1. In Saint-Venant's problem the work done by the system of active forces parallel to the plane (xy) and distributed on the end-section $z=l$ is equal to the work done by the statically equivalent concentrate force and couple along the displacements compatible with the solution, so they are equivalent not only statically but also energetically.

2. The system of reactive forces does work along the displacements compatible with the solution. (But this system of distributed forces is energetically not equivalent to the statically equivalent force and couple in the centroid of the cross section $z=0$.) The work theorems can only be used when taking into consideration this work.

3. Computing work W_{12} and W_{21} in accordance with this condition, on the basis of Trefftz's conception we receive the same formula for the shear center as the one derived analitically.

4. It was furthermore shown, how to determine the coordinates of the shear center, when the origin of the system of coordinates in solving the boundary problem for the warping function is an arbitrary point of the cross section.

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