

ENERGY EQUATIONS OF CRACKED SOLIDS

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Summary

The energy equation of cracked solids is formulated relying on field equations of continuum mechanics and thermodynamics. Under simplifying conditions valid under the test circumstances, the energy balance of elasto-plastic and of elastic solids is determined. An elastic material equation is applied to interpret path-independent line integrals, related, in turn, with other fracture mechanics characteristics.

The mechanical model

Let us consider a continuum of volume V and boundary surface A containing a material discontinuity (crack) of surface A_r^0 (Fig. 1). A_r^0 is assumed to be free of load.

Under load, at a time $t > t_0$ the crack surface increases by $A(t) \cdot A_r(t) = A_r^0 + A(t)$. Crack propagation has to be treated as a non-equilibrium thermodynamical process, its correct description requires the introduction of surface-dependent state characteristics.*

Accordingly, in the following, ρ^* will denote surface mass density, u^* —inherent surface internal energy density, s^* —surface entropy density. Internal points of a solid are those meeting the following equations:

$$T_{ij,j} + \rho f_i - \rho \dot{v}_i = 0 \quad (1)$$

$$T_{ij} = T_{ji}; \quad T_{ij}^e + T_{ij}^d = T_{ij} \quad (2)$$

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad (3)$$

$$T_{ij}^e \dot{\varepsilon}_{ij} + \rho r - h_{k,k} = \rho \dot{u}' \quad (4)$$

* An imperative also in classic continuum mechanics. But since field equations involve material time derivatives, their variation on materially steady-state surfaces may be neglected.

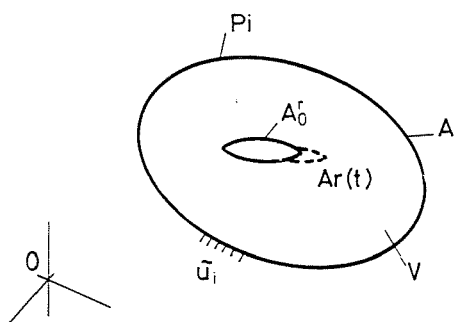


Fig. 1

$$\rho \Theta \dot{s} = \rho r - h_{k,k} \quad (5)$$

$$\rho \left(\dot{s} - \frac{\dot{u}'}{\Theta} \right) + \frac{1}{\Theta} T_{ij}^e \dot{\epsilon}_{ij} - \frac{h_k}{\Theta^2} \Theta_{,k} \geq 0 \quad (6)$$

$$T_{ij}^d \dot{\epsilon}_{ij} \geq 0 \quad (7)$$

The energy equation

According to the first principal theorem of thermodynamics, at any instant of crack propagation, the energy balance is of the form:

$$\dot{W} + \dot{Q} = \frac{d}{dt} [K + U] \quad (8)$$

involved magnitudes being:

$$\dot{W} = \int_A p_k \dot{u}_k dA + \int_V f_k \dot{u}_k \rho dV \quad (9)$$

$$\dot{Q} = \int_V \rho r dV - \int_A h_k n_k dA \quad (10)$$

$$K = \frac{1}{2} \int_V \rho \dot{u}_k \dot{u}_k dV + \frac{1}{2} \int_{A_r} \rho^* \dot{u}_k \dot{u}_k dA + \frac{1}{2} \int_{A(t)} \rho^* \dot{u}_k \dot{u}_k dA \quad (11)$$

$$(12)$$

Timely variation of quantities related to initial crack surface being negligible compared to other terms, the first principal theorem may be put as:

$$\dot{W} + \dot{Q} = \dot{K}[V] + \dot{U}[V] + \frac{d}{dt} \int_{A(t)} \left(u^* + \frac{1}{2} \dot{u}_k \dot{u}_k \right) \rho^* dA =$$

$$\dot{K}[V] + \dot{U}[V] + \frac{d}{dt} \int_A \gamma^* dA \quad (13)$$

where $[v]$ represents the volume dispersion.

The last term is denoted $\dot{\Gamma}$ in publications on fracture mechanics, and called the energy needed to form new surfaces. Concrete formulation of the energy equation needs constitutive equations, to be obtained through the second principal theorem of thermodynamics. Assume a T_{ij}^e part of the stress field h_k , the heat flux vector, u' part of the internal energy, and entropy to be continuous and differentiable function of the following variables at a material point with given coordinates:

$$u' = u'(\varepsilon_{ij}; \Theta_{,k}; \Theta) \quad T_{ij}^e = T_{ij}^e(\varepsilon_{ij}; \Theta_{,k}; \Theta)$$

$$s = s(\varepsilon_{ij}; \Theta_{,k}; \Theta) \quad h_k = h_k(\varepsilon_{ij}; \Theta_{,k}; \Theta) \quad (14)$$

Introducing free energy density function

$$\tau = \rho(u' - \Theta s) \quad (15)$$

deriving and applying (6) and (7) leads to constitutive equations:

$$T_{ij}^e = \frac{\partial \tau}{\partial \varepsilon_{ij}} \quad (16)$$

$$u' = \frac{1}{\rho} \left(\tau - \Theta \frac{\partial \tau}{\partial \Theta} \right) \quad (17)$$

$$\frac{\partial \tau}{\partial \Theta_{,k}} = 0 \quad (18)$$

$$s = - \frac{1}{\rho} \frac{\partial \tau}{\partial \Theta} \quad (19)$$

$$-h_k \Theta_{,k} \geq 0 \quad (20)$$

$$T_{ij}^d \dot{\varepsilon}_{ij} \geq 0 \quad (21)$$

Expanding function $\tau(\varepsilon_{ij}; \Theta)$ with respect to natural condition $\varepsilon_{ij} = 0; \Theta = \Theta_0$ and omitting all but linear terms:

$$\tau(\varepsilon_{ij}; \Theta) = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} - \beta_{ij} \varepsilon_{ij} T + \frac{mT^2}{2} \quad (22)$$

where

$$C_{ijkl} = \frac{\partial^2 \tau(0; \Theta_0)}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \quad (23)$$

$$-\beta_{ij} = \frac{\partial^2 \tau(0; \Theta_0)}{\partial \varepsilon_{ij} \partial \Theta} \quad (24)$$

$$m = \frac{\partial^2 \tau(0; \Theta_0)}{\partial \Theta^2} \quad (25)$$

$$T = |\Theta - \Theta_0| \quad (26)$$

Assuming the linear material to be, in addition, homogeneous and isotropic, free energy density function further simplifies to:

$$\tau(\varepsilon_{ij}; \Theta) = G \varepsilon_{ij} \varepsilon_{ij} + \frac{\lambda}{2} \varepsilon_{kk} \varepsilon_{nn} - \omega \varepsilon_{kk} T - \frac{C_\varepsilon}{\Theta_0} \frac{T^2}{2} \quad (27)$$

Applying linear heat expansion coefficient α_t :

$$\omega = 3\lambda + 2G\alpha_t \quad (28)$$

and

$$C_\varepsilon = \Theta_0 \rho \left. \frac{\partial s}{\partial \Theta} \right|_{\varepsilon_{ij}} \quad (29)$$

Available equations permit to write the so-called equation of heat conduction. For $h_k = -\kappa h_{,k}$

$$\frac{\Theta_0}{\kappa} \dot{\varepsilon}_{kk} - \frac{C_\varepsilon}{\kappa} \dot{T} - \frac{\rho r}{\kappa} = T_{,kk} \quad (30)$$

Again, general form of the first principal theorem:

$$\dot{W} + \int_V (\rho r - h_{k,k}) dV = \dot{K}[V] + \int_V \dot{u} \rho dV + \frac{d}{dt} \int_{A(t)} \gamma^* dA \quad (31)$$

Substituting (4):

$$\dot{W} = \dot{K} + \int_V T_{ij}^e \dot{\varepsilon}_{ij} dV + \int_V T_{ij}^d \dot{\varepsilon}_{ij} dV + \frac{d}{dt} \int_{A(t)} \gamma^* dA \quad (32)$$

From Eqns (16) and (27)

$$\int_V T_{ij}^e \dot{\varepsilon}_{ij} dV = \dot{A} - \int_V \omega T \dot{\varepsilon}_{kk} dV \quad (33)$$

where:

$$A \equiv \int_V \left\{ G \varepsilon_{ij} \varepsilon_{ij} + \frac{\lambda}{2} (\varepsilon_{kk})^2 \right\} dV \quad (34)$$

is deformation energy for a homogeneous temperature field.

Making use of the heat conduction equation and Green's theorem, second term in (33) may obtain the form:

$$\int_V \omega \dot{\varepsilon}_{kk} T dV = \frac{\kappa}{\Theta_0} \int_A T T_{,k} n_k dA + \frac{1}{\Theta_0} \int_V r T \rho dV - \frac{\kappa}{\Theta_0} \int_V T_{,k} T_{,k} dV - \frac{C_\varepsilon}{\Theta_0} \int_V T \dot{T} dV \quad (35)$$

leading to a general energy equation for a cracked solid:

$$\begin{aligned} \dot{W} = \dot{K} + \dot{A} + \int_V \left(T_{ij}^d \dot{\varepsilon}_{ij} + \frac{\kappa}{\Theta_0} T_{,k} T_{,k} \right) dV + \\ + \frac{d}{dt} \left(\frac{C_\varepsilon}{2\Theta_0} \int_V T^2 dV \right) - \frac{\kappa}{\Theta_0} \int_A T T_{,k} n_k dA - \frac{1}{\Theta_0} \int_V T r \rho dV \end{aligned} \quad (36)$$

Introducing functions:

$$\dot{D} \equiv \int_V \left(T_{ij}^d \dot{\varepsilon}_{ij} + \frac{\kappa}{\Theta_0} T_{,k} T_{,k} \right) dV \quad (37)$$

the so-called dissipative power;

$$H \equiv \frac{C_\varepsilon}{2\Theta_0} \int_V T^2 dV \text{ thermal power function} \quad (38)$$

transforms the energy function to:

$$\dot{W} + \frac{1}{\Theta_0} \int_V T r \rho dV + \frac{\kappa}{\Theta_0} \int_A T T_{,k} n_k dA = \dot{K} + \dot{A} + \dot{D} + \dot{H} + \dot{I} \quad (39)$$

Power of outer forces, as well as heat power input from volume and surface are seen to cover the change of the energy, of the dissipation power, of the inherent

heat power of elastic deformations, as well as the energy needed for producing new surfaces in the solid. In compliance with (13), it is seen to depend on the inherent surface energy and on the crack propagation rate as a rule.

The inherent surface energy may be decomposed to free energy density $\tau^*(\varepsilon_{ij}, \Theta_0)$ and sum Θ_S^* . The free surface energy may be produced as linear combination of surface or stable bond energy τ_0^* of an undeformed solid, and a surface deformation energy $A^*(\varepsilon_{ij})$. Hence, effective surface energy γ^* :

$$\gamma^* = \rho^* \left\{ \tau_0^*(0, \Theta_0) + A^*(\varepsilon_{ij}) + \Theta_S^* + \frac{1}{2} \dot{u}_n \dot{u}_k \right\} \quad (40)$$

Simplified equations

Under circumstances of fracture mechanics tests, energy change due to thermal effects, and change of the kinetic energy have been found to be negligible compared to other terms, at a significant simplification of general equation (36).

$$\dot{W} = \dot{A}^e + \dot{D} + \dot{\Gamma} \quad (41)$$

where

$$A^e = \frac{1}{2} T_{ij} \varepsilon_{ij}$$

is energy of elastic deformations.

Neglect of thermal effects makes the dissipation power a purely mechanical phenomenon, possibly equal to the power of plastic deformations.

$$\dot{D} = \int T'_{ij} \dot{\varepsilon}_{ij}^p \equiv \dot{A}^p \quad (42)$$

where $T'_{ij} = T_{ij} - \frac{1}{3} T_{kk} \delta_{ij}$ stress deviator tensor. These expressions make the energy equation for an infinitesimal change:

$$dW = dA^e + dA^p + d\Gamma \quad (43)$$

For a change between two states of a solid differing only by an increased crack surface, the changes may be reduced to it, that is:

$$d[\dots] = \frac{\partial[\dots]}{\partial A_{\underline{r}}} dA_{\underline{r}}; \quad dA_{\underline{r}} \geq 0 \quad (44)$$

In the following, notation $dA_{\underline{r}} \equiv da$ generalized in the literature will be applied.

$$\frac{\partial W}{\partial a} = \left[\frac{\partial A^e}{\partial a} + \frac{\partial A^p}{\partial a} \right] = \frac{\partial \Gamma}{\partial a} \quad (45)$$

Analysis of an elastic solid

For an elastic solid $A^p = 0$. The energy balance is:

$$\frac{\partial W}{\partial a} - \frac{\partial A^e}{\partial a} = \frac{\partial \Gamma}{\partial a} \equiv \gamma \quad (46)$$

Introducing the term of total potential energy,

$$P = A^e - W = \frac{1}{2} \int_V T_{ij} \varepsilon_{ij} dV - \int_A p_k u_k dA \quad (47)$$

(46) becomes:

$$\gamma = \frac{\partial \Gamma}{\partial a} = - \frac{\partial P}{\partial a} \quad (48)$$

That is, energy resulting from the reduction of potential energy covers that needed for producing new surfaces.

From the aspect of crack propagation, two typical boundary conditions are possible. One is the case of the so-called "fixed grip" where boundary displacement is assumed to be zero for a small value of crack propagation. Then:

$$- \frac{\partial P}{\partial a} = - \frac{\partial A^e}{\partial a} \Big|_{\tilde{u}_k=0} = \frac{\partial \Gamma}{\partial a} = G \quad (49)$$

G is called the strain energy release rate, namely the energy needed for crack propagation has to be taken from the energy field of elastic deformations.

The second case is that of so-called "dead load" where the applied load is constant for a small value of crack propagation.

Now,

$$- \frac{\partial P}{\partial a} = \frac{\partial A^e}{\partial a} \Big|_{p_k = \text{const.}} = \frac{\partial \Gamma}{\partial a} \quad (50)$$

Defining $\frac{\partial A^e}{\partial a}$ as the strain energy release rate may be physically misleading.

Absolute values of term $\frac{\partial A^e}{\partial a}$ have been found to be equal in either case, (explaining) why the term strain energy release rate is applied in either case.

Equations (49) and (50) are equivalent to Griffith's criterion on brittle crack propagation, stating the crack propagation to start at $\frac{\partial}{\partial a} [P + \Gamma] = 0$.

Equation (46) also underlies the Sih theory of strain energy density.

Path-independent line integrals

Let us consider a plate of unit thickness, with a crack of size a at time $t = t_0$ (Fig. 2). At any internal point of the solid, equations

$$T_{ij,j} + \rho f_i = \rho v_i \quad (51)$$

$$T_{ij} = T_{ji} \quad (52)$$

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad (53)$$

$$r = h_k = 0 \quad (54)$$

are assumed to hold.

The continuum material is said to be elastic if there is an elastic potential $A^e(\varepsilon_{ij}, x_k)$ yielding the stress field as:

$$T_{\alpha\beta} = \frac{\partial A^e}{\partial \varepsilon_{\alpha\beta}} \quad (55)$$

Deriving A^e with respect to x_γ :

$$\frac{\partial A^e}{\partial x_\gamma} = \frac{\partial A^e}{\partial \varepsilon_{\alpha\beta}} \varepsilon_{\alpha\beta,\gamma} \quad (56)$$

Let \otimes denote some operation symbol (algebraic multiplication, scalar multiplication, vector multiplication, tensor multiplication) and $C_{\delta\dots q}$ some tensor of order q at any internal point:

$$A^e_{,\gamma} \otimes C_{\delta\dots q} = \frac{\partial A^e}{\partial \varepsilon_{\alpha\beta}} \varepsilon_{\alpha\beta,\gamma} \otimes C_{\delta\dots q} = T_{\alpha\beta} \varepsilon_{\alpha\beta,\gamma} \otimes C_{\delta\dots q} \quad (57)$$

Arranging with regard to (51) and (53):

$$(A^e \otimes C_{\delta\dots q})_{,\beta} - (A^e \otimes C_{\delta\dots q,\beta}) \delta_{\beta\gamma} = (T_{\alpha\beta} u_{\alpha,\gamma} \otimes C_{\delta\dots q})_{,\beta} - T_{\alpha\beta} u_{\alpha,\gamma} \otimes C_{\delta\dots q,\beta} \quad (58)$$

Integrating it over a volume V^* of the solid containing only internal points, and applying the Gauss-Ostrogradsky theorem:

$$\int_{\hat{r}} (A^e \delta_{\beta\gamma} \otimes C_{\delta\dots q} - T_{\alpha\beta} u_{\alpha,\gamma} \otimes C_{\delta\dots q}) n_\beta ds - \int_{\hat{A}} (A^e \otimes C_{\delta\dots q,\gamma} + T_{\alpha\beta} u_{\alpha,\gamma} \otimes C_{\delta\dots q,\beta}) dA = 0 \quad (59)$$

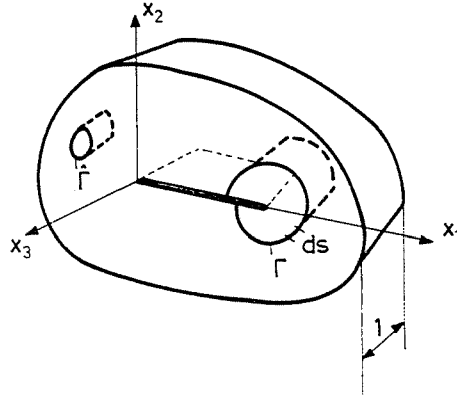


Fig. 2

Integrating along a curve starting at one, and ending at the other, surface of the crack will generally yield a non-zero integral. Such integrals are fundamental in fracture mechanics, and their various forms are obtained by aptly selecting \otimes and $C_{\delta\dots q}$.

J-Integral

Be \otimes an algebraic multiplication and $C_{\delta\dots q} \equiv 1$. Then from (59):

$$J_\gamma = \int_\Gamma (A^e \delta_{\beta\gamma} - T_{\alpha\beta} u_{\alpha,\gamma}) n_\beta ds \tag{60}$$

Taking $\gamma = 1$ leads to the integral according to Rice:

$$J = \int_\Gamma A^e dx_2 - \int T_{\alpha\beta} n_\beta u_{\alpha,1} ds \tag{61}$$

M-Integral

Let \otimes indicate vector multiplication and $C_{\delta\dots q} \equiv x_\gamma$. Then from (59), since $x_{\alpha,\beta} = \delta_{\alpha\beta}$

$$M = \int_\Gamma (A^e x_\gamma n_\beta + p_\beta u_\gamma - p_\rho u_{\rho,\beta} x_\gamma) e_{\gamma\beta} ds \tag{62}$$

where

$$p_\beta = T_{\beta\alpha} n_\alpha$$

Assuming again the solid to be linear elastic yields:

$$A^e = \frac{1}{2} T_{\alpha\beta} \epsilon_{\alpha\beta} \tag{63}$$

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