ON REDUCTIVE STRUCTURES AND THEIR APPLICATIONS

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Received: June 16, 1977

Presented by Prof. Dr. Gy. STROMMER

The concept of reductive structure appeared for the first time in a basic paper by K. NOMIZU [7] where the up-to-date foundations for the theory of invariant affine connections have been laid. The general importance of this concept was recognized before long; in fact the first comprehensive work on homogeneous manifolds by A. LICHNEROWICZ [6] systematically utilized the reductive structures. Since then results on and applications of reductive structures have considerably increased. It seems therefore justified to make an attempt at a methodical account of some basic facts concerning this important concept. Such an attempt is made below which in turn will bring about some extensions of known results as well.

1. The basic concepts

Let G be a connected Lie group, $H \subset G$ a closed subgroup and consider the set $M = G/H$ of left cosets aH, $a \in G$. Let now

$$\pi : G \to M \quad \text{and} \quad \alpha : G \times M \to M$$

be respectively the canonical projection and the natural group action, then M admits a unique analytic manifold structure such that both π and α are analytic. Thus M becomes a homogeneous analytic manifold and all the homogeneous manifolds are obtained this way up to isomorphisms ([10] 316—327). Consider now the Lie algebra $\mathfrak{g}$ of G which will be identified with the tangent space T_eG of G at the identity element $e \in G$ as usual. Then the subalgebra $\mathfrak{h} \subset \mathfrak{g}$ corresponding to H will be identified with the subspace T_eH of T_eG. Let $\mathfrak{m} \subset \mathfrak{g}$ be a subspace such that the decomposition

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$$

into direct sum of subspaces is valid. Then the linear tangent map

$$T_e\, ad(h) : \mathfrak{g} \to \mathfrak{g}$$

of the automorphism $ad(h) : G \to G$ maps $\mathfrak{h}$ onto itself for any $h \in H$ but, in general, $\mathfrak{m}$ is not left invariant by these linear tangent maps. If in particular

$$T_e\, ad(h)\mathfrak{m} = \mathfrak{m}$$

holds for every $h \in H$ then the vector space direct sum decomposition

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$$

is called a *reductive structure* of the homogeneous space $M = G/H$. Thus the existence of a reductive structure is obviously assured in the case where representation of H on $\mathfrak{g}$ given by

$$T_e\, ad(h) : \mathfrak{g} \to \mathfrak{g},\ h \in H$$

is completely reducible. Consequently, provided G is a connected Lie group and $H \subset G$ a closed connected subgroup which is reductive in G, the corresponding homogeneous space $M = G/H$ admits a reductive structure.

Let $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ be a reductive structure of a homogeneous space $M = G/H$ then by an obvious standard argument:

$$[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}.$$

Conversely assume that G is a connected Lie group, $H \subset G$ a closed subgroup and $\mathfrak{m} \subset \mathfrak{g}$ a subspace such that both

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h} \quad \text{and} \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$$

are valid then $\mathfrak{m}$ defines a reductive structure of the homogeneous space $M = G/H$ provided H is connected ([4] II, 190—191).

Some particular reductive structures with advantageous properties can be introduced in special cases. In fact, let G be a connected reductive Lie group and $H \subset G$ a closed subgroup. Since G is reductive its Lie algebra $\mathfrak{g}$ has a commutative ideal $\mathfrak{c}$ and a semisimple ideal $\mathfrak{s}$ such that

$$\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{s}$$

is valid in consequence of a result of J. H. C. Whitehead [9]. Let now A be a negative definite bilinear form on $\mathfrak{c}$ and K the Killing form of $\mathfrak{s}$, then

$$B = A \oplus K$$

is a non-degenerate invariant bilinear form on $\mathfrak{g}$ which will be called a *quasi-*

Killing form of the reductive Lie algebra $\mathfrak{g}$ ([6] 66—70). Now a decomposition

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$$

into direct sum of subspaces is called a *Killing structure* of the homogeneous manifold $M = G/H$ provided

$$B(\mathfrak{m}, \mathfrak{h}) = 0.$$

is valid. Some of the results concerning Killing structures below are extensions of observations due to A. FLEISCHER [11].

The fact that a Killing structure is a special reductive structure is established by the following theorem.

THEOREM 1.1. *Let G be a connected reductive Lie group, $H \subset G$ a closed subgroup and $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ a Killing structure of the homogeneous manifold $M = G/H$. Then this Killing structure is a reductive structure too of the homogeneous manifold M.*

Proof. Let B be the quasi-Killing form of $\mathfrak{g}$ which defines the given Killing structure. The restriction of B to $\mathfrak{m}$ is non-degenerate since $X \in \mathfrak{m}$ and $B(X, \mathfrak{m}) = 0$ imply

$$B(X, \mathfrak{g}) = B(X, \mathfrak{m} \oplus \mathfrak{h}) = B(X, \mathfrak{m}) + B(X, \mathfrak{h}) = 0$$

and thus $X = 0$ is obtained since B is non-degenerate. Consequently

$$\mathfrak{m} = \{X \mid B(X, \mathfrak{h}) = 0, X \in \mathfrak{g}\}.$$

On the other hand, the invariance of B yields that

$$B(\mathfrak{m}, \mathfrak{h}) = B(T_e\, ad(h)\mathfrak{m}, T_e\, ad(h)\mathfrak{h}) = B(T_e\, ad(h)\mathfrak{m}, \mathfrak{h}) = 0$$

holds for $h \in H$. Therefore the preceding two observations imply now

$$T_e\, ad(h)\mathfrak{m} = \mathfrak{m}$$

for $h \in H$ and consequently the given Killing structure is a reductive structure as well.

2. Some results concerning the existence and uniqueness of reductive structures

The fact that there are homogeneous manifolds of geometric interest which do not admit reductive structures at all was recognized by NOMIZU soon after he had introduced these structures ([7] II, 190—200.) However, the existence of a reductive structure of a homogeneous manifold $M = G/H$ can be directly verified in such important cases where H is compact or H is connected and semisimple ([6] 50—51.) Recently the existence of reductive structures has been discussed by W. KAMBER and P. TONDEUR using methods of algebraic topology [3].

The problem of the uniqueness of reductive structures seems to have been relatively neglected in spite of the fact that even homogeneous spaces $M = G/M$ with H compact may possess more than one reductive structure. In what follows a necessary and sufficient condition will be given for the uniqueness of reductive structures by generalizing a result of B. KOSTANT [5].

Some well-known basic concepts will be mentioned first. In fact let V, W be finite-dimensional real vector spaces, H a group, and consider representations

$$v \mapsto \varphi(h)v \text{ where } v \in V \text{ and } h \in H,$$

$$w \mapsto \psi(h)w \text{ where } w \in W \text{ and } h \in H$$

of H on V and W, respectively. Then a linear map $f: V \to W$ is called an $\mathbf{R}$(H)-*map* ([1] 22—24) or an *operator intertwining* φ with ψ [5] provided the following diagram is commutative for every $h \in H$:

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle \varphi(h)}\downarrow & & \downarrow{\scriptstyle \psi(h)} \\ V & \xrightarrow[f]{} & W \end{array}$$

The representations φ, ψ are said to be *equivalent* if f is an isomorphism. Take now representations φ, ψ which are completely reducible and consider their irreducible components; the representations φ, ψ are said to be *disjoint* if there is no irreducible component of φ equivalent to an irreducible component of ψ.

The following lemma contains a simple but useful observation.

LEMMA 2.1. *Let G be a connected Lie group, $H \subset G$ a closed subgroup and $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ a reductive structure of the homogeneous space $M = G/H$. Consider the adjoint representation*

$$T_e ad(h) : \mathfrak{g} \to \mathfrak{g}, \; h \in H$$

then its restrictions φ and ψ to $\mathfrak{m}$ and $\mathfrak{h}$, respectively, yield the representations

$$\varphi(h) : \mathfrak{m} \to \mathfrak{m}, \; h \in H,$$
$$\psi(h) : \mathfrak{h} \to \mathfrak{h}, \; h \in H.$$

The reductive structures of M are in one-to-one correspondence with the operators intertwining the representation φ with the representation ψ.

P r o o f. Let $\mathfrak{g} = \mathfrak{x} \oplus \mathfrak{h}$ be a reductive structure of the homogeneous space. Then for $X \in \mathfrak{m}$ the unique decomposition $X = Y + Z$, where $Y \in \mathfrak{x}$ and $Z \in \mathfrak{h}$, exists. Consequently a map $\xi : \mathfrak{m} \to \mathfrak{h}$ is defined by

$$X \mapsto \xi(X) = Z \text{ for } X \in \mathfrak{m} .$$

The map ξ is obviously linear; moreover the validity of $T_e ad(h)X \in \mathfrak{m}$, $T_e ad(h)Y \in \mathfrak{x}$, $T_e ad(h)Z \in \mathfrak{h}$ and

$$T_e ad(h)X = T_e ad(h)Y + T_e ad(h)Z$$

imply $\xi(T_e ad(h)X) = T_e ad(h)\ \xi(X)$. Therefore ξ is an operator intertwining φ with ψ.

Assume conversely that an operator ξ intertwining φ with ψ is given. Consider now the set $\mathfrak{x} \subset \mathfrak{g}$ defined by

$$\mathfrak{x} = \{X - \xi(X) \mid X \in \mathfrak{m}\} .$$

By an obvious argument $\mathfrak{x}$ is a subspace which yields a reductive structure $\mathfrak{g} = \mathfrak{x} \oplus \mathfrak{h}$ of the homogeneous manifold M.

A necessary and sufficient condition for the uniqueness of reductive structures is given by the following theorem.

THEOREM 2.2. *Let G be a connected Lie group, $H \subset G$ a closed subgroup which is reductive in G and $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ a reductive structure of the homogeneous manifold $M = G/H$. Let φ and ψ be the restrictions of the representation*

$$T_e ad(h) : \mathfrak{g} \to \mathfrak{g},\ h \in H$$

to $\mathfrak{m}$ and $\mathfrak{h}$, respectively, Then the given reductive structure of M is unique if and only if the representations φ and ψ are disjoint.

P r o o f. In consequence of the hypothesis that H is reductive in G the subspaces $\mathfrak{m}$, $\mathfrak{h}$ admit decompositions

$$\mathfrak{m} = \oplus \{\mathfrak{m}_i \mid i = 1, \ldots, k\} \text{ and } \mathfrak{h} = \oplus \{\mathfrak{h}_j \mid j = 1. \ldots, l\}$$

into direct sums of irreducible components. Assume first that there is a reductive structure $\mathfrak{g} = \mathfrak{x} \oplus \mathfrak{h}$ different from the given one. Then on account of the preceding lemma there is a non-trivial operator

$$\xi : \mathfrak{m} \to \mathfrak{h}$$

intertwining φ with ψ. Let ξ_i be the restriction of ξ to $\mathfrak{m}_i$ for $i = 1, \ldots, k$ then there is at least one of these maps, say ξ_1, which is not trivial. The subspace

$$\xi(\mathfrak{m}_1) \subset \mathfrak{h}$$

is irreducible since $\mathfrak{m}_1 \subset \mathfrak{m}$ is irreducible. Consequently there is no loss of generality by assuming $\xi(\mathfrak{m}_1) = \mathfrak{h}_1$. But then on account of Schur's lemma the map

$$\xi_1 : \mathfrak{m}_1 \to \mathfrak{h}_1$$

is an isomorphism. Consequently the tepresentations φ and ψ are not disjoint. Assume secondly that φ and ψ are not disjoint. Then there is no loss of generality by assuming their restrictions to $\mathfrak{m}_1$, and $\mathfrak{h}_1$ to be equivalent. Let now

$$\xi_1 : \mathfrak{m}_1 \to \mathfrak{h}_1$$

be the intertwining operator which defines this equivalence. Let further ξ_i be the trivial map of $\mathfrak{m}_i$ into $\mathfrak{h}$ for $i = 2, \ldots, k$. Then there is a unique linear map

$$\xi : \mathfrak{m} \to \mathfrak{h}$$

such that the restriction of ξ to $\mathfrak{m}_i$ is equal to ξ_i for $i = 1, \ldots, k$. It is obvious that ξ is a non-trivial operator intertwining φ with ψ. But then by the preceding lemma there is a reductive structure $\mathfrak{g} = \mathfrak{x} \oplus \mathfrak{h}$ of M which is different from the given one.

The problem of the existence and uniqueness of Killing structures is much more accessible. In fact the uniqueness of a Killing structure of a homogeneous manifold $M = G/H$ is an obvious consequence of the fact already established in the proof of Theorem 1.1 that

$$\mathfrak{m} = \{X \mid B(X, \mathfrak{h}) = 0, X \in \mathfrak{g}\}$$

is valid where B is the quasi-Killing form of $\mathfrak{g}$ applied in defining the Killing structure. Concerning the existence of Killing structures the following theorem is valid:

THEOREM 2.3. *Let G be a connected reductive Lie group, $H \subset G$ a closed connected subgroup and B a quasi-Killing form of G. Then the homogeneous space $M = G/H$ admits a Killing structure if and only if the restriction of B to $\mathfrak{h}$ is non-degenerate.*

Proof. Suppose first that a Killing structure $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ of $M = G/H$ exists. In order to prove by contradiction assume that there is a $Z \in \mathfrak{h}$ such that $Z \neq 0$ and $B(Z, \mathfrak{h}) = 0$. Since $B(Z, \mathfrak{m}) = 0$ the assumption implies that

$$B(Z, \mathfrak{g}) = B(Z, \mathfrak{m} \oplus \mathfrak{h}) = B(Z, \mathfrak{m}) + B(Z, \mathfrak{h}) = 0.$$

But then B is degenerate which is in contradiction with the definition of B. Suppose secondly that the restriction of B to $\mathfrak{h}$ is non-degenerate. Consider now the subspace $\mathfrak{h}^\perp$ defined by

$$\mathfrak{h}^\perp = \{X \mid B(X, \mathfrak{h}) = 0, \ X \in \mathfrak{g}\} .$$

Since the restriction of B to $\mathfrak{h}$ is non-degenerate, $\mathfrak{h} \cap \mathfrak{h}^\perp = \{0\}$ is valid. Consider therefore the subspace $\mathfrak{h}^\perp \oplus \mathfrak{h}$ of $\mathfrak{g}$. It is sufficient to show that

$$\mathfrak{g} = \mathfrak{h}^\perp \oplus \mathfrak{h} .$$

In order to prove by contradiction assume now that there is a $Y \in \mathfrak{g}$ such that $Y \neq 0$ and $Y \notin \mathfrak{h}^\perp \oplus \mathfrak{h}$. Thus a linear form is defined on $\mathfrak{h}$ by

$$Z \mapsto B(Y, Z), \ Z \in \mathfrak{h}$$

which is non-degenerate since $B(Y, \mathfrak{h}) \neq 0$. Since the restriction of B to $\mathfrak{h}$ is supposed to be non-degenerate there is a $V \in \mathfrak{h}$ such that

$$B(V, Z) = B(Y, Z) \ \text{ for } \ Z \in \mathfrak{h} .$$

This implies that $Y - V \in \mathfrak{h}^\perp$ and therefore $Y \in \mathfrak{h}^\perp \oplus \mathfrak{h}$, which is in contradiction with the definition of Y. Thus $\mathfrak{g} = \mathfrak{h}^\perp \oplus \mathfrak{h}$ is valid.

According to a well-known definition a homogeneous manifold $M = G/H$ is said to be *effective* if the only invariant subgroup of G contained in H is the trivial one. In that case where G is reductive a necessary condition can be easily derived in order that M be effective.

THEOREM 2.4. *Let G be a connected reductive Lie group, $H \subset G$ a closed subgroup and $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ a reductive structure of the homogeneous manifold $M = G/H$. If M is effective then $\mathfrak{h} \subset [\mathfrak{m}, \mathfrak{m}]$ is valid.*

Proof. Put $[\mathfrak{m}, \mathfrak{m}]_\mathfrak{h} = \mathfrak{h} \cap [\mathfrak{m}, \mathfrak{m}]$, then according to an observation due to B. KOSTANT ([4] II, 200—216), an ideal $\mathfrak{a}$ of $\mathfrak{g}$ is given by

$$\mathfrak{a} = \mathfrak{m} \oplus [\mathfrak{m}, \mathfrak{m}]_\mathfrak{h} .$$

Since $\mathfrak{g}$ is reductive there is an ideal $\mathfrak{b}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ holds. But the existence of the reductive structure yields $\mathfrak{b} \subset \mathfrak{h}$ and the assumption that M is effective implies $\mathfrak{b} = \{0\}$. Therefore $\mathfrak{h} = [\mathfrak{m}, \mathfrak{m}]_\mathfrak{h} \subset [\mathfrak{m}, \mathfrak{m}]$ is valid.

3. Homogeneous submanifolds and reductive structures

Homogeneous submanifolds are studied subsequently with the aid of reductive structures. The methods applied and the results achieved are obtained by generalizing earlier methods and results for 1-dimensional homogeneous submanifolds, i.e. orbits of 1-parameter subgroups [8].

Let G be a connected Lie group, $H \subset G$ a closed subgroup, $M = G/H$ the corresponding homogeneous manifold and $\alpha : G \times M \to M$ the canonical action of G. A submanifold N of M is called a *homogeneous submanifold* if there is a Lie subgroup K of G such that elements of K map N onto itself and they are transitive on N. Let N be a homogeneous submanifold and consider the set $\mathfrak{H}$ of Lie subgroups K of G which leave N invariant and act transitively on N. If $K', K'' \in \mathfrak{H}$ and $K \subset G$ is the subgroup generated by elements of K' and K'' then $K \in \mathfrak{H}$ is valid; in fact it is obvious that elements of K leave N invariant and that K acts transitively on N; moreover, K is a Lie subgroup of G in consequence of a theorem of H. YAMABE ([4] I, 275—276.) Considering the partial ordering of $\mathfrak{H}$ defined by inclusion the Kuratowski-Zorn lemma applies and yields that $\mathfrak{H}$ has a maximal element K_N. The Lie subgroup K_N of G will be called the *subgroup corresponding to the homogeneous submanifold* N in the Lie group G. The Lie algebra $\mathfrak{k}_N$ of K_N will be called the *Lie subalgebra corresponding to the homogeneous submanifold* in the Lie algebra $\mathfrak{g}$. Assume now that $o \in N$ holds, then obviously $N = \alpha(K_N, o) = \pi(K_N)$ and consequently $T_oN = T_e\pi\mathfrak{k}_N$. Therefore a subspace $\mathfrak{n} \subset \mathfrak{k}_N$ exists such that the decomposition

$$\mathfrak{k}_N = \mathfrak{n} \oplus \mathfrak{q}$$

into direct sum of subspaces exists where $\mathfrak{q}$ is a subalgebra defined by $\mathfrak{q} = \mathfrak{k}_N \cap \mathfrak{h}$. Such a decomposition will be called an *isotropy decomposition* of the subalgebra $\mathfrak{k}_N$ corresponding to the homogeneous submanifold N. The subspace $\mathfrak{l}_N \subset \mathfrak{g}$ defined by $\mathfrak{l}_N = T_e\pi^{-1}(T_oN)$ will also be applied in what follows.

The following lemma contains a simple but useful observation which will lead to essential consequences.

LEMMA 3.1. *Let $N \subset M$ be a homogeneous submanifold with $o \in N$ and*

$$\mathfrak{k}_N = \mathfrak{n} \oplus \mathfrak{q}$$

an isotropy decomposition of the corresponding subalgebra. If $\mathfrak{a} \subset \mathfrak{g}$ is a subalgebra such that $\mathfrak{n} \subset \mathfrak{a} \subset \mathfrak{l}_N$ holds then even $\mathfrak{a} \subset \mathfrak{k}_N$ is valid.

P r o o f. Let $A \subset G$ be the Lie subgroup defined by $\mathfrak{a}$. Since obviously $\mathfrak{l}_N = \mathfrak{n} \oplus \mathfrak{h}$ there is a canonical coordinate system of the second kind of A given by

$$\varphi_1(\sigma_1) \ldots \varphi_k(\sigma_k)\zeta_1(\tau_1) \ldots \zeta_l(\tau_l)$$

on a neighborhood U of e in A such that a canonical coordinate system of the second kind of $A \cap H$ is given by

$$\zeta_1(\tau_1) \dots \zeta_l(\tau_l)$$

on a neighborhood of e in $A \cap H$ ([10] 302—307). There is a neighborhood V of o in N and U' of e in A such that if $x \in V$ and $a \in U'$ then there is a $g \in U$ such that $x = \pi(g)$ and $ag \in U$. Assume therefore

$$a = \varphi_1(\sigma_1) \dots \varphi_k(\sigma_k)\zeta_1(\tau_1) \dots \zeta_l(\tau_l) \text{ and } g = \varphi_1(\alpha_1) \dots \varphi_k(\alpha_k),$$

then on account of the preceding stipulations the following equalities hold

$$\begin{aligned}\alpha(a, x) &= \alpha(a, \pi(g)) = \pi(ag) = \\ &= \pi(\varphi_1(\sigma_1) \dots \varphi_k(\sigma_k)\zeta_1(\tau_1) \dots \zeta_l(\tau_l)\varphi_1(\alpha_1) \dots \varphi_k(\alpha_k)) = \\ &= \pi(\varphi_1(\sigma_1') \dots \varphi_k(\sigma_k')\zeta_1(\tau_1') \dots \zeta_l(\tau_l')) = \pi(\varphi_1(\sigma_1') \dots \varphi_k(\sigma_k')) \in N.\end{aligned}$$

Consequently $\alpha(U', V) \subset N$ is valid and this implies that $A \subset K_N$ is true as well, by definition of K_N.

COROLLARY. *Let $N \subset M$ be a homogeneous submanifold with $o \in N$* and

$$\mathfrak{k}_N = \mathfrak{n} \oplus \mathfrak{q}$$

an isotropy decomposition of the corresponding subalgebra. Then $\mathfrak{k}_N$ is equal to the maximal subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ such that $\mathfrak{n} \subset \mathfrak{a} \subset \mathfrak{l}_N$ holds.

The following theorem which yields another description of subalgebras corresponding to homogeneous submanifolds exhibits some structural properties of these subalgebras.

THEOREM 3.2. *Let $N \subset M$ be a homogeneous submanifold with $o \in N$ and*

$$\mathfrak{k}_N = \mathfrak{n} \oplus \mathfrak{q}$$

an isotropy decomposition of the corresponding subalgebra $\mathfrak{k}_N$. Consider further the decreasing sequence

$$\mathfrak{h} = \mathfrak{q}^0 \supset \mathfrak{q}^1 \supset \dots \supset \mathfrak{q}^j \supset \mathfrak{q}^{j+1} \supset \dots$$

of sets which are given by the following successive definition:

$$\mathfrak{q}^j = \{Z \mid ad(Z)\mathfrak{n} \subset \mathfrak{n} \oplus \mathfrak{q}^{j-1} \text{ and } Z \in \mathfrak{q}^{j-1}\}$$

where $j = 1, 2, \dots$. Then these $\mathfrak{q}^j$ are all subalgebras of $\mathfrak{h}$ and if k is the first j such that $\mathfrak{q}^j = \mathfrak{q}^{j+1}$ then $\mathfrak{q}^k = \mathfrak{q}$ is valid.

P r o o f. The assertion that the $\mathfrak{q}^j$ are subalgebras will be verified by induction. Since $\mathfrak{q}^0$ is a subalgebra by definition assume $\mathfrak{q}^{j-1}$ to be a subalgebra for $j = 1, 2, \ldots$. In order to show that $\mathfrak{q}^j$ is a subalgebra consider arbitrary elements $Z' Z'' \in \mathfrak{q}^j$ and $\xi, \eta \in \mathbf{R}$. Then $Z', Z'' \in q^{j-1}$ and $\xi Z' + \eta Z'' \in \mathfrak{q}^{j-1}$, $[Z', Z''] \in \mathfrak{q}^{j-1}$ since q^{j-1} is supposed to be a subalgebra. Moreover

$$ad(\xi Z' + \eta Z'')\,\mathfrak{n} \subset \xi ad\,(Z')\,\mathfrak{n} + \eta\, ad(Z'')\,\mathfrak{n} \subset \mathfrak{n} \oplus \mathfrak{q}^{j-1}$$

by definition of $\mathfrak{q}^j$ and furthermore, in consequence of the Jacobi identity even

$$ad([Z', Z''])\mathfrak{n} = -[\mathfrak{n},[Z', Z'']] \subset [Z'', [\mathfrak{n}, Z']] + [Z', [Z'', \mathfrak{n}]] \subset \mathfrak{n} \oplus \mathfrak{q}^{j-1}$$

holds by definition of $\mathfrak{q}^j$ and since $\mathfrak{q}^{j-1}$ is supposed to be a subalgebra. These observations yield that $\xi Z' + \eta Z'' \in \mathfrak{q}^j$, $[Z', Z''] \in \mathfrak{q}^j$ holds. Therefore $\mathfrak{q}^j$ is a subalgebra. Let now k be the first integer j such that $\mathfrak{q}^j = \mathfrak{q}^{j+1}$ is valid. Since $\mathfrak{k}_N = \mathfrak{n} \oplus \mathfrak{q} \subset \mathfrak{n} \oplus \mathfrak{q}^j$ holds for every j, by the above construction $\mathfrak{k}_N \subset \mathfrak{n} \oplus \mathfrak{q}^k$ is obviously valid. This yields

$$[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{n} \oplus \mathfrak{q}^k$$

since $\mathfrak{k}_N$ is a subalgebra. Consequently $\mathfrak{n} \oplus \mathfrak{q}^k$ is a subalgebra such that

$$\mathfrak{n} \subset \mathfrak{n} \oplus \mathfrak{q}^k \subset \mathfrak{l}_N$$

is satisfied. Thus by the preceding lemma $\mathfrak{n} \oplus \mathfrak{q}^k \subset \mathfrak{k}_N$ holds. Therefore

$$\mathfrak{n} \oplus \mathfrak{q} = \mathfrak{k}_N = \mathfrak{n} \oplus \mathfrak{q}^k.$$

Further $\mathfrak{q}, \mathfrak{q}^k \subset \mathfrak{h}$ implies that $\mathfrak{q}^k = \mathfrak{q}$ is valid as well.

Let $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ be a reductive structure of a homogeneous manifold $M = G/H$ where the closed subgroup $H \subset G$ is connected. If $\mathfrak{n} \subset \mathfrak{m}$ is a subspace and $\mathfrak{q} \subset \mathfrak{h}$ a subalgebra such that $\mathfrak{k} = \mathfrak{n} \oplus \mathfrak{q}$ is a subalgebra of $\mathfrak{g}$ then

$$\mathfrak{k} = \mathfrak{n} \oplus \mathfrak{q}$$

is a *reductive substructure* of the given reductive structure $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. In particular, a reductive substructure $\mathfrak{k} = \mathfrak{n} \oplus \mathfrak{q}$ is said to be *symmetric* if $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{q}$ holds.

The following lemma shows the importance of reductive substructures in studying homogeneous submanifolds.

LEMMA 3.3. *Let $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ be a reductive structure of a homogeneous manifold $M = G/H$ such that the closed subgroup $H \subset G$ is connected. If $N \subset M$ is a homogeneous submanifold with $o \in N$ such that an isotropy decomposition $\mathfrak{k}_N = \mathfrak{n} \oplus \mathfrak{q}$ of the corresponding subalgebra with $\mathfrak{n} \subset \mathfrak{m}$ exists then $\mathfrak{k}_N = \mathfrak{n} \oplus \mathfrak{q}$ is a reductive substructure. Conversely, if $\mathfrak{k} = \mathfrak{n} \oplus \mathfrak{q}$ is a reductive substructure of* *the given reductive structure and* $K \subset G$ *is the Lie subgroup defined by* $\mathfrak{k}$ *then* $N = \pi(K)$ *is a homogeneous submanifold and for the corresponding subgroup* K_N *if* N *is effective* $K_N = K$ *is valid.*

P r o o f. Consider first a homogeneous submanifold $N \subset M$ with $o \in N$ such that an isotropy decomposition $\mathfrak{k}_N = \mathfrak{n} \oplus \mathfrak{q}$ of the corresponding subalgebra with $\mathfrak{n} \subset \mathfrak{m}$ exists. Since $\mathfrak{k}_N$ is a subalgebra, $\mathfrak{k}_N = \mathfrak{n} \oplus \mathfrak{q}$ is a reductive substructure. Consider secondly a reductive substructure $\mathfrak{k} = \mathfrak{n} \oplus \mathfrak{q}$. Let $K \subset G$ be the Lie subgroup defined by $\mathfrak{k}$ and N the homogeneous submanifold given by $N = \pi(K)$. Then

$$\mathfrak{n} \subset \mathfrak{n} \oplus \mathfrak{q} \subset \mathfrak{l}_N$$

and therefore $\mathfrak{k} = \mathfrak{n} \oplus \mathfrak{q} \subset \mathfrak{k}_N$ is consequence of Lemma 3.1. Consider now the isotropy decomposition

$$\mathfrak{k}_N = \mathfrak{n} \oplus \mathfrak{q}_N$$

of $\mathfrak{k}_N$. Then $\mathfrak{q} \subset \mathfrak{q}_N$ by the preceding observation. Since $N = \pi(K)$ is effective $\mathfrak{q}_N \subset [\mathfrak{n}, \mathfrak{n}]$ is valid on account of Theorem 2.4 and consequently $\mathfrak{q}_N \subset \mathfrak{q}$ holds. Therefore $\mathfrak{q} = \mathfrak{q}_N$ is obtained and this implies that $K = K_N$ is valid.

The following theorem shows the importance of reductive substructures in classifying homogeneous submanifolds by more geometric terms.

THEOREM 3.4. *Let* $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ *be a reductive structure of a homogeneous manifold* $M = G/H$ *where the closed subgroup* $H \subset G$ *is connected. If*

$$\mathfrak{k} = \mathfrak{n} \oplus \mathfrak{q}$$

is a reductive substructure then the corresponding homogeneous submanifold is totally geodesic with respect to the natural torsion-free connection of the given reductive structure. Conversely if $N \subset M$ *is a homogeneous submanifold with* $o \in N$ *which is totally geodesic with respect to the natural torsion-free connection of the given reductive structure and all geodesics of* N *are trajectories of* 1*-parameter subgroups defined by elements of* $\mathfrak{m}$ *then an isotropy decomposition* $\mathfrak{k}_N = \mathfrak{n} \oplus \mathfrak{q}$ *with* $\mathfrak{n} \subset \mathfrak{m}$ *exists.*

P r o o f. Assume first that a reductive substructure $\mathfrak{k} = \mathfrak{n} \oplus \mathfrak{q}$ is given and consider the corresponding homogeneous submanifold N defined according to the proof of the preceding lemma. If $X \in \mathfrak{n} - \{0\}$ then the corresponding orbit

$$\tau \to \pi \circ \exp(\tau X), \quad \tau \in \mathbf{R}$$

is a curve of N since $\pi \circ \exp(\mathfrak{n}) \subset N$ holds. On the other hand this orbit is a geodesic of the natural torsion-free connection of the given reductive structure ([4] II, 190—200). Since $T_0N = T_e\pi\mathfrak{n}$ this implies that the submanifold N is geodesic at the point o. But the submanifold N is homogeneous and the natural torsion-free connection is invariant, therefore N is geodesic at all of its points.

Consequently N is totally geodesic. The proof of the converse of this assertion is now evident.

Let $\mathfrak{g}$ be a finite dimensional real Lie algebra and $\mathfrak{n} \subset \mathfrak{g}$ a subspace such that

$$[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] \subset \mathfrak{n}.$$

Then the subspace $\mathfrak{n}$ is called a *Lie triple system* in the Lie algebra $\mathfrak{g}$.

The geometric significance of Lie triple systems for Lie groups and symmetric spaces is a well-known fact ([2] 189—191). The following lemma shows that some of the concerned results generalize to Killing structures.

LEMMA 3.5. *Let G be a reductive connected Lie group, $H \subset G$ a closed connected subgroup such that a Killing structure $\mathfrak{g} = m \oplus \mathfrak{h}$ of the homogeneous manifold $M = G/H$ exists and that the restriction of the representation*

$$T_e ad(h) : \mathfrak{g} \to \mathfrak{g}, \quad h \in H$$

to $\mathfrak{m}$ is irreducible. Then a reductive substructure $\mathfrak{k} = \mathfrak{n} \oplus \mathfrak{q}$ is symmetric if and only if $\mathfrak{n}$ is a Lie triple system.

P r o o f. The assumption concerning the irreducibility of the representation obviously implies $[\mathfrak{h}, \mathfrak{n}] = \mathfrak{m}$ for any subspace $\mathfrak{n} \subset \mathfrak{m}$. Let B be the quasi-Killing form of $\mathfrak{g}$ which defines the given Killing structure. Then the invariance of B yields

$$B(\mathfrak{m}, [\mathfrak{n}, \mathfrak{n}]) = B([\mathfrak{h}, \mathfrak{n}], [\mathfrak{n}, \mathfrak{n}]) = B(\mathfrak{h}, [\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]])$$

for any subspace $\mathfrak{n} \subset \mathfrak{m}$. But these equalities yield that $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{h}$ is equivalent to $[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] \subset \mathfrak{m}$. Assume now $\mathfrak{n}$ to be given by a reductive substructure $\mathfrak{k} = \mathfrak{n} \oplus \mathfrak{q}$. Since $\mathfrak{k}$ is a subalgebra now $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{q}$ is equivalent to $[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] \subset \subset \mathfrak{n}$.

THEOREM 3.6. *Let G be a connected reductive Lie group, $H \subset G$ a closed connected subgroup such that a Killing structure $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ of the homogeneous manifold $M = G/H$ exists and such that the restriction of the representation*

$$T_e ad(h) : \mathfrak{g} \to \mathfrak{g}, \quad h \in H$$

is irreducible. If $\mathfrak{n} \subset \mathfrak{m}$ is a Lie triple system then $N = \pi \circ \exp(\mathfrak{n})$ is a homogeneous submanifold which is totally geodesic and symmetric with respect to the natural torsion-free connection corresponding to the given Killing structure. Conversely if $N \subset M$ is a homogeneous submanifold with $o \in N$ which is totally geodesic and symmetric with respect to the natural torsion-free connection of the given Killing structure then there is a Lie triple system $\mathfrak{n} \subset \mathfrak{m}$ with $N = \pi \circ exp(\mathfrak{n})$.

P r o o f. Assume first that a Lie triple system $\mathfrak{n} \subset \mathfrak{m}$ is given. Then

$$\mathfrak{k} = \mathfrak{n} \oplus [\mathfrak{n}, \mathfrak{n}]$$

is a subalgebra and even a reductive substructure. Consequently the homogeneous submanifold defined by $\mathfrak{k}$ is totally geodesic and even symmetric since $\mathfrak{k}$ is symmetric ([4] II, 222—238). Assume secondly that a homogeneous submanifold $N \subset M$ with $o \in N$ exists which is totally geodesic and symmetric with respect to the natural torsion-free connection of the given Killing structure. Then geodesics in N passing through o are orbits of 1-parameter subgroups in K_N and defined by elements of $\mathfrak{m}$. Thus assumptions in Theorem 3.4. are satisfied and therefore a decomposition

$$\mathfrak{k}_N = \mathfrak{n} \oplus \mathfrak{q}$$

with $\mathfrak{n} \subset \mathfrak{m}$ exists. But $\mathfrak{k}_N = \mathfrak{n} \oplus \mathfrak{q}$ is symmetric since N is symmetric. Therefore $\mathfrak{n}$ is a Lie triple system by Lemma 3.5.

4. Decompositions of homogeneous manifolds and reductive structures

The decompositions of homogeneous manifolds are considered generally under the existence of an invariant Riemannian metric ([4] II, 210—216). Here the problem of decomposition of a homogeneous manifold will be considered first in entire generality and then under the existence of a reductive structure. Different kinds of decompositions will be introduced and studied.

The following lemma summarizes some simple facts basic for the decomposition of homogeneous manifolds into homogeneous submanifolds.

LEMMA 4.1. *Let G be a connected Lie group H a closed connected subgroup*

$$\pi : G \to M = G/H \text{ and } \alpha : G \times M \to M$$

the canonical projection and the natural action respectively. If $G', G'' \subset G$ are closed connected subgroups and $M' = \pi(G')$, $M'' = \pi(G'')$ the corresponding homogeneous submanifolds then the following assertions hold:

1. $M = \bigcup \{\alpha(g', M'') \mid g' \in G'\}$ *if and only if* $G = G'G''H$;
2. *The validity of $\alpha(a,' o) = \alpha(b', o)$ implies that of $\alpha(a', x'') = \alpha(b', x'')$ for every $x'' \in M''$ if and only if $g''^{-1}(H \cap G')g'' = H \cap G'$ for $g'' \in G''$;*
3. *The orbits $\alpha(G', x'')$ of different $x'' \in M''$ are disjoint if and only if $g''^{-1}G'g'' \cap G''H \subset H$ for $g'' \in G''$.*

P r o o f. 1. Assume $M = \bigcup \{\alpha(g', M'') \mid g' \in G'\}$ to be valid then any $x \in M$ can be obtained as $\pi(g) = x = \alpha(g', x'') = \pi(g'g'')$ where $g \in G$, $g' \in G'$, $x'' = \pi(g'')$ and $g'' \in G''$; consequently $g = g'g''h$ holds for any g with $h \in H$. This argument yields the proof of the converse assertion, too.

2. Assume $g''^{-1}(H \cap G')\, g'' = H \cap G'$ for $g'' \in G''$ and $\alpha(a', o) = \alpha(b', o)$ for some $a', b' \in G'$. Then $a' = b'h$ with some $h \in H$. Consider $x'' \in M''$ with $x'' = \pi(g'')$ where $g'' \in G''$. Then $\alpha(a', x'') = \pi(a'g'') = \pi(b'hg'') =$

$= \pi(b'g''(g''^{-1}hg'')) = \pi(b'g'')$. Assume in turn that $\alpha(a', o) = \alpha(b', o)$ implies $\alpha(a', x'') = \alpha(b', x'')$ for every $x'' \in M''$. This means in other words that $a' = b'h$ with $h \in H$ implies $a'g'' = b'g''h$ for every $g'' \in G''$ with some $h \in H$. Then $a'g'' = b'hg'' = b'g''(g''^{-1}hg'') = b'g''h$ yields $h = g''^{-1}hg''$ and consequently $g''^{-1}(H \cap G')g'' = H \cap G'$ for $g'' \in G''$.

3. Assume $g''^{-1}G'g'' \cap G''H \subset H$ for $g'' \in G''$. In order to show that orbits of different $x'' \in M''$ are disjoint it is sufficient to see that $y'' = \alpha(g', x'')$ implies $y'' = x''$. Consider therefore $a'', b'' \in G''$ with $x'' = \pi(a'')$ and $y'' = \pi(b'')$, then $b'' = g'a''h$ with $h \in H$. Consequently $g' = b''h^{-1}a''^{-1} \in G' \cap G''Ha''^{-1} \subset a''H''^{-1}$ and $b'' = a''h^* a''^{-1}a''h = a''h^*h$ therefore $y'' = x''$ holds. Assume in turn that $y'' = \alpha(g', x'')$ with $g' \in G'$ and $x'', y'' \in M''$ implies $y'' = x''$. Then $g'a'' = b''h$ with $a'', b'' \in G''$ and $h \in H$ implies $a'' = b''h$ with some $h \in H$. Consequently $G'a'' \cap G''H \subset a''H$ and therefore $a''^{-1}G'a'' \cap G''H \subset H$ for every $a'' \in G''$.

The following theorem prepares for the introduction of a decomposition of homogeneous manifolds into homogeneous submanifolds.

THEOREM 4.2. *Let G be a connected Lie group, $H \subset G$ a closed connected subgroup*

$$\pi : G \to M = G/H \quad \text{and} \quad \alpha : G \times M \to M$$

the canonical projection and the natural action respectively. Let $G', G'' \subset G$ be closed connected subgroups such that the following conditions are satisfied:

1. $G = G'G''H$;
2. $g''^{-1}(H \cap G')g'' = H \cap G'$ *for* $g'' \in G''$;
3. $g''^{-1}G'g'' \cap G''H \subset H$ *and* $G' \cap g''Hg''^{-1} \subset H$ *for* $g'' \in G''$. *Then a map $\mu : M' \times M'' \to M$ is defined by the following prescription:*

$(x', x'') \mapsto \mu(x', x'') = \alpha(g', x'')$ *for* $(x', x'') \in M' \times M''$ *where* $g' \in G'$ *is such that $x' = \pi(g')$. The map μ is a diffeomorphism.*

P r o o f. In order to see that the above definition of μ justified consider $x' = \pi(a') = \pi(b')$ with $a', b' \in G'$. Then $a' = b'h$ with $h \in H \cap G'$ and consequently $a'g'' = b'hg'' = \alpha(b', x'')$ for $x'' \in M''$. The map μ is obviously surjective since $G = G'G''H$. In order to see that μ is injective assume $\mu(x', x'') = \mu(y', y'')$ and therefore $\alpha(a', x'') = \alpha(b', y'')$ with $a', b' \in G'$ such that $x' = \pi(a')$ and $y' = \pi(b')$. But then $x'' = y''$ must be valid on account of the preceding lemma and consequently $x'' = \alpha(a'^{-1}b', x'')$. Consider now $g'' \in G''$ such that $x'' = \pi(g'')$ then $a'^{-1}b'g'' = g''h$ with some $h \in H$. But then $G' \cap g''Hg''^{-1} \subset H$ implies $a'^{-1}b' \in H$ and consequently $x' = y'$ is valid.

In order to see that the map μ is differentiable consider a point (x', x'') of $M' \times M''$ and a differentiable local cross-section $\sigma : U \to G'$ of the fibration $G' \to M'$ defined on a neighborhood U of x'_0 in M'. Then

$$\mu(x', x'') = \alpha(\sigma(x'), x'')$$

holds for $(x', x'') \in U \times M''$ consequently μ is differentiable at the **point** considered. In order to show that μ is a diffeomorphism now it is obviously **sufficient** to show that

$$T_{(x',x'')}\mu\,(T_{(x',x'')}\,M' \times M'') = T\mu_{(x',x'')}\,M$$

is valid. Consider therefore the restriction μ' of μ to $M' \times \{x''\}$ und μ'' the restriction of μ to $\{x'\} \times M''$. It is obviously sufficient to verify the following three assertions:

1. $T_{x'}\,\mu' : T_{x'}\,M' \to T_{(x',x'')}\,M$ is injective;
2. $T_{x''}\mu'' : T_{x''}M'' \to T_{(x',x'')}\,M$ is injective;
3. $T_{x'}\,\mu'\,(T_{x'}, M') \cap T_{x''}\,\mu''\,(T_{x''}\,M'') = \{0\}$

which are obvious consequences of the assumptions of the theorem.

On account of the preceding theorem the following definition can be introduced: Let G be a connected Lie group $H \subset G$ a closed connected subgroup and $M = G/H$ the corresponding homogeneous manifold. Let $G', G'' \subset G$ be closed connected subgroups such that the following conditions are satisfied:

1. $G = G'G''H$;
2. $g''^{-1}(H \cap G')g'' = H \cap G'$ for $g'' \in G''$;
3. $g''^{-1}Hg'' \cap G' \subset H \cap G'$ for $g'' \in G''$;
4. $G''H \cap g''^{-1}G'g'' \subset H$ for $g'' \in G''$.

Then the map $\mu : M' \times M''$ is called a *decomposition of the first kind of the homogeneous manifold* M into homogeneous submanifolds.

The following corollary yields a less general kind of decomposition of homogeneous manifolds into homogeneous submanifolds.

COROLLARY. *Let G be a connected Lie group, $H \subset G$ a closed connected subgroup*

$$\pi : G \to M = G/H \quad \text{and} \quad \alpha : G \times M \to M$$

the canonical projection and the natural action respectively. Let $G', G'' \subset G$ be closed connected subgroups and $M' = \pi(G')$, $M'' = \pi(G'')$ the corresponding homogeneous submanifolds, such that the following conditions are satisfied:

1. $G = G'G''H = G''G'H$;
2. $g''^{-1}(H \cap G')g'' = H \cap G'$ for $g'' \in G''$ and
   $g'^{-1}(H \cap G'')g' = H \cap G''$ for $g' \in G'$;
3. $g''^{-1}Hg'' \cap G' \subset H \cap G'$ for $g'' \in G''$ and
   $g'^{-1}Hg' \cap G'' \subset H \cap G''$ for $g' \in G'$;

4. $G''H \cap g''^{-1}G'g'' \subset H$ for $g'' \in G''$ and
$G'H \cap g'^{-1}G''g' \subset H$ for $g' \in G'$.

Consider the diffeomorphisms μ, ν: $M' \times M'' \to M$ *given by*

$$\mu(x', x'') = \alpha(g', x'') \text{ and } \nu(x', x'') = \alpha(g'', x')$$

respectively, where $g' \in G'$ *and* $g'' \in G''$ *with* $x' = \pi(g')$ *and* $x'' = \pi(g'')$*. Then* $\mu = \nu$ *holds if and only if the commutators of elements of* G' *and* G'' *are contained in* H.

Proof. Since $\mu(x', x'') = \nu(x', x'')$ holds if and only if $g'g'' = g''g'h$ is satisfied with some $h \in H$, the assertion of the corollary obviously follows.

On account of the preceding corollary a more special concept for decompositions of homogeneous manifolds into homogeneous submanifolds seems to be justified as well. Let G be a connected Lie group, $H \subset G$ a closed connected subgroup and $M = G/H$ the corresponding homogeneous manifold. If $G', G'' \subset G$ are closed connected subgroups such that

1. $G = G'G''H = G''G'H$;
2. $g''^{-1}(H \cap G')g'' = H \cap H'$ for $g'' \in G''$ and
$g'^{-1}(H \cap G'')g' = H \cap G''$ for $g' \in G'$;
3. $g''^{-1}Hg'' \cap G' \subset H \cap G'$ for $g'' \in G''$ and
$g'^{-1}Hg' \cap G'' \subset H \cap G''$ for $g' \in G'$;
4. $G''H \cap g''^{-1}G'g'' \subset H$ for $g'' \in G$ and
$G'H \cap g'^{-1}G''g' \subset H$ for $g' \in G''$;
5. $g'g''g'^{-1}g''^{-1} \in H$ for $g' \in G'$ and $g'' \in G''$

then the corresponding diffeomorphism $\mu : M' \times M'' \to M$ is called a *decomposition of second kind of the homogeneous manifold* M into homogeneous submanifolds.

In order to see that in spite of the abundant collection of conditions the above concept is rather general a discussion for the case where a reductive structure exists seems to be useful. Let $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ be a reductive structure of a homogeneous manifold and $\mathfrak{k}' = \mathfrak{n}' \oplus \mathfrak{q}'$, $\mathfrak{k}'' = \mathfrak{n}'' \oplus \mathfrak{q}''$ reductive substructures such that $\mathfrak{m} = \mathfrak{n}' \oplus \mathfrak{n}''$ and $\mathfrak{n}', \mathfrak{n}''$ are invariant subspaces of the representation

$$T_e ad(h) : \mathfrak{g} \to \mathfrak{g}, \; h \in H.$$

Then $\mathfrak{g} = \mathfrak{n}' \oplus \mathfrak{n}'' \oplus \mathfrak{h}$ is called a *decomposition of the reductive structure.* A more special concept is obtained by stronger requirements. If the reductive substructures $\mathfrak{k}' = \mathfrak{n}' \oplus \mathfrak{q}'$, $\mathfrak{k}'' = \mathfrak{n}'' \oplus \mathfrak{q}''$ are such that $\mathfrak{k}', \mathfrak{k}''$ are ideals of $\mathfrak{g}$

and the decompositions

$$\mathfrak{g} = \mathfrak{k}' \oplus \mathfrak{k}'', \quad \mathfrak{h} = \mathfrak{q}' \oplus \mathfrak{q}''$$

into direct sums of ideals are valid then a *decomposition of the reductive structure into direct sum of reductive substructures* is said to be given. Such decompositions are obtained in case of naturally reductive homogeneous Riemannian manifolds ([4] II, 210—216).

The following theorem yields another justification for the concept of decomposition of the second kind.

THEOREM 4.3. *Let G be a connected Lie group, $H \subset G$ a closed connected subgroup and $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ a reductive structure of the homogeneous manifold $M = G/H$. If $G', G'' \subset G$ are closed connected subgroups such that their Lie algebras $\mathfrak{g}', \mathfrak{g}''$ with their reductive substructures $\mathfrak{g}' = \mathfrak{n}' \oplus \mathfrak{q}'$, $\mathfrak{g}'' = \mathfrak{n}'' \oplus \mathfrak{q}''$ define a decomposition of the reductive structure $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}$ into direct sum of these reductive substructures then $M' = \pi(G')$ and $M'' = \pi(G'')$ define a decomposition of the second kind of M provided the occurring sets are connected.*

Proof. The assumption that $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}''$ yields a decomposition into direct sum of reductive substructures implies that the following conditions are satisfied:

1. $\mathfrak{g} = \mathfrak{g}' + \mathfrak{g}'' + \mathfrak{h}$;
2. $[\mathfrak{g}'', \mathfrak{h} \cap \mathfrak{g}'] = \{0\} \subset \mathfrak{h} \cap \mathfrak{g}'$ and
   $[\mathfrak{g}', \mathfrak{h} \cap \mathfrak{g}''] = \{0\} \subset \mathfrak{h} \cap \mathfrak{g}''$;
3. $[\mathfrak{g}'', \mathfrak{h}] \cap \mathfrak{g}' = \{0\} \subset \mathfrak{h} \cap \mathfrak{g}'$ and
   $[\mathfrak{g}', \mathfrak{h}] \cap \mathfrak{g}'' = \{0\} \subset \mathfrak{h} \cap \mathfrak{g}''$;
4. $(\mathfrak{g}'' + \mathfrak{h}) \cap [\mathfrak{g}'', \mathfrak{g}'] = \{0\} \subset \mathfrak{h}$ and
   $(\mathfrak{g}' + \mathfrak{h}) \cap [\mathfrak{g}', \mathfrak{g}''] = \{0\} \subset \mathfrak{h}$;
5. $[\mathfrak{g}', \mathfrak{g}''] = \{0\} \subset \mathfrak{h}$.

Since the sets occurring are supposed to be connected the validity of these conditions implies that the conditions on decompositions of the second kind are satisfied too.

A reduction of the hypotheses in the above theorem is obviously possible.

Summary

A systematic account of some basic facts concerning reductive structures is given. In particular, the importance of reductive structures for classification of homogeneous submanifolds and for the decomposition of homogeneous manifolds is shown. Different kinds of decompositions are introduced and studied.

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