THE ROLE OF MATHEMATICS IN TEACHING FLUID MECHANICS FOR MECHANICAL ENGINEERS

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Introduction

The fast increase of the subject matter in Fluid Mechanics on one hand, and the limited, in some cases even decreasing time available for teaching it, calls for the reconsideration of its teaching methods. Teaching methods are certainly connected with the topics discussed, the background knowledge of students, pedagogical and educational aspects, hence without a clear and detailed check list in those fields the selection of the appropriate teaching method might become a difficult job. This paper intends to list the different aspects and tries to outline a suitable teaching method somewhat different from the prevailing one.

Objectives

The subject "Fluid Mechanics for Mechanical Engineers" should deal with the fundamentals (continuity, momentum and energy equation), their application in various fields (flow in closed conduits, turbomachinery, environmental engineering etc.) including non-Newtonian and compressible fluids as well. This part may be called the physics-side of the subject. Its fast development (turbulent boundary layer theory, fluidics, multiphase flow etc.) in quantity alone calls for the reconsideration of the teaching methods.

Fluid Mechanics — over and above its "physics" side offers an excellent opportunity to use mathematics, sometimes even to clear the students' only formal understanding of the higher mathematical apparatus. In fact the mathematical side of the subject has a big pedagogical value not only because of the frequent use of mathematics, but because of the construction of mathematical models, also an engineer's task.

Correctly balancing the physics and mathematics side is the third, educational aim of this subject. By pushing it to extremes one may end up in a course of descriptive presentation, rules of thumb and table or graph readings fit for conventional routine jobs only. It is then far from what one might expect from a graduate engineering course. On the other hand, little application and a very big mathematical apparatus may feature a kind of theoretical physics which should not be the goal when training mechanical engineers. It is indeed an educational aim to demonstrate when and how deep an engineer is bound to dive into the problem, where he should use exact mathematical methods and where approximations. It should be an educational goal as well to use the proper numerical apparatus, a pocket or desk calculator if a calculator is justified, or a thoroughly checked computer programme if the problem requires it. To this end Fluid Mechanics offers ample opportunities and one has to utilise them consciously.

The use of mathematics

Concerning the information part of the training the use of the proper mathematical apparatus is advantageous because it offers a more global viewpoint, shortens the time required to present the matter and avoids advanced, not always definite physical explanations. Opposing this concept one may say — in some cases justly — that the mathematical apparatus diverts the students' attention from the physical reality and has adverse educational results especially when forming an engineer's way of thinking. Experience shows that the latter happens only if the use of the mathematical apparatus is not looked upon as a link between the physical model and the physically interpreted result, but serves to exhibit the beauty of the mathematical apparatus only. Numerical methods — nowadays so important — may lead to serious errors without the control of rigorous analytical check; hence it does not seem possible to reduce drastically the classical higher mathematics syllabus. Fluid Mechanics faces similar problems and can offer good examples especially in two-dimensional flow. The method of finite differences (relaxation method), apart from calling for big computer capacity, may require boundary conditions which are difficult to define, hence in some cases the traditional conformal mapping or singularity method leads to a faster and easier interpretable result from the educational point of view. Dropping classical chapters because of new methods seems to be unwise in this field too.

The presentation of fundamentals

Following the pedagogical principle, to start with simple matters and turn later to the complicated ones, Fluid Mechanics traditionally starts with hydrostatics.

$$\operatorname{grad} U + \frac{1}{\varrho} \operatorname{grad} p = 0$$

or

$$p = -\varrho U + \text{const.}$$

are the usually desired results picturing the connection between pressure p, conservative external force field potential U and density ϱ.

Although lectures start in a very simple way and students may enjoy immediate pratical applications, it has been experienced that they store the expression as a singular theorem and will hardly consider it as a special case of Bernoulli's-equation, not even after their attention has been drawn to the fact. It is a troublesom observation from the pedagogical point of view because in this case the particular does not emerge from the general but intends to live an independent life belying so the fundamental concept of university teaching — the drive towards the general rather than towards the particular. The correction, namely to quit tradition and discuss hydrostatics as a particular case, will result in considerable time saving as well.

By fundamental equations only those of continuity, of motion (Euler, Bernoulli's, Navier-Stokes equation) of momentum and energy should be understood. In each case an arbitrary closed moving surface is being chosen in which mass (continuity) momentum equation's of motion or energy change in time has to be considered. Although the same mathematical procedure is being performed when developing the left hand side of the equations, usually a lengthy, questionably correct physical explanation of the matter serves as backing. It is time consuming and checks with students show that their understanding is far less than expected. If so, why not to try a more general rigorous mathematical procedure with the promise of physical interpretation at the start and at the end.

The general mathematical problem is materialised in the expression:

$$\frac{d}{dt}\int\limits_{V(t)} f(\mathbf{x}, t)\, dV = ?$$

where $f(\mathbf{x}, t)$ may be a continuously differentiable scalar (e.g. density ϱ) or vector (e.g. momentum $\varrho\mathbf{c}$) and $V(t)$ is a measurable, simple connected fluid volume moving with the fluid. t stands for time. It will be useful to know that the closed surface over V is denoted by A.

Since the controlled volume moves with the fluid it will be advantageous to transform the problem to a steady volume $V_0(\xi)$ by the so called Jacobi's determinant — discussed usually in mathematics.

$$J = \det\left(\frac{\partial x_i}{\partial \xi_j}\right) \neq 0 \qquad (i, j = 1, 2, 3)$$

with which

$$\frac{d}{dt}\int\limits_{V(t)} f(\mathbf{x}, t)\, dV = \frac{d}{dt}\int\limits_{V_0} F(\xi, t) J dV_0 = \int\limits_{V_0}\left(J\frac{dF}{dt} + F\frac{dJ}{dt}\right) dV_0$$

assuming that F is regular.

Using Euler's formula, assumed also to have been discussed in mathematics, **c** denoting the velocity

$$\frac{dJ}{dt} = J \operatorname{div} \mathbf{c}$$

hence

$$\int_{V_0} \left(J \frac{dF}{dt} + F \frac{dJ}{dt}\right) dV_0 = \int_{V_0} \left(\frac{dF}{dt} + F \operatorname{div} \mathbf{c}\right) J dV_0$$

and because

$$J dV_0 = dV$$

hence

$$\frac{d}{dt} \int_{V(t)} f(\mathbf{x}, t)\, dV = \int_{V(t)} \left(\frac{df}{dt} + f \operatorname{div} \mathbf{c}\right) dV.$$

Students' attention should be drawn now to the fact, that as they see it, it is incorrect to differentiate behind the integral straightaway when the integrate volume is time-dependent.

This equation is a fundamental one and should be underlined by the lecturer. It is his duty to point towards other fields as well (thermodynamics, mass transfer etc.) and advise students that the expression is the base of all "transport" phenomena, hence its application does not stop with Fluid Mechanics. Having stressed the general aspect, an immediate application will enforce the newly acquired information.

Let us regard $f(\mathbf{x}, t)$ as a scalar, hence

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_i \frac{\partial f}{\partial x_i} \frac{dx_i}{dt} = \frac{\partial f}{\partial t} + \mathbf{c} \operatorname{grad} f.$$

But because

$$\operatorname{div}(f\mathbf{c}) = \mathbf{c} \cdot \operatorname{grad} f + f \operatorname{div} \mathbf{c}$$

hence

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \operatorname{div}(f\mathbf{c}) - f \operatorname{div} \mathbf{c}$$

yielding

$$\frac{d}{dt} \int_{V(t)} f dV = \int_{V(t)} \left(\frac{df}{dt} + f \operatorname{div} \mathbf{c}\right) dV = \int_{V(t)} \left(\frac{\partial f}{\partial t} + \operatorname{div}(f\mathbf{c})\right) dV$$

or using Gauss' theorem

$$\frac{d}{dt} \int_{V(t)} f dV = \int_{V(t)} \frac{\partial f}{\partial t} dV + \int_{A(t)} f\mathbf{c} \cdot d\mathbf{A}.$$

It is now only to substitute ϱ for f, and as a start explain the physical meaning of

$$\frac{d}{dt}\int\limits_{V(t)} \varrho dV = 0$$

from which one arrives immediately to the well-known expressions of continuity:

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \mathbf{c}) = 0$$

or

$$\int\limits_{V(t)} \frac{\partial \varrho}{\partial t} dV + \int\limits_{A(t)} \varrho \mathbf{c}\cdot d\mathbf{A} = 0.$$

This is the right moment for physically interpreting the "result", introducing the concepts of source and sink, convection etc.

In a very similar way one may show that if f is a vector then

$$\frac{d}{dt}\int\limits_{V(t)} \mathbf{f} dV = \int\limits_{V(t)} \left(\frac{d\mathbf{f}}{dt} + \mathbf{f}\operatorname{div}\mathbf{c}\right) dV.$$

But because of

$$\frac{d\mathbf{f}}{dt} + \mathbf{f}\operatorname{div}\mathbf{c} = \frac{\partial \mathbf{f}}{\partial t} + (\operatorname{Grad}\mathbf{f})\,\mathbf{c} + \mathbf{f}\operatorname{div}\mathbf{c}$$

and

$$\operatorname{div}(\mathbf{f}\circ\mathbf{c}) = (\operatorname{Grad}\mathbf{f})\mathbf{c} + \mathbf{f}\operatorname{div}\mathbf{c}$$

therefore

$$\frac{d\mathbf{f}}{dt} + \mathbf{f}\operatorname{div}\mathbf{c} = \frac{\partial \mathbf{f}}{\partial t} + \operatorname{div}(\mathbf{f}\circ\mathbf{c})$$

or by using Gauss' theorem

$$\int\limits_{V(t)} \left(\frac{\partial \mathbf{f}}{\partial t} + \operatorname{div}(\mathbf{f}\circ\mathbf{c})\right) dV = \int\limits_{V(t)} \frac{\partial \mathbf{f}}{\partial t} dV + \int\limits_{A(t)} (\mathbf{f}\circ\mathbf{c})\, d\mathbf{A}.$$

Turning immediately towards physics, Newton's law about the change of momentum applied in Fluid Mechanics should be demonstrated. Here — physically interpreted — $\varrho\mathbf{c}$ will take now the place of $\mathbf{f}$, hence

$$\frac{d}{dt}\int\limits_{V(t)} \varrho\mathbf{c} dV = \int\limits_{V(t)} \left(\frac{d(\varrho\mathbf{c})}{dt} + \varrho\mathbf{c}\operatorname{div}\mathbf{c}\right) dV$$

but since

$$\frac{d(\varrho\mathbf{c})}{dt} + \varrho\mathbf{c}\,\mathrm{div}\,\mathbf{c} = \varrho\frac{d\mathbf{c}}{dt} + \mathbf{c}\frac{d\varrho}{dt} + \varrho\mathbf{c}\,\mathrm{div}\,\mathbf{c} =$$

$$= \varrho\frac{d\mathbf{c}}{dt} + \mathbf{c}\left(\frac{\partial\varrho}{\partial t} + \mathbf{c}\,\mathrm{grad}\,\varrho + \varrho\,\mathrm{div}\,\mathbf{c}\right) .$$

Remembering that from the continuity equation

$$\frac{\partial\varrho}{\partial t} + \mathrm{div}\,(\varrho\mathbf{c}) = \frac{\partial\varrho}{\partial t} + \mathbf{c}\cdot\mathrm{grad}\,\varrho + \varrho\,\mathrm{div}\,\mathbf{c} = 0$$

$$\frac{d}{dt}\int\limits_{V(t)} \varrho\mathbf{c}\,dV = \int\limits_{V(t)} \varrho\frac{d\mathbf{c}}{dt}\,dV = \int\limits_{V(t)} \varrho\left[\frac{\partial\mathbf{c}}{\partial t} + (\mathrm{Grad}\,\mathbf{c})\,\mathbf{c}\right]dV$$

which is actually the left-hand side of the equation of motion called either Euler, Bernoulli, or Navier-Stokes equations, the different forms of the same concept.

Similarly, making use of

$$\frac{d}{dt}\int\limits_{V(t)} \mathbf{f}\,dV = \int\limits_{V(t)} \left(\frac{\partial\mathbf{f}}{\partial t} + \mathrm{div}\,(\mathbf{f}\circ\mathbf{c})\right)dV = \int\limits_{V(t)} \frac{\partial\mathbf{f}}{\partial t}\,dV + \int\limits_{A(t)} (\mathbf{f}\circ\mathbf{c})\,dA$$

and inserting $\mathbf{f} = \varrho\mathbf{c}$ will result in the expression

$$\frac{d}{dt}\int\limits_{V(t)} \varrho\mathbf{c}\,dV = \int\limits_{V(t)} \left(\frac{\partial(\varrho\mathbf{c})}{\partial t} + \mathrm{div}\,(\varrho\,\mathbf{c}\circ\mathbf{c})\right)dV = \int\limits_{V(t)} \frac{\partial(\varrho\mathbf{c})}{\partial t}\,dV + \int\limits_{A(t)} \varrho(\mathbf{c}\circ\mathbf{c})\,dA$$

or

$$\frac{d}{dt}\int\limits_{V(t)} \varrho\mathbf{c}\,dV = \int\limits_{Vt} \frac{\partial(\varrho\mathbf{c})}{\partial t}\,dV + \int\limits_{A(t)} \varrho\mathbf{c}(\mathbf{c}\cdot d\mathbf{A})$$

known as the left-hand side of the momentum equation.

It has been tested that if the speed is checked by the understanding of students, the discussed part of the subject matter can be easily lectured on in two or two and a half academic hours (45 minutes each). Not only time, but important pedagogical advantages are gained as well. Students having the fundamentals in "one go" will clearly see that e.g. the momentum law is nothing but another, sometimes more suitable form of the equation of motion. They will also see that this form of the equation of motion is not valid e.g. in aeroacoustics where the equation of continuity is often inhomogenous.

Summary

The subject Fluid Mechanics is an important and fast developing chapter of engineering physics. Restricted time available for teaching it calls for rational selection of subject matter and methods of presentation. The use of higher mathematics throughout the subject will not only shorten the time required when discussing e.g. fundamentals but will open up a more general view, goal of university teaching. The time saved may be turned to tackle more and more physical phenomena widening the scope of the subject matter. Mathematics as a bridge between physical model and physically interpreted results should connect both ends. By no means should the explanation of physics be shortened at start and at the end; but it is neither advisable to replace some mathematical steps with visual physics during the mathematical procedure.

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