GENERALIZATION OF A THEOREM BY P. HARTMAN AND A. WINTNER

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Theorems of standard differential geometry are usually stated under strong sufficient conditions. In order to reduce conditions in the fundamental theorem of 3-dimensional space curves a modified treatment was given by P. HARTMAN and A. WINTNER [1]. Their result will be generalized in this paper for higher dimensions.

1. Basic facts and definitions

The foundation of differential geometry of curves in n-dimensional spaces was given by W. BLASCHKE [2].

The definitions of higher curvatures will be reviewed here as they were introduced by H. GLUCK [3] and [4] and earlier but in a less concise form by EGERVÁRY [5]. A bit of modification will be given here too by introducing signed higher curvatures.

Let V^n be the oriented n-dimensional Euclidean vector space formed by vectors of the n-dimensional Euclidean space E^n and $\Lambda^p(V^n)$ for $p = 1, \ldots, n$ the $\binom{n}{p}$-dimensional Euclidean vector space formed by the p-vectors over V^n with the inner product induced by that of V^n.

Let $\mathbf{x} = \mathbf{x}(\tau)$ be a vector-valued function which represents a curve $C \subset E^n$ in a well-known sense, i.e. $\mathbf{x} = \mathbf{x}(\tau) = \overrightarrow{OP}(\tau)$, where τ runs over an interval I of real numbers, $P(\tau)$ is the point on C which belongs to τ and $O \in E^n$ is a fixed point considered as the origin of the vectors.

The p-dimensional osculating subspace L_p to the curve at a point $P(\tau)$ is defined as spanned by the linearly independent derivative vectors $\mathbf{x}^{(i)}(\tau)$, $i = 1, \ldots, p$, and the p-th higher curvature $\varkappa_p$, as the measure of the rate of turning of the appropriate p-dimensional osculating subspaces with respect to the arc length where $p = 1, \ldots, n-1$.

This latter definition can be written therefore in the following form

$$\varkappa_p = \frac{1}{\|\mathbf{x}'(\tau)\|} \left\| \frac{d}{d\tau} \mathbf{n}_p^0 \right\|, \; p = 1, \ldots, n-1$$

where

$$\mathbf{n}_p^0 = \frac{\mathbf{x}'(\tau) \wedge \mathbf{x}''(\tau) \wedge \ldots \wedge \mathbf{x}^{(p)}(\tau)}{\|\mathbf{x}'(\tau) \wedge \mathbf{x}''(\tau) \wedge \ldots \wedge \mathbf{x}^{(p)}(\tau)\|}.$$

Evidently the above definitions work for $p = 1, \ldots, n-1$ if $\mathbf{x} = \mathbf{x}(\tau)$ is of class C^n and if the derivative vectors $\mathbf{x}^{(i)}(\tau)$ $(i = 1, \ldots, n-1)$ are linearly independent at every $\tau \in I$. The first assumption can, however, be replaced by a little more general one as the following lemma shows.

LEMMA. Let $\mathbf{x} = \mathbf{x}(\tau)$ be a vector-valued function of class C^{n-1} such that its derivatives $\mathbf{x}^{(i)}(\tau)$ for $i = 1, \ldots, n-1$ are linearly independent vectors at every $\tau \in I$. Assume further $\mathbf{v}_{n-1}$ to be differentiable at every $\tau \in I$, where $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is the so-called Frenet frame, which is obtained at τ by applying the Gram-Schmidt orthonormalization process to the linearly independent vectors $\mathbf{x}'(\tau), \mathbf{x}''(\tau), \ldots, \mathbf{x}^{n-1}(\tau)$ and the last element $\mathbf{v}_n$ is chosen so that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ form a right-hand orthonormal base for V^n. Then all the higher curvatures exist at every $\tau \in I$.

PROOF. The differentiability of $\mathbf{n}_p^0$ is evident for $p = 1, \ldots, n-2$. In order to compute $\varkappa_{n-1}$, the derivative vectors of $\mathbf{x}(\tau)$ will be given by linear combinations

$$\mathbf{x}^{(i)} = \sum_{j=1}^{i} \lambda_{ij} \mathbf{v}_j \quad \text{for} \quad i = 1, \ldots, n-1,$$

where $\lambda_{ii} > 0$ $(i = 1, \ldots, n-1)$ because of the uniqueness of the orthonormalization process.

On account of well-known properties of the exterior product it can be written:

$$\mathbf{n}_{n-1}^0 = \frac{\mathbf{x}' \wedge \mathbf{x}'' \wedge \ldots \wedge \mathbf{x}^{(n-1)}}{\|\mathbf{x}' \wedge \mathbf{x}'' \wedge \ldots \wedge \mathbf{x}^{(n-1)}\|} = \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \ldots \wedge \mathbf{v}_{n-1}.$$

On the other hand the differentiability of all $\mathbf{v}_j$ $(j = 1, \ldots, n)$ is easily seen to be assured by the assumptions of the lemma.

Thus

$$\varkappa_{n-1} = \frac{1}{\|\mathbf{x}'(\tau)\|} \left\| \frac{d}{d\tau} (\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \ldots \wedge \mathbf{v}_{n-1}) \right\|$$

exists as well and the lemma is proved.

In addition, the convenient expressions of the higher curvatures due to H. GLUCK will be applied. Considering the orthonormality relations among $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, the following derivational formulae (the so-called Frenet equations) hold at every

$\tau \in I$

$$\mathbf{v}_1' = \frac{c_2}{c_1} \mathbf{v}_2, \; \mathbf{v}_i' = -\frac{c_i}{c_{i-1}} \mathbf{v}_{i-1} + \frac{c_{i+1}}{c_i} \mathbf{v}_{i+1}, \text{ for } i = 2, \ldots, n-2$$

as well as

$$\mathbf{v}'_{n-1} = -\frac{c_{n-1}}{c_{n-2}}\mathbf{v}_{n-2} + c\mathbf{v}_n \quad \text{and} \quad \mathbf{v}'_n = -c\mathbf{v}_{n-1}$$

with suitably chosen coefficient c.

Notice that $c_i = \lambda_{ii} > 0$ $(i = 1, \ldots, n-1)$ holds in these equations.

Now let us differentiate $\mathbf{n}^0_p$. The validity of the ordinary rule for differentiating a product, the alternating character of the exterior multiplication and the above derivational formulae give:

$$\frac{d}{d\tau}\mathbf{n}^0_p = \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \ldots \wedge \mathbf{v}'_p = \frac{c_{p+1}}{c_p}(\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \ldots \wedge \mathbf{v}_{p-1} \wedge \mathbf{v}_{p+1})$$

for $p = 1, \ldots, n-2$ and

$$\frac{d}{d\tau}\mathbf{n}^0_{n-1} = \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \ldots \wedge \mathbf{v}'_{n-1} = c(\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \ldots \wedge \mathbf{v}_{n-2} \wedge \mathbf{v}_n).$$

The preceding observations yield now:

$$\varkappa_p = \left|\frac{1}{c_1}\cdot\frac{c_{p+1}}{c_p}\right|, \quad \text{where } p = 1, \ldots, n-2 \text{ and } \varkappa_{n-1} = \left|\frac{c}{c_1}\right|.$$

Consider now the quantities μ_p, $p = 1, \ldots, n-2$ defined by

$$\mu_p = \frac{1}{c_1}\frac{c_{p+1}}{c_p} = \varkappa_p > 0 \text{ and } \mu_{n-1} = \frac{c}{c_1}$$

as signed higher curvatures in accordance with the standard theory of 3-dimensional space curves.

In the case where the vector-valued function representing the curve is the parametrization by arc length, i.e. $c_1 = 1$ the coefficients in the Frenet equations are equal to the signed higher curvatures.

2. The generalized theorem

The following theorem is the generalization of a fundamental theorem of 3-dimensional space curves [1].

THEOREM. Let $k_1(s), k_2(s), \ldots, k_{n-2}(s)$ be on a closed interval I positive real-valued functions of class $C^{n-3}, C^{n-4}, \ldots, C^0$ respectively and $k_{n-1}(s)$ an arbitrary real-valued C^0-function on the same interval.

Then there exists a curve $C \subset E^n$ represented by $\mathbf{x} = \mathbf{x}(s)$ of class C^{n-1} on I for which s means arc-length and $k_1(s), k_2(s), \ldots, k_{n-1}(s)$ are the signed higher curvatures at every $s \in I$. The curve is uniquely determined up to an orientation preserving isometry of E^n.

Proof. It is a generalization of the classical one. The following system of linear differential equations

$$\mathbf{u}_1' = k_1(s)\,\mathbf{u}_2\,,$$
$$\mathbf{u}_j' = -k_{j-1}(s)\,\mathbf{u}_{j-1} + k_j(s)\,\mathbf{u}_{j+1} \text{ for } j = 2, \ldots, n-1$$

and

$$\mathbf{u}_n' = -k_{n-1}(s)\,\mathbf{u}_{n-1}$$

consists of n^2 equations for the co-ordinates $u_j^i, (i,j = 1, \ldots, n)$ after introducing a fixed right-hand orthonormal base $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ of V^n, i.e. $\mathbf{u}_j = u_j^1\mathbf{e}_1 + u_j^2\mathbf{e}_2 + \ldots + u_j^n\mathbf{e}_n$ holds for $j = 1, 2, \ldots, n$.

Let $\mathbf{W} = w_{ij}$ denote the n-rowed skew-symmetric matrix where $w_{ij} = k_i(s)$ for $j = i + 1$, $i = 1, \ldots, n-1$ and $w_{ij} = 0$ for $j = i + 2, \ldots, n$, $i = 1, \ldots, n-2$.

If $\mathbf{U}$ denotes the matrix whose consecutive columns are $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ then the above system of differential equations can be written in the simple matrix form

$$\mathbf{U}' = -\mathbf{UW} \text{ or } (\mathbf{U}^*)' = \mathbf{WU}^*$$

where asterisks denote transposition.

Let the initial conditions be chosen so that, at an initial value $s = s_0$ the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ form a right-hand orthonormal system. Let now $\mathbf{u}_j(s_0) = \mathbf{e}_j$ for $j = 1, \ldots, n$.

Since the function $\mathbf{UW}$ is continuous on a closed $(n^2 + 1)$-dimensional square domain where $s \in I$, $|u_j^i| \leq 1$ for $i, j = 1, \ldots, n$, the existence of a solution satisfying the given initial conditions is assured ([6], pp. 85—86).

On the other hand

$$(\mathbf{UU}^*)' = -\mathbf{UWU}^* + \mathbf{UWU}^* = 0\,.$$

Hence $\mathbf{UU}^* = \delta_{ij} = \text{const.}$ holds at every $s \in I$, that is, the vector-functions $\mathbf{u}_1(s), \mathbf{u}_2(s), \ldots, \mathbf{u}_n(s)$ satisfying the differential equations and the initial conditions form an orthonormal system at every $s \in I$.

According to the usual proof of the uniqueness let $\mathbf{u}_j(s)$ and $\tilde{\mathbf{u}}_j(s)$ be two solutions of the Frenet equations for which $\mathbf{u}_j(s_0) = \tilde{\mathbf{u}}_j(s_0)$ $(j = 1, \ldots, n)$.

It is easy to see that the derivative of the scalar-function $f(s) = \sum_{j=1}^{n} \langle \mathbf{u}_j(s), \tilde{\mathbf{u}}_j(s) \rangle$ is identically zero.

Since $f(s) = f(s_0) = n$, and $|\langle \mathbf{u}_j(s), \tilde{\mathbf{u}}_j(s) \rangle| \leq 1$ $\mathbf{u}_j(s) = \tilde{\mathbf{u}}j(s)$ holds for every $s \in I$ and $j = 1, \ldots, n$.

Let now $\mathbf{x}(s) = \int_{s_0}^{s} \mathbf{u}_1(s)\, ds$ be considered on I as a representation of a curve $C \subset E^n$ after having fixed the point O. The curve passes through O at the value s_0.

Evidently the arc-length between two points $P(s_1)$ and $P(s_2)$ on the curve is $| s_1 - s_2 |$.

Let us compute now the higher curvatures in points of C using vector-function representation. The higher derivatives of $\mathbf{x}(s)$ can be obtained recursively as linear combinations of $\mathbf{u}_1(s), \mathbf{u}_2(s), \ldots, \mathbf{u}_n(s)$.

Applying the given conditions for $k_j(s)$ and $\mathbf{u}_j(s)$ $(j = 1, \ldots, n)$ the following are deduced

$$\begin{aligned} \mathbf{x}' &= \mathbf{u}_1 \\ \mathbf{x}^{(r)} &= (\lambda'_{r-1,1} - k_1 \lambda_{r-1,2})\, \mathbf{u}_1 + \\ &+ \sum_{j=2}^{n-1} (\lambda'_{r-1,j} + k_{j-1} \lambda_{r-1,\, j-1} - k_j \lambda_{r-1,j+1})\, \mathbf{u}_j + \\ &+ (\lambda'_{r-1,n} + k_{n-1} \lambda_{r-1,\, n-1})\, \mathbf{u}_n \quad \text{for} \quad r = 2, \ldots, n-1 \end{aligned}$$

where λ_{ij} is the coefficient of $\mathbf{u}_j$ in the linear combination which gives $\mathbf{x}^{(i)}$.

Notice that $\lambda_{11} = 1$ and $\lambda_{ii} = k_1 k_2 \ldots k_{i-1} > 0$ for $i = 2, \ldots, n-1$.

Returning to the previous notations $\lambda_{ii} = c_i$ for $i = 1, \ldots, n-1$ the Frenet-frame can be made uniquely at every s since $\mathbf{x}(s)$ is of class C^{n-1} and the vectors $\mathbf{x}'(s), \mathbf{x}''(s), \ldots, \mathbf{x}^{(n-1)}(s)$ are linearly independent at $s \in I$.

Moreover $\mathbf{v}_j(s) = \mathbf{u}_j(s)$ holds for $j = 1, \ldots, n$.

Now the higher curvatures according to their definitions can be given by

$$\varkappa_p(s) = \left| \frac{1}{c_1} \cdot \frac{c_{p+1}}{c_p} \right| = k_p(s) \text{ for } p = 1, \ldots, n-2$$

and

$$\varkappa_{n-1}(s) = | k_{n-1}(s) |$$

or

$$\mu_p(s) = k_p(s) \quad \text{for} \quad p = 1, \ldots, n-1.$$

At last it is easy to see that a change of the initial conditions in the above differential equations represents only a motion, i.e. an orientation preserving isometry of E^n.

Summary

The fundamental theorem of 3-dimensional Euclidean space curves states that a curve can be uniquely determined up to an orientation preserving isometry by two prescribed functions $k_1(s)$ and $k_2(s)$ which are the curvature and the torsion of the curve, respectively. The weakening of the differentiability requirements on the functions $k_1(s)$ and $k_2(s)$ has been studied by P. HARTMAN and A. WINTNER. Their result has been generalized here for curves in higher dimensional Euclidean spaces.

References

1. HARTMAN, P., WINTNER, A.: On the fundamental equations of differential geometry. Amer. J. Math, 72 (1950), 757—774.
2. BLASCHKE, W.: Frenets Formeln für den Raum von Riemann. Math. Zeitschr. 6 (1920), 94—99.
3. GLUCK, H.: Higher curvatures of curves in Euclidean spaces I. Amer. Math. Monthly 73 (1966), 699—704.
4. GLUCK, H.: Higher curvatures of curves in Euclidean spaces II. Amer. Math. Monthly 74 (1967), 1049—1056.
5. EGERVÁRY, E.: Über die Kurven des n-dimensionalen euklidischen Raumes. Math. Term. Tud. Értesítő 59 (1940), 787—797.
6. KAMKE, E.: Differentialgleichungen I. Leipzig 1962.

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