ON SOME METRICS AND CONNECTIONS IN THE TANGENT BUNDLE*

By

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K. YANO has given the global description of the relationships between a manifold M and its tangent bundle $T(M)$. He has defined derivational mappings of the algebra $\mathfrak{T}(M)$ of tensor fiels of a manifold M into the algebra $\mathfrak{T}(T(M))$ of tensor fields of the tangent bundle $T(M)$ of M. First the definition of the vertical lift and the complete lift [2], then that of the horizontal lift of tensor fields [3] have been given. In [4] metrics and connections in the tangent bundle have been considered.

In this paper three further metrics will be discussed, which can be defined on $T(M)$. The base manifold, the tensor fields and the affine connection will be supposed to be of class C^∞, further M will be assumed to be a connected Riemannian space. Notations and terminology of the quoted papers will be followed.

Let (x^i) $(i = 1, 2, \ldots, n)$ be a local co-ordinate system in a co-ordinate neighbourhood U in M and let π be the projection $T(M) \to M$. Denote the induced local co-ordinate system is $\pi^{-1}(U) \subset T(M)$ by (x^i, y^i) $(i = 1, 2, \ldots, n)$ or by (ξ^I) $(I = 1, 2, \ldots, n)$, where $\xi^i = x^i$ and $\xi^{\bar{i}} = \xi^{i+n} = y^i$. That is, if $\bar{x} = b^i \frac{\partial}{\partial x^i} \in T_x(M)$ and $x \in M$ is a point with co-ordinates $a^1, a^2, \ldots a^n$ with respect to (x^i), then $\bar{x}$ has co-ordinates $a^1, a^2, \ldots, a^n, b^1, b^2, \ldots b^n$ with respect to (x^i, y^i). If $\mathfrak{X} = X^i \frac{\partial}{\partial x^i}$ is a local vector field on M, then its vertical, complete and horizontal lifts in terms of the induced local co-ordinate system are

$$\begin{aligned} \mathfrak{X}^V &= X^i \frac{\partial}{\partial y^i} \\ \mathfrak{X}^C &= X^i \frac{\partial}{\partial x^i} + \frac{\partial X^i}{\partial x^j} y^j \frac{\partial}{\partial y^i} \\ \mathfrak{X}^H &= X^i \frac{\partial}{\partial x^i} - \Gamma^i_{js} y^j X^s \frac{\partial}{\partial y^i} \end{aligned} \tag{1}$$

* The results of this paper are taken from the author's doctoral dissertation, submitted to the Eötvös Loránd University in 1973.

respectively, where Γ^i_{jk} denote components of the affine connection defined on M with respect to a local co-ordinate system (x^i). Be $\frac{\partial X^i}{\partial x^j} y^j = \partial X^i$ and $\Gamma^i_{js} y^j = \Gamma^i_s$, then the local expressions for the various lifts of the vector field are in short:

$$\mathfrak{X}^V : \begin{bmatrix} 0 \\ X^i \end{bmatrix}, \quad \mathfrak{X}^C : \begin{bmatrix} X^i \\ \partial X^i \end{bmatrix}, \quad \mathfrak{X}^H : \begin{bmatrix} X^i \\ -\Gamma^i_s X^s \end{bmatrix}. \tag{1'}$$

Let G be a Riemannian metric on M that is a tensor field of type (0,2) with components with respect to (x^i) given by g_{ij} $(i, j = 1, 2, \dots n)$. It has been customary to write $ds^2 = g_{ij}\, dx^i\, dx^j$. The complete lift of G may be expressed in terms of the induced local co-ordinate system (x^i, y^i) of $T(M)$ by a $(2n \times 2n)$ matrix [2]:

$$G^C : \begin{bmatrix} \frac{\partial g_{ij}}{\partial x^k} y^k & g_{ij} \\ g_{ij} & 0 \end{bmatrix}. \tag{2}$$

In 1958 [5] S. Sasaki defined on the manifold $T(M)$ the Riemannian metric G^L as follows:

$$ds^2 = g_{ij}\, dx^i\, dx^j + g_{ij}\, \delta y^i\, \delta y^j \tag{3}$$

with respect to the lifted base vectors $e_i = \left(\frac{\partial}{\partial x^i}\right)^H$ $e_{\bar{i}} = \left(\frac{\partial}{\partial x^i}\right)^V$ $(\bar{i} = i + n)$, where $\delta x^i = (dx^i)^V$ and $\delta y^i = dy^i + \Gamma^i_{sm} y^s\, dx^m = (dx^i)^H$. Then G^L can be expressed by a $(2n \times 2n)$ matrix

$$\begin{bmatrix} g_{ij} & 0 \\ 0 & g_{ij} \end{bmatrix}. \tag{3'}$$

The components of this metric with respect to (x^i, y^i) are

$$\bar{G}_{IJ} : \begin{bmatrix} g_{ij} + \Gamma^s_i \Gamma^t_s g_{st} & \Gamma^s_i g_{sj} \\ \Gamma^s_j g_{si} & g_{ij} \end{bmatrix}. \tag{4}$$

The metric G^L satisfies the following conditions [4]:

$$\begin{aligned} G^L(X^H, Y^H) &= g(X, Y) \circ \pi, \\ G^L(X^V, Y^V) &= g(X, Y) \circ \pi, \\ G^L(X^H, Y^V) &= G^L(X^V, Y^H) = 0, \end{aligned} \tag{5}$$

where $g(X, Y)$ represents the inner product in M itself. S. Sasaki calculated the corresponding Christoffel symbols:

$$\begin{aligned}
&\bar{\Gamma}^q_{ij} = \frac{1}{2}(R^q_{0is}\Gamma^s_j + R^q_{0js}\Gamma^s_i) + \Gamma^q_{ij} \\
&\bar{\Gamma}^q_{\bar{i}j} = \frac{1}{2}R^q_{0ji} \\
&\bar{\Gamma}^q_{i\bar{j}} = \frac{1}{2}R^q_{0ij} \\
&\bar{\Gamma}^q_{\bar{i}\bar{j}} = 0 \\
&\bar{\Gamma}^{\bar{q}}_{ij} = \partial\Gamma^q_{ij} - \frac{1}{2}\Gamma^q_r(R^r_{0is}\Gamma^s_j + R^r_{0js}\Gamma^s_i) \\
&\bar{\Gamma}^{\bar{q}}_{\bar{i}j} = \Gamma^q_{ij} + \frac{1}{2}R^m_{0ij}\Gamma^q_m \\
&\bar{\Gamma}^{\bar{q}}_{i\bar{j}} = \Gamma^q_{ij} + \frac{1}{2}R^m_{0ji}\Gamma^q_m \\
&\bar{\Gamma}^{\bar{q}}_{\bar{i}\bar{j}} = 0
\end{aligned} \tag{6}$$

with respect to the induced local co-ordinate system (x^i, y^i) of $T(M)$, where $\bar{i} = i + n$ refer to y^i, $\Gamma^q_i = y^j\Gamma^q_{ji}$ and if R^q_{kij} denote the components of the curvature tensor then $R^q_{0ij} = y^k R^q_{kij}$.

In a similar manner as S. Sasaki did in (3′), we define three metrics on $T(M)$ by

$$\begin{bmatrix} 0 & g_{ij} \\ g_{ij} & 0 \end{bmatrix}, \begin{bmatrix} g_{ij} & g_{ij} \\ g_{ij} & 0 \end{bmatrix}, \begin{bmatrix} 0 & g_{ij} \\ g_{ij} & g_{ij} \end{bmatrix}$$

with respect to the basis e_i, $e_{\bar{i}}$.

Proposition 1. *Suppose a Riemannian metric on $T(M)$ to be given by*

$$ds^2 = g_{ij}\,dx^i\,\delta y^j + g_{ij}\,\delta y^i\,dx^j \tag{7}$$

or else by the matrix

$$\begin{bmatrix} 0 & g_{ij} \\ g_{ij} & 0 \end{bmatrix} \tag{7′}$$

with respect to the basis $e_i = \left(\frac{\partial}{\partial x^i}\right)^H$, $e_{\bar{i}} = \left(\frac{\partial}{\partial x^i}\right)^V$ $(\bar{i} = i + n)$. *Then this metric coincides with the complete lift of metric G defined on M.*

Proof. Substituting $\delta y^i = dy^i + \Gamma^i_{tm}y^t dx^m$ into (7), an expression of the form $ds^2 = \tilde{G}_{IJ}dx^I dx^J$ $(I, J = 1, 2, \ldots 2n)$ is obtained. In terms of the induced local co-ordinate system, the components $\tilde{G}_{IJ}$ are seen immediately to be:

$$\tilde{G}_{IJ} : \begin{bmatrix} g_{is}\Gamma^s_j + g_{sj}\Gamma^s_i & g_{ij} \\ g_{ij} & 0 \end{bmatrix} \tag{8}$$

where $\Gamma^s_j = y^t \Gamma^s_{tj}$. For the Riemannian connection $\nabla g = 0$ implies

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma^m_{ki} g_{mj} + \Gamma^m_{kj} g_{im}$$

therefore by (2) the matrix (8) coincides with the matrix of G^C.

The metric G^C has the following properties [4]:

$$\begin{aligned} &G^C(X^V, Y^V) = 0 \\ &G^C(X^V, Y^H) = G^C(X^H, Y^V) = g(X, Y) \circ \pi, \\ &G^C(X^C, Y^C) = ((g(X, Y))^C, \\ &G^C(X^H, Y^H) = 0 . \end{aligned} \tag{9}$$

Proposition 2. *The components of the torsion-free Riemannian connection of the metric G^C on the manifold $T(M)$ coincide with the components of the complete lift of the linear connection defined on the manifold M.*

Proof. Components of the connection are calculated with respect to the local co-ordinate system (x^i, y^i) of the manifold $T(M)$

$$\Gamma^Q_{IJ} = \frac{1}{2} G^{KQ} \left(\frac{\partial G_{KJ}}{\partial \xi^I} + \frac{\partial G_{IK}}{\partial \xi^J} - \frac{\partial G_{IJ}}{\partial \xi^K} \right) \tag{10}$$

where $I, J, K, Q = 1, 2, \ldots, 2n$. G_{IJ} is replaced by $\tilde{G}_{IJ}$ from (8), and G^{KQ} by the contravariant components of the metric G^C, obtained from $\tilde{G}_{JQ}\tilde{G}^{QK} = \delta^K_J$. Hence

$$\tilde{G}^{QK} : \begin{bmatrix} 0 & g^{qk} \\ g^{qk} & -\Gamma^q_s g^{sk} - \Gamma^k_s g^{sq} \end{bmatrix}, \tag{11}$$

where g^{qk} are the contravariant components of the metric given on the manifold M.

Using the expression for the components of the curvature tensor

$$R^q_{0ij} = \frac{\partial \Gamma^q_j}{\partial x^i} - \frac{\partial \Gamma^q_i}{\partial x^j} + \Gamma^m_j \Gamma^q_{im} - \Gamma^m_i \Gamma^q_{jm} \tag{12}$$

where $R^q_{0ij} = y^k R^q_{kij}$ and $\Gamma^q_i = y^j \Gamma^q_{ji}$.

For the indices $q, \bar{i}, j$ (10) becomes

$$2\tilde{\Gamma}^q_{\bar{i}j} = g^{kq} \left[\frac{\partial}{\partial y^i} g_{kj} - \frac{\partial}{\partial y^k} g_{ij} \right] = 0$$

and similarly

$$2\Gamma^{q}_{i\bar{j}} = g^{kq}\left[\frac{\partial}{\partial y^j} g_{ik} - \frac{\partial}{\partial y^k} g_{ij}\right] = 0 .$$

Substituting the components of the indices $\bar{q}$, $\bar{i}$, j into (10), we have

$$2\widetilde{\Gamma}^{\bar{q}}_{\bar{i}j} = g^{kq}\left[\frac{\partial}{\partial y^i}(g_{ks}\,\Gamma^s_j + g_{sj}\,\Gamma^s_k) + \frac{\partial}{\partial x^j} g_{ik} - \frac{\partial}{\partial x^k} g_{ij}\right] -$$

$$- \left(\Gamma^q_s g^{sk} + \Gamma^k_s g^{sq}\right)\left(\frac{\partial}{\partial y^i} g_{kj} - \frac{\partial}{\partial y^k} g_{ij}\right) = g^{kq}\left(g_{ks}\Gamma^s_{ji} + g_{sk}\,\Gamma^s_{ij}\right) = 2\Gamma^q_{ij}.$$

For the indices $\bar{q}$, i, $\bar{j}$ we obtain

$$2\widetilde{\Gamma}^{\bar{q}}_{i\bar{j}} = g^{kq}\left[\frac{\partial}{\partial x^i} g_{kj} + \frac{\partial}{\partial y^j}(g_{is}\,\Gamma^s_k + g_{sk}\,\Gamma^s_i) - \frac{\partial}{\partial x^k} g_{ij}\right] -$$

$$- (\Gamma^q_s\, g^{sk} + \Gamma^k_s\, g^{sq})\left[\frac{\partial}{\partial y^j} g_{ik} - \frac{\partial}{\partial y^k} g_{ij}\right] =$$

$$= g^{kq}(g_{kr}\,\Gamma^r_{ji} + g_{sk}\,\Gamma^s_{ij}) = 2\Gamma^q_{ij}\,,$$

Now $\widetilde{\Gamma}^q_{ij}$ will be calculated as follows:

$$2\widetilde{\Gamma}^q_{ij} = g^{kq}\left[\frac{\partial}{\partial x^i} g_{kj} + \frac{\partial}{\partial x^j}\, g_{ik} - \frac{\partial}{\partial y^k}(g_{is}\,\Gamma^s_j + g_{sj}\,\Gamma^s_i)\right] =$$

$$= g^{kq}(g_{ks}\,\Gamma^s_{ji} + g_{sk}\,\Gamma^s_{ij}) =$$

$$= 2\Gamma^q_{ij}\,.$$

In a similar way:

$$2\widetilde{\Gamma}^{\bar{q}}_{ij} = g^{kq}\left[\frac{\partial}{\partial x^i}(g_{ks}\,\Gamma^s_j + g_{sj}\,\Gamma^s_k) + \frac{\partial}{\partial x^j}(g_{is}\,\Gamma^s_k + g_{sk}\,\Gamma^s_i) -\right.$$

$$\left. - \frac{\partial}{\partial x^k}(g_{is}\,\Gamma^s_j + g_{sj}\,\Gamma^s_i)\right] +$$

$$+ (-\Gamma^q_s g^{sk} - \Gamma^k_s g^{sq})\left[\frac{\partial}{\partial x^i} g_{kj} + \frac{\partial}{\partial x^j} g_{ik} - \frac{\partial}{\partial y^k}(g_{im}\,\Gamma^m_j + g_{mj}\,\Gamma^m_i)\right] =$$

$$= g^{kq}\left[\left(\frac{\partial\Gamma^s_k}{\partial x^i} - \frac{\partial\Gamma^s_i}{\partial x^k} + \Gamma^r_k\,\Gamma^s_{ri} - \Gamma^r_i\,\Gamma^s_{rk}\right) g_{sj} +\right.$$

$$\left. + \left(\frac{\partial\Gamma^s_k}{\partial x^j} - \frac{\partial\Gamma^s_j}{\partial x^k} + \Gamma^r_k\,\Gamma^s_{rj} - \Gamma^r_j\,\Gamma^s_{rk}\right) g_{is}\right] +$$

$$+ g^{kq}\left(g_{kr}\,\Gamma^r_{si}\,\Gamma^s_j + g_{rk}\,\Gamma^r_{sj}\,\Gamma^s_i + \frac{\partial\Gamma^s_j}{\partial x^i} g_{ks} + \frac{\partial\Gamma^s_i}{\partial x_j} g_{sk}\right) -$$

$$- \Gamma^q_s g^{sk}\, 2g_{kr}\,\Gamma^r_{ij} =$$

$$= R^q_{0ij} + R^q_{0ji} + 2\Gamma^q_{ij}\,.$$

Summarizing

$$\overline{\Gamma}^q_{ij} = \widetilde{\Gamma}^{\bar q}_{i\bar j} = \widetilde{\Gamma}^{\bar q}_{\bar i j} = \Gamma^q_{ij},\ \widetilde{\Gamma}^{\bar q}_{ij} = \partial \Gamma^q_{ij}, \tag{13}$$
$$\widetilde{\Gamma}^q_{\bar i j} = \widetilde{\Gamma}^q_{i\bar j} = \widetilde{\Gamma}^q_{\bar i \bar j} = \widetilde{\Gamma}^{\bar q}_{\bar i \bar j} = 0\ .$$

These components coincide with the components of the connection ∇^C defined in [2] as:

$$\nabla^C_{\mathfrak{X}^C} \mathfrak{Y}^C = (\nabla_{\mathfrak{X}} \mathfrak{Y})^C.$$

Proposition 3. *Suppose the metric G^S on the manifold $T(M)$ to be given by*

$$ds^2 = g_{ij}\, dx^i\, dx^j + g_{ij}\, dx^i\, \delta y^j + g_{ij}\, \delta y^i\, dx^i \tag{14}$$

or else by the matrix

$$\begin{bmatrix} g_{ij} & g_{ij} \\ g_{ij} & 0 \end{bmatrix}. \tag{14'}$$

With respect to the basis $e_i = \left(\dfrac{\partial}{\partial x^i}\right)^H$, $e_{\bar i} = \left(\dfrac{\partial}{\partial x^i}\right)^V$. *Then the covariant and contravariant components of G^S in terms of the local co-ordinate system (x^i, y^i) are:*

$$\hat{G}_{IJ}\ ; \begin{bmatrix} g_{ij} + \partial g_{ij} & g_{ij} \\ g_{ij} & 0 \end{bmatrix} \tag{15}$$

and

$$\hat{G}^{QK}\colon \begin{bmatrix} 0 & g^{qk} \\ g^{qk} & -g^{qk} + \partial g^{qk} \end{bmatrix} \tag{16}$$

respectively, where $\partial g_{ij} = \dfrac{\partial g_{ij}}{\partial x^k} y^k$ *and* $\partial g^{qk} = \dfrac{\partial g^{qk}}{\partial x^i} y^i$.

Proof. The covariant components $\hat{G}_{IJ}$ can be calculated in a similar way as the $\widetilde{G}_{IJ}$ in Proposition 1, and the contravariant components $\hat{G}^{QK}$ from $\hat{G}_{JQ}\, \hat{G}^{QK} = \delta^K_J$.

Proposition 4. *The metric G^S defined on the manifold $T(M)$ has the following properties:*

$$\begin{aligned} &G^S(X^V, Y^V) = 0 \\ &G^S(X^V, Y^H) = G^S(X^H, Y^V) = g(X, Y) \circ \pi \\ &G^S(X^H, Y^H) = g(X, Y) \circ \pi \\ &G^S(X^C, Y^C) = g(X, Y) \circ \pi + (g(X, Y))^C. \end{aligned} \tag{17}$$

Proof. To prove (17) the left sides in a local coordinate system (x^i, y^i) of $T(M)$ will be calculated, e.g. the last inner product is the following:

$$[X^i \quad \partial X^i] \begin{bmatrix} g_{ij} + \partial g_{ij} & g_{ij} \\ g_{ij} & 0 \end{bmatrix} \begin{bmatrix} Y^j \\ \partial Y^j \end{bmatrix} = g_{ij} X^i Y^j + \partial(g_{ij} X^i Y^j)\,.$$

Making use of $f^C = f_i y^i$, $f \in \mathfrak{F}^0_0(M)$ (see [2]), leads to the stated result.

According to Proposition 4, in the metric G^S the vertical vectors are zero. The vertical lift of a vector is orthogonal to the horizontal lift of another if and only if the vectors themselves are orthogonal with respect to the metric (g_{ij}) in M. The length of a horizontal vector equals the length of the vector obtained by the projection π.

Proposition 5. *The components of the torsion-free Riemannian connection of the metric G^S on the manifold $T(M)$ are as follows:*

$$\begin{aligned} &\hat{\Gamma}^q_{ij} = \Gamma^q_{ij}, \quad \hat{\Gamma}^q_{\bar{i}j} = \hat{\Gamma}^q_{i\bar{j}} = \hat{\Gamma}^q_{\bar{i}\bar{j}} = 0\,, \\ &\hat{\Gamma}^{\bar{q}}_{ij} = \partial\Gamma^q_{ij}, \quad \hat{\Gamma}^{\bar{q}}_{\bar{i}j} = \hat{\Gamma}^{\bar{q}}_{i\bar{j}} = \Gamma^q_{ij}\,, \quad \hat{\Gamma}^{\bar{q}}_{\bar{i}\bar{j}} = 0. \end{aligned} \tag{18}$$

Proof. The components can be calculated from (10) similarly as in the proof of Proposition 2.

By Proposition 5, components of the connections of the metric G^S and G^C coincide, consequently the geodetics are the same in both cases.

Proposition 6. *Suppose the metric G^Z on the manifold $T(M)$ to be given by*

$$ds^2 = g_{ij}\, dx^i\, \delta y^j + g_{ij}\, \delta y^i\, dx^j + g_{ij}\, \delta y^i\, \delta y^j \tag{19}$$

or else by the matrix

$$\begin{bmatrix} 0 & g_{ij} \\ g_{ij} & g_{ij} \end{bmatrix} \tag{19'}$$

with respect to the basis $e_i = \left(\dfrac{\partial}{\partial x^i}\right)^H$, $e_{\bar{i}} = \left(\dfrac{\partial}{\partial x^i}\right)^V$. *Then the covariant and contravariant components of G in terms of the local co-ordinate system (x^i, y^i) are*

$$\breve{G}_{IJ}: \begin{bmatrix} \partial g_{ij} + \Gamma^s_i \Gamma^t_j g_{st} & g_{ij} + \Gamma^s_i g_{sj} \\ g_{ij} + \Gamma^s_j g_{is} & g_{ij} \end{bmatrix} \tag{20}$$

and

$$\breve{G}^{QK}: \begin{bmatrix} -g^{qk} & g^{qk} + \Gamma^k_s g^{sq} \\ g^{qk} + \Gamma^q_s g^{sk} & \partial g^{qk} - \Gamma^q_s \Gamma^k_t g^{st} \end{bmatrix} \tag{21}$$

respectively.

Proof. Similar to that of Proposition 3.

Proposition 7. *The metric G^Z defined on the manifold $T(M)$ has the following properties:*

$$\begin{aligned} &G^Z(X^V, Y^V) = g(X, Y) \circ \pi, \\ &G^Z(X^H, Y^V) = G^Z(X^V, Y^H) = g(X, Y) \circ \pi, \\ &G^Z(X^H, Y^H) = 0. \end{aligned} \tag{22}$$

Proof. Similar to that of Proposition 4.

By words, in the metric G^Z the length of a vertical vector equals the length of the vector obtained by the projection π. The horizontal lift of a vector is orthogonal to the vertical lift of another if and only if the vectors themselves are orthogonal in the metric (g_{ij}). Moreover, each horizontal vector is zero.

Proposition 8. *The components of the torsion-free Riemannian connection of the metric G^Z on the manifold $T(M)$ are as follows:*

$$
\begin{aligned}
\breve{\Gamma}^{q}_{ij} &= \Gamma^{q}_{ij} - \frac{1}{2}\left(R^{q}_{0it}\,\Gamma^{t}_{j} + R^{q}_{0jt}\,\Gamma^{t}_{i}\right)\\
\breve{\Gamma}^{q}_{\bar{i}j} &= \frac{1}{2} R^{q}_{0ij}\\
\breve{\Gamma}^{q}_{i\bar{j}} &= \frac{1}{2} R^{q}_{0ji}\\
\breve{\Gamma}^{q}_{\overline{ij}} &= 0\\
\breve{\Gamma}^{\bar{q}}_{ij} &= \partial\Gamma^{q}_{ij} + \frac{1}{2}\Gamma^{q}_{m}\left(\Gamma^{t}_{j}R^{m}_{0it} + \Gamma^{t}_{i}R^{m}_{0jt}\right) + \frac{1}{2}\left(R^{q}_{0it}\,\Gamma^{t}_{j} + R^{q}_{0jt}\,\Gamma^{t}_{i}\right)\\
\breve{\Gamma}^{\bar{q}}_{\bar{i}j} &= \Gamma^{q}_{ij} + \frac{1}{2}\left(R^{q}_{0ji} + \Gamma^{q}_{m} R^{m}_{0ji}\right)\\
\breve{\Gamma}^{\bar{q}}_{i\bar{j}} &= \Gamma^{q}_{ij} + \frac{1}{2}\left(R^{q}_{0ij} + \Gamma^{q}_{m} R^{m}_{0ij}\right)\\
\breve{\Gamma}^{\bar{q}}_{\overline{ij}} &= 0.
\end{aligned}
\tag{23}
$$

Proof. Similar to that of Proposition 2 by a somewhat longer calculation.

Summary

Three metrics are discussed which can be defined on the tangent bundle $T(M)$, where M is a connected Riemannian space.

Components of the torsion-free Riemannian connection of these metrics G^C, G^S and G^Z on the manifold $T(M)$ are calculated.

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