A MODEL OF BALANCING

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Summary

Place the end of a rod on your fingertip and move this lowest point of the rod to a degree that the upper vertical position of it should be stable. It is obvious, that it is not possible to equilibrate it if one's reflexes are slow. The paper shows the determination of the critical delay of the reflexes where this balancing is still possible.

Let us try to balance a rod in the following way: the end of the rod is placed on the fingertip and this lowest point of the rod is moved to a degree that its upper position should be stable. It is obvious that the rod cannot be balanced if our reflexes are slow. This paper shows how to determine the critical delay of the reflexes where a balancing is still possible.

The first figure shows the simplest model of the above described dynamical system (m is the mass, J_S is the moment of inertia, g denotes the gravitational acceleration). The coordinate x of point A is controlled by force **F** (i.e. by the balancing person) according to the delayed value of angle φ.

The linearized equation of the small motions is:

$$(J_S + ml^2)\ddot{\varphi} + ml\ddot{x} - mgl\varphi = 0 \tag{1}$$

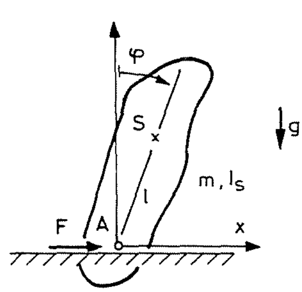

Fig. 1

Because of the constraint between x and φ, φ may be considered as the only coordinate of the system, so it has only one degree of freedom. Naturally, before the stability investigation of (1), a suitable linearized constraint has to be chosen. But first, let us see the stability criterion for these delayed (or retarded) dynamical systems according to [1].

Let us consider the retarded functional differential equation

$$\mathbf{A}\ddot{\mathbf{q}}+\int_{-r}^{0} d\mathbf{B}(\vartheta)\,\dot{\mathbf{q}}(t+\vartheta)+\int_{-r}^{0} d\mathbf{C}(\vartheta)\,\mathbf{q}(t+\vartheta)=0 \tag{2}$$

where $\mathbf{A}$ is a positive, definit constant matrix of dimension n, $\mathbf{B}$ and $\mathbf{C}_{n\times n}$ are functions of bounded variations, r is the length of time lag, t denotes the time. Let

$$D(\lambda)=\det\left(\mathbf{A}\lambda^2+\int_{-r}^{0}\lambda e^{\lambda\vartheta}\,d\mathbf{B}(\vartheta)+\int_{-r}^{0}e^{\lambda\vartheta}\,d\mathbf{C}(\vartheta)\right) \tag{3}$$

where λ is a complex number and let

$$M(y)=\operatorname{Re}D(iy),\qquad S(y)=\operatorname{Im}D(iy),\qquad i=\sqrt{-1} \tag{4}$$

where y is a real number. $y_1\geq y_2\geq\ldots\geq y_m\geq 0$ denote the non-negative real zeroes of M. The $\mathbf{q}\equiv\mathbf{0}$ solution of (2) is asymptotically stable if, and only if:

$$S(y_k)\neq 0,\qquad k=1,\ldots,m \tag{5}$$

and

$$\sum_{k=1}^{m}(-1)^{k+1}\operatorname{sign}S(y_k)+(-1)^n n=0\,. \tag{6}$$

With the help of (5) and (6), a lot of statements can be proved for retarded dynamical systems. For example, the theorems published in [2] imply that the simplest possibility to balance the rod is to make the acceleration of point A proportional to the sum of the delayed values of angle φ and the angular velocity. By supposing that the delay of our reflexes is a distinct constant, τ, we get the constraint:

$$\ddot{x}(t)=b_1\dot{\varphi}(t-\tau)+b_0\varphi(t-\tau)\,. \tag{7}$$

Of course, the parameters b_0 and b_1 have to be chosen in a suitable way. Substitute (7) into (1) and use time T of the low oscillation of the rod when it is a pendulum hung up at point A:

$$T=2\pi\sqrt{\frac{l}{g}\,\frac{J_S+ml^2}{ml^2}}\,, \tag{8}$$

then the equation of motion is:

$$\ddot{\varphi}(t)+\frac{4\pi^2}{T^2g}b_1\dot{\varphi}(t-\tau)+\frac{4\pi^2}{T^2g}b_0\varphi(t-\tau)-\frac{4\pi^2}{T^2}\varphi(t)=0\,. \tag{9}$$

This equation can be transformed into (2) if $r>\tau$ and

$$A=1,$$

$$B(\vartheta)=\begin{cases}0, & -r\le\vartheta<-\tau\\ \dfrac{4\pi^2}{T^2g}b_1, & -\tau\le\vartheta\le 0,\end{cases}$$

$$C(\vartheta)=\begin{cases}0, & -r\le\vartheta<-\tau\\ \dfrac{4\pi^2}{T^2g}b_0, & -\tau\le\vartheta<0\\ \dfrac{4\pi^2}{T^2g}(b_0-g), & \vartheta=0,\end{cases} \tag{10}$$

so (5) and (6) can be used for the stability investigation of the position $\varphi\equiv 0$ of the rod:

$$D(\lambda)=\lambda^2+\frac{4\pi^2}{T^2g}\tau b_1\lambda e^{-\lambda}+\frac{4\pi^2}{T^2g}\tau^2 b_0 e^{-\lambda}-\frac{4\pi^2}{T^2}\tau^2, \tag{11}$$

$$M(y)=-y^2+\frac{4\pi^2}{T^2g}\tau b_1 y\sin y+\frac{4\pi^2}{T^2g}\tau^2 b_0\cos y-\frac{4\pi^2}{T^2}\tau^2, \tag{12}$$

$$S(y)=\frac{4\pi^2}{T^2g}\tau b_1 y\cos y-\frac{4\pi^2}{T^2g}\tau^2 b_0\sin y\,. \tag{13}$$

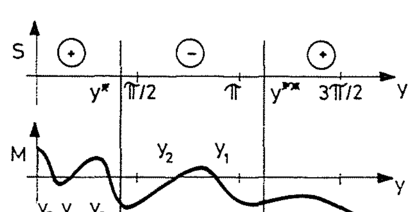

Fig. 2

In this case, the degree of freedom $n=1$, thus (6) becomes

$$\sum_{k=1}^{m} (-1)^{k+1} \operatorname{sign} S(y_k)=1 . \tag{14}$$

With the help of the second figure it can be proved that (14) is equivalent to inequalities

$$M(0)>0 , \tag{15}$$

$$M(y^*)<0 , \tag{16}$$

where y^* is the zero point of S in the interval $\left(0, \frac{\pi}{2}\right)$. It can also be proved that $M(y^*)<0$ implies $M(y^{**})<0$, and so on. Let us see what (15) and (16) mean:

$$M(0)>0 \quad \Rightarrow \quad b_0>g \tag{17}$$

$$\left.\begin{array}{l} M(y^*)<0 \\ \\ S(y^*)=0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} b_0<\left(\dfrac{T^2}{4\pi^2\tau^2}\, y^{*2}+1\right) g \cos y^* \\ \\ b_1=b_0\tau\dfrac{\tan y^*}{y^*} \end{array}\right\} \quad 0<y^*<\frac{\pi}{2} \tag{18}$$

If the rod is $2l=1$ meter long, Fig. 3 shows the stability regions on the plane of parameters b_0 and b_1 according to inequalities (17) and (18). For example, if the delay of our reflexes is 0.2 second, then we have to choose point (b_0, b_1) from the shaded area. As it can be seen in the figure, the greater the time lag is, the smaller the stability regions are; moreover if $\tau=0.4$ second, there is no region at all, so

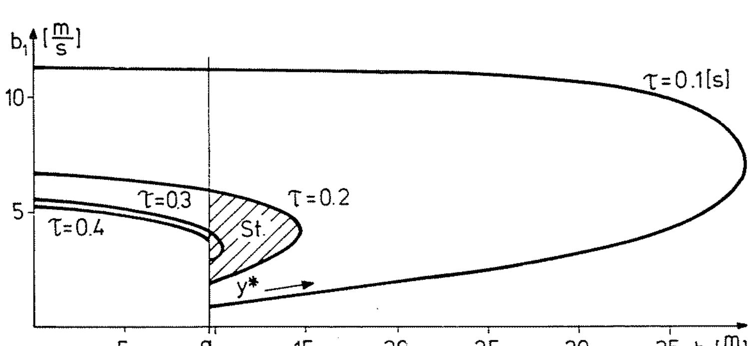

Fig. 3

the rod cannot be equilibrated. It is natural that the critical delay, where this balancing is still possible, can be computed with the help of the equation (18):

$$\left.\frac{db_1}{db_0}\right|_{y^*=0} > 0 . \tag{19}$$

After a short computation, we come to the conclusion that (19) is fulfilled if

$$\left.\frac{d^2 b_1}{dy^{*2}}\right|_{y^*=0} > 0 \qquad \text{and} \qquad \left.\frac{d^2 b_0}{dy^{*2}}\right|_{y^*=0} > 0 . \tag{20}$$

This means that

$$\tau < \frac{\sqrt{2}}{2\pi} T. \tag{21}$$

Let us see the result! For example, if the rod is 0.3 meter long, the critical delay is 0.2 second according to (21), so after a short practice, anyone is able to equilibrate such a rod because the delay of our reflexes is about 0.1 second. Anybody can see that the longer the rod, the simpler it is to equilibrate it, according to (21), because in this case oscillation time T of the pendulum is greater, so the critical delay is also greater. Finally, if one is a bit tipsy, the rod can not be equilibrated, because the delay of the reflexes becomes greater than the critical value.

References

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