BOOK REVIEW

H. Brezis and J. L. Lions

"Nonlinear partial differential equations and their applications"

Collége de France Seminar

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The volume consists of written versions of the 21 lectures held during the year 1979–1980 at the weekly seminars on Applied Mathematics at the College de France. They mostly deal with various aspects of the theory of non-linear partial differential equations.

10 papers are written in English and the other ones in French. The papers in French have abstracts in English.

The papers quote detailed references.

S. Alinhac discusses strong unique continuation from a point for higher order operators with C^∞ (complex) coefficients. An improvement of the classical results are presented. These results are obtained using Carleman-type inequalities for evolution in Hilbert space. These inequalities are of interest in themselves, and are discussed independently from their use in the uniqueness problem. Some counter-examples (for example for elliptic operators) show the optimality of the conditions obtained.

In 1948, Hille and Yosida applied semi-group for Cauchy problems in linear case. *J. B. Baillon* studies a version of the theorem of Hille and Yosida for non-linear continuous semi-group. The semi-group is defined on Babach spaces.

Various mathematical problems associated with the control of nonlinear partial differential equations and in particular with control systems described by variational inequalities have been studied in the past decade. *V. Barbu* illustrates on some typical problems a different method to obtain necessary conditions of optimality. The idea to approximate a non-linear control problem by a family of differentiable control problems is not new and it is frequently used. What is new in this paper, and in fact represents the main difficulty of the method, is the convergence of the approximating optimality system.

The lecture of *C. Bardos, J. C. Guillot* and *J. Ralston* is divided into two parts. The first part is mainly historical. It is here presented a survey on the relations existing between the eigenvalues for the Laplacian in a bounded domain of $\mathbb{R}^n$ or on a compact Riemann manifold without boundary. In the second part it is discussed how analogous methods can be used for the wave equation in the exterior of a compact obstacle.

In the paper of *L. Boccardo* and *F. Murat* we can find new results in the convergence of the solutions of variational inequalities of obstacle type when the obstacle (and simultaneously the operator) varies.

H. Brézis describes results concerning the non-linear Schrödinger equation

$$i\frac{\partial u}{\partial t} - \Delta u + k|u|^2 u = 0 \quad \text{on } \Omega x(0, +\infty).$$

This kind of problem occurs in nonlinear optics (laser beams) and has been extensively studied when $\Omega \subset \mathbb{R}^2$, but nothing seems to have been known for $\Omega \subset \mathbb{R}^n$. In order to solve the problem it will be relied on limiting cases of Sobolev type inequalities in dimension $n \geq 2$.

Let Ω be an open bounded set of $\mathbb{R}^n$ $(n \geq 2)$. For each $\varepsilon > 0$, it is made some holes (obstacles) in Ω, and so is obtained an open set Ω^ε.

D. Cioranescu and *F. Murat* discuss the Dirichlet problem in Ω^ε. The typical case they consider is the following: they cover $\mathbb{R}^n$ with cube of size ε; in the middle of each they make a hole of size $\varepsilon^{n/(n-2)}$ (if $n > 2$) or $\exp\left(-\frac{1}{\varepsilon^2}\right)$ (if $n = 2$). Let u^ε be the solution of the Dirichlet problem in Ω^ε. They show that u^ε goes weakly to u where u is the solution of the corresponding Dirichlet problem in Ω.

A. Douglis (1979) has used an approximate layering method for multidimensional nonlinear parabolic systems of a certain type. The layering method which is described in this book by *A. Douglis* provides an approach to fluid flow in which convective effects — the larger scale motions and the microscopic — are treated separately and by different way. The layering method is expected to be helpful in constructing, improving, and justifying calculational procedures. Here the layering method is adapted to the Navier–Stokes equations of incompressible flow. The work reported on is confined to the pure initial-value problem, in which the fluid occupies a full space $\mathbb{R}^n$ $(n \geq 2)$ and its velocities at an initial instant are prescribed. An extension of the layering method to rather general boundary-value problems for the Navier–Stokes equations is at an early stage.

J. Ginibre and *G. Velo* review some recent results on the nonlinear Schrödinger equation with non local interaction (also known as the Hattree equation). They concentrate on solving the following problems: the existence and uniqueness of global solutions of the Cauchy problem, and asymptotic behaviour of the solutions.

In his paper *Gu Chao-Hao* gives a brief review of the results on the boundary value problems for mixed partial differential equations, i.e. for partial differential equations which are elliptic in some domain and are hyperbolic or parabolic in another one.

J. Heyvaerts, J. M. Lasry, M. Schatzman and *P. Witomski* model the phenomenon of solar flares by an elliptic semilinear equation. They give the physical criterion of stability and the instability; the later models the flare. Numerical computations are presented.

R. Jensen presents a method generalizing the interior $W^{2,\infty}$ estimates of Brezis and Kinderlehrer for the linear obstacle problem, i.e. he states a theorem of global $W^{2,\infty}$ regularity for variational inequalities.

Consider the equation $F(x) = 0$ in a Banach space. A solution $x = x^0$ of this problem can be isolated or nonisolated. Such situations occur very frequently in applications, for example at limit point and bifurcation computations. *H. B. Keller* gives the definition of geometrically isolated solutions. If a solution is isolated it is also geometrically isolated. However, a nonisolated solution can also be geometrically isolated. The author derives sufficient conditions for this. Only geometrically isolated solutions can reasonably be approximated by the usual computational techniques. H. B. Keller considers the cases of scalar equations and of equations in Banach spaces. The problem of the second case is reduced to the one dimensional scalar case.

In their talk, *R. J. Knops* and *B. Straughan* study two forced systems. The first is a linear elastic body subject to a sublinear body force, while the second is a rigid heat conductor occupying the whole space and subject to a sublinear heat sink. In these semilinear problems of the continuum mechanics, the paper studies the case where the solution is controlled to zero in a finite time or ceases to exist within this time.

L. Ta-Tsien gives conditions which allows to solve locally quasilinear hyperbolic system of order 1. Applications are given to supersonic flow around a curved profile, to the piston problem, and to the Cauchy problem for a system of conservation laws with discontinuous initial data.

P. L. Lions investigates a minimization problem in L^1 arising in astrophysics in the determination of equilibrium configurations of axisymmetric rotating fluids (for instance stars) and in quantum mechanics (in Thomas–Fermi theory).

T. P. Liu and *J. A. Smoller* consider the vacuum problem for the system of isentropic gas dinamic.

Let $N(\lambda)$ be the number of eigenvalues of an elliptic operator smaller then λ. *R. B. Melrose* investigates the asymptotic behaviour of $N(\lambda)$ when the set of closed geodesics is empty.

D. Sattinger studies the bifurcation equation $G(\lambda, u) = 0$ where G is a smooth mapping from $\mathbb{R} \times \varepsilon$ to F, ε and F are Banach spaces. Assuming that G is differentiable in the sense of Frechet and the derivative $Gu(\lambda, u)$ is an operator of Fredholm-type, the bifurcation equation can be reduced to a finite dimensional problem; if the mapping $G(\lambda, \cdot)$ is covariant with a group representation T_g, we obtain more information on the equation. This idea is applied to the case when the group is SO (3) (i.e. is invariant to rotations), with explicit computations for the representations of lower dimensions.

S. Ukai and *K. Asano* investigates the existence and stability of stationary solutions of the Boltzmann equation for a gas flow past an obstacle. So far the problem of the gas flow past an object has been investigated exclusively within the fluid mechanics based on the compressible Euler equation with the assumption of irrotation and isentropy. However the Boltzmann equation in the kinetic theory of gases gives a better description of the gas motion.

Consider the discretisation

$$y_{n+1} - y_n = \Delta t f(y_n)$$

$$t_n = t_0 + nt, \qquad n = 0, 1, \ldots, N,$$

$$N = \frac{T - t_0}{\Delta t}$$

of the differential equation

$$\frac{dy}{dt} = f(y); \quad t_0 \leq t < T$$

$$y(t_0) = y_0.$$

The aim of the paper of *M. Yamaguti* is to provide an exploration of chaotic phenomena that occur in discrete dynamic problems.

V. Kertész