SOME FEATURES OF ELEMENTS GENERATING AN ALTERNATING GROUP

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The structure of elements generating a so-called alternating group, and the element relations will be considered from group construction aspects. Three problems will be involved:

1. It will be demonstrated that elements of form $(ij)(kl)$ generating the n-degree alternating group A_n are contained exactly $2 \cdot \left[\binom{n-4}{2} + 3 \cdot \binom{n-4}{4} + 1\right]$ times in the main diagonal of the Cayley table where $i \neq j \neq k \neq l$ and $i, j, k, l = 1, 2, 3, \ldots, n$.

2. It will be proven that the number of elements under 1 is $3\binom{n}{4}$.

3. Finally, it will be verified — by introducing the concept of the expanding set — that the n-degree alternating group A_n may also be produced by products of elements in the n-degree symmetric group S_n with exactly $(n-4)$ fixed elements.

One generator system of the n-degree alternating group A_n is the set of all elements of form $(ij)(kl)$ where $i \neq j \neq k \neq l$ and $i, j, k, l = 1, 2, 3, \ldots, n$ (see p. 243 in [1]). These elements are very important for the examination of the subgroups of the alternating group, and so are triple cycles, also generating the alternating group. This is why a detailed analysis of their properties is advisable.

Examinations below will affect generating elements of form $(ij)(kl)$, to see products of what elements of n-degree symmetric group S_n they come from; number of these elements in the main diagonal of the Cayley table of the n-degree symmetric group S_n — hence of the n-degree alternating group A_n; — how many generating elements exist; finally, what subset of the n-degree symmetric group S_n of an odd inverted number of elements the n-degree alternating group A_n results from.

The general formula of the number of alien cycles in the n-degree symmetric group S_n is known to be:

$$\frac{1}{k}\left[n(n-1)\ldots(n-k+1)\right],$$

where n is the degree of S_n, and k the cycle length (see p. 40 in [3]).

For instance, elements (1 2 3 4) of S_5 number 30, namely $n = 5$ and $k = 4$, hence:

$$30 = \frac{5 \cdot 4 \cdot 3 \cdot 2}{4}.$$

Or, elements (1 2 3) (4 5) number 20, namely:

$$20 = \frac{5 \cdot 4 \cdot 3}{3} \cdot \frac{2 \cdot 1}{2}.$$

This formula also lends itself to determine the number of cycles generating the n-degree alternating group A_n. Here the numbers of cycles of form $(ij)(kl)$ and (ijk) will be determined by formulae $3\binom{n}{4}$ and $2\binom{n}{3}$, respectively. Demonstration will refer to the $(ij)(kl)$ cycles, since these will be treated alone. In the following, elements of form $(ij)(kl)$ will be called the preferential, and those (ijk) the principal elements. Their number of occurrence in the main diagonal of the Cayley table makes them preferential.

I

The identical, so-called preferential elements will be demonstrated to occur exactly $2\left[\binom{n-4}{2}+3\binom{n-4}{4}+1\right]$ times in the main diagonal of the Cayley table of the n-degree alternating groups A_n, where $n \geqq 4$.

1.1 Definition

Preferential elements are generating elements of the form $(ij)(kl)$ of the n-degree alternating group A_n, where $i \neq j \neq k \neq l$ and $i, j, k, l = 1, 2, \ldots n$, their set will be denoted by K.

1.2 Theorem

Identical elements K occur exactly $2\left[\binom{n-4}{2}+3\binom{n-4}{4}+1\right]$ times in the main diagonal of the Cayley table of the n-degree symmetric group S_n — hence of the n-degree alternating group A_n, where $n \geqq 4$.

This theorem will be demonstrated by means of two lemmas. The first lemma will point to these elements in n-degree symmetric group S_n the product of which yields elements K, and the second to these elements the squares of which generate K.

Before stating the first lemma, two subsets of the n-degree symmetric group S_n will be needed.

Selecting out of S_n elements the cycles with at least $(n-4)$ fixed elements and denoting their set by H, those elements of H will be taken the products of which are squares of cycles consisting of four different elements. Denote their set by M.

Cyclic form of elements H:

$$\alpha = (i\,j\,k\,l)$$
$$\beta = (a\,b\,c)$$
$$\gamma = (r\,s).$$

The multiplication table of S_n of elements each containing at least $(n-4)$ fixed elements is the same as the product of elements H. According to the rule of permutation, these will include product pairs the factors of which have four different elements in all.

Then obviously,

$$\{\alpha_p\} = \{\alpha_q\}$$

for any p and q.

Furthermore:

$$\{\gamma_q\} \subset \{\alpha_p\}$$

so that only non-adjacent elements of $\{\alpha_p\}$ are identical to $\{\gamma_q\}$.

For instance, if:

$$\alpha = (i\,j\,k\,l) \quad \text{then} \quad \gamma_1 = (i\,k) \quad \text{and} \quad \gamma_2 = (j\,l)$$

and finally:

$$\{\beta_p\} \cap \{\beta_q\} = \begin{cases} \{i\,j\}, \quad \text{or} \quad \{j\,k\}. \\ \{i\,k\} \end{cases}$$

(Here in the first case ϱ the element in the odd place is different, and in the second case ϱ the middle one — in the cycle.)

For instance:

$$(i\,j\,k)\,(a\,j\,k)\,; \quad \text{and} \quad (i\,j\,k)\,(i\,b\,k)\,;$$

Thus, elements in M are:

$$\alpha^2 = (i\,j\,k\,l)^2 = (i\,k)\,(j\,l)$$

$$\alpha \cdot \gamma = (i\,j\,k\,l)(i\,k) = (i\,j)(k\,l) = (i\,k\,j\,l)^2$$

$$\beta_p \cdot \beta_q = (i\,j\,k)(a\,j\,k) = (i\,k)(j\,a) = (i\,j\,k\,a)^2 \qquad (p \neq q)$$

$$\beta_p \cdot \beta_q = (i\,j\,k)(i\,b\,k) = (i\,j)(k\,b) = (i\,k\,j\,b)^2 \qquad (p \neq q).$$

1.2.1. *Lemma:*

A generator system of K is the entity of elements M.

Proof:

The well-known relationship states that if δ and ϱ are elements of S_n, then the cyclic structure of the result of $\delta\varrho\delta^{-1}$ equals that of ϱ (see p. 39 in [3]). Hence:

$$S_n \cdot M \cdot S_n^{-1} = K$$

that is, for

$$\delta \in S_n \quad \text{and} \quad \varrho \in M$$

the structure of $\delta\varrho\delta^{-1}$ equals that of ϱ where ϱ means a cycle square $(i\;j\;k\;l)^2$ for any elements in S_n. Namely, be

a) $\delta_1 = \ldots (i\,j\,k\,l) \ldots$

$\varrho = (i\,k)(j\,l)$,

then

$\ldots (i\,j\,k\,l) \ldots [(i\,k)(j\,l)] \ldots (l\,k\,j\,i) \ldots = (i\,k)(j\,l)$

b) $\delta_2 = \ldots (i\,j\,k) \ldots$

$\ldots (i\,j\,k) \ldots [(i\,k)(j\,l)] \ldots (k\,j\,i) \ldots = (i\,l)(j\,k)$

c) $\delta_3 = \ldots (j\,i) \ldots$

$\ldots (i\,j) \ldots [(i\,k)(j\,l)] \ldots (i\,j) \ldots = (i\,l)(j\,k)$

d) $\delta_4 = \ldots (i) \ldots$

$\ldots (i\,p) \ldots [(i\,k)(j\,l)] \ldots (p\,i) \ldots = (p\,k)(j\,l).$

Hence, in fact, $M = K$.

1.2.2. *Lemma:*

The generator system of K is at the same time square of any element of the n-degree symmetric group S_n with exactly $(n-4)$ fixed elements.

Proof:

These elements are of type α^2, where $\alpha^2 \in M$, for $M \subset S_n$. The main diagonal of the Cayley table contains only squares α^2, has really the desired property. Namely:

$$\alpha^2 = (i\,k\,j\,l)^2 = (i\,j)\,(k\,l) \in K\,.$$

It can be proven that any element in S_n other than α the square of which is again an element of K, is of type δ, where δ is the product of alien cycles of form $(i\,j\,k\,l)(a_1a_2)(a_3a_4)\ldots(a_{n-5}a_{n-4})$ and cycle products have no common part. Namely here absolutely:

$$\delta^2 \in K$$

and so it is since

$$\delta \equiv \alpha \quad \text{and} \quad \delta \equiv \alpha^{-1}$$

squares of all elements other than α, α^{-1}, consisting of products of transpositions other than $(i\,j)(k\,l)$, where $i \neq j \neq k \neq l$.
Thus:

$$\text{either} \quad \delta \cdot \alpha = \alpha^2$$

$$\text{or} \quad \delta \cdot \alpha = e\,,$$

e being the unit element of the group.

Now, the theorem is simple to demonstrate.

Proof:

α and δ shaped cycles are contained exactly once in rows and columns of the Cayley table, and so are their squares in its main diagonal but:

$$\alpha^2 = \delta^2,$$

hence identical elements K are contained exactly $2\left[\binom{n-4}{2}+3\binom{n-4}{4}+1\right]$ times in the main diagonal of the Cayley table, where $n \geqq 4$.

II

Now, the formula $3\binom{n}{4}$ furnishing the number of preferential elements generating the n-degree alternating group A_n will be demonstrated combinatorically and geometrically.

2.1 *Theorem*

The generator system of preferential elements of the n-degree alternating group A_n numbers

$$3\binom{n}{4},$$

where $n \geqq 4$.

The demonstration will involve an auxiliary lemma.

2.1.1. *Lemma:*

The main diagonal of the Cayley table of the n-degree symmetric group S_n – hence, of the n-degree alternating group A_n – contains all preferential elements.

Proof:

This statement is a direct conclusion of Lemma 1.2.1. namely:

$$N \cdot N \cong \{(i\,j\,k\,l)\}^2 \in M,$$

where:

$$(i\,j\,k\,l) \in N,$$

where $M = K$ has been demonstrated.

A. Combinatorical proof of the theorem

According to Lemma 1.2.2.

$$\alpha^2 \in K,$$

on the other hand, according to Lemma 2.1.1., all elements K are contained in the main diagonal of the Cayley table of the n-degree alternating group A_n.

Thus, 4 out of n elements have to be selected in all ways possible, hence in $\binom{n}{4}$ ways. To obtain the desired configuration, the selected elements have to be paired, possible in $\binom{4}{2}$ ways. To the selection of each $\binom{n}{4}\binom{4}{2}$ configurations belong, making up the number of all selections to:

$$\binom{4}{2} \cdot \binom{n}{4}.$$

Now, according to $\alpha \cdot \delta = 1$ each form $(i\,j)(k\,l)$ is contained twice (see Lemma 1.2.2.) hence K is of a number:

$$3 \cdot \binom{n}{4},$$

namely:

$$\frac{1}{2}\binom{4}{2}\binom{n}{4} = 3\binom{n}{4}.$$

B. Geometrical proof of the theorem

2.2 *Definition*

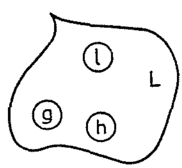
Be parallelness defined as follows:

Be L the set of straight lines in the plane, i.e., L (where Π is a plane set and $\Pi = \{1, 2, \ldots, n\}$). Now, if $l \in L$ and $P \notin l$ there exist one, and only one straight line $g (g \in L)$ such as:

$$(P \in g) \wedge (l \,||\, g).$$

A parallel relation is an equivalence relation, hence for

$$\left.\begin{matrix} l \cap g = \Phi \\ g \cap h = \Phi \end{matrix}\right\} \Longrightarrow l \cap h = \Phi$$



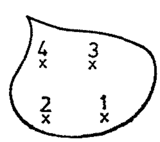
This relation permits to classify elements into so-called parallel classes.

2.B.1. For instance, if:

$$\Pi = \{1, 2, 3, 4\},$$

then

$$L = \{\{1, 2\}; \{1, 3\}; \{1, 4\}; \{2, 3]; \{2, 4\}; \{3, 4\}\}$$



In this case L has three classes with no common part each pair, i.e.:

$$\{1,2\} \parallel \{3,4\}$$
$$\{1,3\} \parallel \{2,4\}$$
$$\{1,4\} \parallel \{2,3\}.$$

Obviously, there is an imaging φ helping to image isomorphically the set of parallel straight lines on a subgroup of the n-degree alternating group A_n:

$$\{1,2\} \cdot \{3,4\} \xrightarrow{\varphi} (12)\ (34)$$
$$\{1,3\} \cdot \{2,4\} \xrightarrow{\varphi} (13)\ (24)$$
$$\{1,4\} \cdot \{2,3\} \xrightarrow{\varphi} (14)(23).$$

Thereafter this relationship will be generalized for any n, where $n \geqq 4$.

Take an arbitrary set of points, of them n are fixed, and classify the parallel straight lines meeting the relation in the obtained plane set (set of planes)

$$\Pi = \{1, 2, 3, \ldots, n\}.$$

Let us fix two of the n points as in example 2.B.2. and select 2 points of the remaining $(n-2)$ points in any possible hence $\binom{n-2}{2}$ different ways. The obtained expression has still to be multiplied by $(n-1)$, and performing all further selection yields the number of classes:

$$\binom{n-2}{2}(n-1) + \binom{n-3}{2}(n-2) + \ldots + \binom{3}{2} \cdot 4 + \binom{2}{2} \cdot 3 =$$

$$= \sum_{k=4}^{n} \binom{k-2}{2} k - 1).$$

Full induction demonstrates that:

$$\sum_{k=4}^{n} \binom{k-2}{2}(k-1) = 3\binom{n}{4}.$$

For $n = 4$, this statement is valid, namely:

$$\binom{2}{2} \cdot 3 = 3,$$

complying with example 2.B.2.

Assume the statement is valid for k:

$$\binom{k-2}{2}(k-1) + \binom{k-3}{2}(k-2) + \ldots + \binom{3}{2} \cdot 4 + \binom{2}{2} \cdot 3 = 3\binom{k}{4}.$$

It will be proven that the statement also holds for $k+1$:

$$3\binom{k}{4} + \binom{k-1}{2}k = 3\binom{k}{4} + \binom{k-1}{2}\binom{k}{1} = 3\binom{k}{4} + \frac{(k-1)!}{2!\,(k-3)!} \cdot \frac{k!}{(k-1)!} =$$

$$= 3\binom{k}{4} + \frac{k!}{2!\,(k-3)!} = 3\binom{k}{4} + 3\frac{k!}{3!\,(k-3)!} = 3\left[\binom{k}{4} + \binom{k}{3}\right] = 3 \cdot \binom{k+1}{4}$$

making use of $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$.

It follows directly that in case of any n, the set of parallel straight line classes corresponds to, hence is isomorphic with a subgroup of the group A_n. Thus, there is a $K \subset A_n$ with elements in K numbering:

$$3\binom{n}{4}$$

III

It will be demonstrated that

$$N \cdot N \cong A_n$$

where

$$N \subset H \subset S_n$$

and elements of N are of type α.

Prior to stating the theorem, let us introduce definitions:

3.1 *Definition*

A set H is termed an expanding one if its product is a group G, so that $H \cap G = \varnothing$.

3.2 *Definition*

Sets are termed semi-expanding ones if their product is a group, and their elements are not absolutely elements of that group $\geqq$.

3.3 *Definition*

A cycle is termed odd (even) if it can be decomposed into products of transpositions of odd (even) number.

3.1.1 *Theorem*

The set of cycles containing $(n-4)$ fixed elements in the n-degree symmetric group S_n is an expanding one.

Proof:

Cycles containing $(n-4)$ fixed elements are odd in number, their product is even.

The resulting group will be seen to be exactly the n-degree alternating group A_n.

Take odd cycles with exactly $(n-4)$ fixed elements of the n-degree symmetric group S_n. Their product is of type α^2, where:

$$\alpha = (i\,j\,k\,l)$$

and set of their elements is N, i.e.,

$$\alpha \in N .$$

It is stated that any

$$\alpha_p \cdot \alpha_q \in A_n \quad (p \neq q) .$$

A trivial statement is $\alpha_p \cdot \alpha_q \in A_n$, namely both α_p and α_q are odd, thus their product is even, thereby A_n fully contains $\alpha_p \cdot \alpha_q$. (Namely A_n contains all even cycles of S_n.) Validity of statement $N \cdot N = A_n$, needs still to prove that the outcome of any product $\alpha_p \cdot \alpha_q$ is a triple cycle or product of triple cycles. Now, it is sufficient to demonstrate that $N \cdot N$ contains all elements of form $(i\,j)\,(k\,l)$ generating A_n, already done in Lemma 2.1.1. hence, in fact,

$$N \cdot N = A_n .$$

Products $\alpha_p \cdot \alpha_q$ yield in fact a triple cycle or a product of triple cycles if α_p and α_q are cycles with four elements, all of them being different.

Namely, then the following opportunities arise:

1. $\alpha_p \cap \alpha_q = \varnothing$

Be now:

$$\alpha_p = (i\,j\,k\,l)\,; \quad \alpha_q = (a\,b\,c\,d)$$

$$\alpha_p \cdot \alpha_q = (i\,j\,k\,l) \cdot (a\,b\,c\,d) = (i\,j\,k)\,(i\,l)\,(a\,b\,c)\,(a\,d) =$$

$$= (i\,j\,k)\,(a\,b\,c)\,(i\,l)\,(a\,d) = (i\,j\,k)\,(a\,b\,c)\,(i\,l\,d)\,(i\,a\,d)$$

2. $\alpha_p \cap \alpha_q = \{j\}$

$$\alpha_p \cdot \alpha_q = (i\,j\,k\,l)\,(a\,b\,j\,d) = (i\,d\,a\,b\,j\,k\,l) = (i\,d\,a)\,(i\,b\,j)\,(i\,k\,l)$$

3. $\alpha_p \cap \alpha_q = \{i\,l\}$

$$\alpha_p \cdot \alpha_q = (i\,j\,k\,l)\,(a\,i\,c\,l) = (i\,j\,k\,a)\,(l\,c) = (i\,j)\,(i\,k\,a)\,(l\,c) =$$

$$= (i\,j)\,(l\,c)\,(i\,k\,a) = (i\,j\,c)\,(i\,l\,c)\,(i\,k\,a)$$

4. $\alpha_p \cap \alpha_q = \{j\,k\,l\}$

$$\alpha_p \cdot \alpha_q = (i\,j\,k\,l)\,(a\,j\,k\,l) = (i\,k\,a\,j\,l) = (i\,k\,a)\,(i\,j\,l)$$

5. $\alpha_p \cap \alpha_q = \{i\,j\,k\,l\}$

$$\alpha_p \cdot \alpha_q = (i\,j\,k\,l)^2 = (i\,k)\,(j\,l) = (i\,j\,k)\,(j\,k\,l)$$

Summary

Three problems of elements of form $(ij)\,(kl)$ generating the n-degree alternating group A_n will be examined.

First, the number of elements of form $(ij)\,(kl)$ in the main diagonal of the Cayley table of the n-degree alternating group A_n is examined.

The second one demonstrates geometrically and combinatorically the well-known statement that the number of elements in group A_n is $3\binom{n}{4}$.

Third, it will be considered, by introducing the concept of expanding set, the product of what elements of the n-degree symmetric group S_n composes the n-degree alternating group A_n.

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