ANALOGUE SIMULATION OF STOCHASTIC EXCITED NON-LINEAR VIBRATION SYSTEMS

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Movements of vibration systems are excited in various practical cases by a random effect (e.g. vehicles). If the system is linear and statistical characteristics of the random signal are known, the problem can be solved by using one of the procedures found in the literature ([1] to [6]).

Several authors examined non-linear vibration systems, too (e.g. [7] to [9]). Some of them described the solution of special problems (in general, single degree-of-freedom systems), while others gave methods for the determination of the equivalent linear system. Solution of the problem along this way is difficult and supplies only approximative results.

Reality can much better be approximated by modelling. In the case of deterministic excitation, non-linear systems with several degrees of freedom can well be examined by the help of an electronic analogue computer ([10] to [13]). Also in the case of a stochastic excitation, the analogue computer can be employed if signal $q(t)$ given by statistical characteristics can be realized in the form of an electric voltage. In the case of an excitation of this kind the output signal of the analogue computer, that is, the change of motion characteristics of the modelled non-linear system is similarly a stochastic function of time. The analogue computer permits also the statistical processing of simulation results.

In the followings the main steps of the computer modelling process elaborated at the Depertment of Engineering Mechanics, Technical University Budapest are presented. The utility of the method is shown on an example. Motion of the non-linear vibrating system of an autobus travelling on a road given by its statistical characteristics was examined.

1. Simulation of stochastic excitation effect by analogue computer

For the description and characterization of steady stochastic processes, the application of correlation functions is advisable [2]. The autocorrelation function of a determined set of signals $q(t)$ depends only on displacement time

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τ. For ergodic processes, the autocorrelation function is, by definition [2]

$$R_{qq}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T q(t) \cdot q(t+\tau)\, dt . \tag{1}$$

Its value for displacement time $\tau = 0$ is the square mean of the signal:

$$D = R_{qq}(0) = \overline{q^2(t)}.$$

In the theory of vibrations the mean value of steady signals is zero, consequently D is, at the same time, also the variance of the signal.

The autocorrelation function of a given steady, stochastic signal, determined by measurement, can well be approximated in most cases by the formula [5]:

$$R_{qq}(\tau) \approx D \sum_{k=1}^{n} A_k e^{-\alpha_k|\tau|} \cdot \cos \beta_k \tau . \tag{2}$$

Here D, A_k, α_k and β_k are quantities to be determined by measurement. The following auxiliary condition has also to be satisfied:

$$\sum_{k=1}^{n} A_k = 1 .$$

The more members are taken, the more accurate is the approximation. In practice (e.g. specifying quality of public roads) the specification of $n = 2$ or 3 members is considered as satisfactory [5]. In the case of individual members, factor β_k is zero, a simplification in our problem.

In modelling the excitation specified by its autocorrelation function an analogue signal $q(t)$ has to be produced the autocorrelation function of which is identical with, or close to that of the excitation to be modelled.

The random signal can be produced by a noise generator. Output voltage $u(t)$ of such devices is in general a limited white noise. This is not suited for directly exciting the model, since its autocorrelation function is mostly not identical with that of the signal to be produced.

If the signal of the noise generator is let to pass filters having transfer characteristics corresponding to members of autocorrelation function (2), and summarizing their output signals (Fig. 1), we obtain voltage $q(t)$ to be produced. In the figure $H_k(i\omega)$ is the transfer function of the filter producing the k-th member of the autocorrelation function. In the case of a linear transfer member, the correlation of power density spectra of output and input signals is given by formula [2]:

$$\Phi_{qq}(\omega) = |H(i\omega)|^2\, \Phi_{uu}(\omega) . \tag{3}$$

Power density spectrum of the white noise limited by lower and upper angular frequencies ω_a and ω_f, respectively, is constant in the range

$$\Phi_{uu}(\omega) = N^2 .$$

Its power, taking also the negative range into consideration, is:

$$P = 2\frac{1}{2\pi}\int_{\omega_a}^{\omega_f} N^2 \mathrm{d}\omega = \frac{N^2}{\pi}(\omega_f - \omega_a) \tag{4}$$

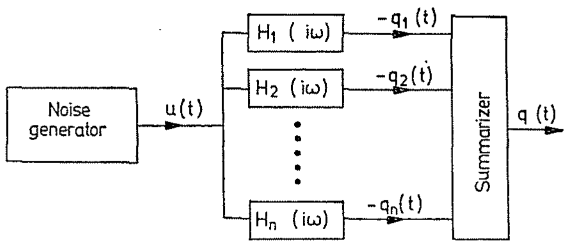

Fig. 1. Principle of modelling random signals

The momentary value of output voltage of the noise generator is $u(t)$. The square of the effective value of voltage of normal distribution is given by

$$\tilde{u}^2 = \lim_{T\to\infty}\frac{1}{T}\int_0^T u^2(t)\,\mathrm{d}t . \tag{5}$$

Equalizing (4) and (5):

$$N^2 = \pi\frac{\tilde{u}^2}{\omega_f - \omega_a} = \Phi_{uu} . \tag{6}$$

The k-th member of the autocorrelation function (2), for $\beta_k = 0$:

$$[R_{qq}(\tau)]_k = DA_k e^{-\alpha_k|\tau|} \tag{7}$$

Its power density spectrum ([2]):

$$[\Phi_{qq}(\omega)]_k = 2DA_k\frac{\alpha_k}{\alpha_k^2 + \omega^2} . \tag{8}$$

According to (3), (6), and (8), the transfer function of the filter member to be realized:

$$|H(i\omega)|_k^2 = \frac{4DA_k}{\tilde{u}^2}\,\frac{\omega_f - \omega_a}{2\pi}\,\frac{\alpha_k}{\alpha_k^2 + \omega^2} . \tag{9}$$

Solving the first-order inhomogeneous differential equation using Laplace transformation:

$$\dot{q}_k + a_k q_k = k_k u(t). \tag{10}$$

Denoting $s = i\omega$ and the transforms of variables by the same capitals, then

$$Q_k(i\omega) \cdot (i\omega + a_k) = k_k U(i\omega). \tag{11}$$

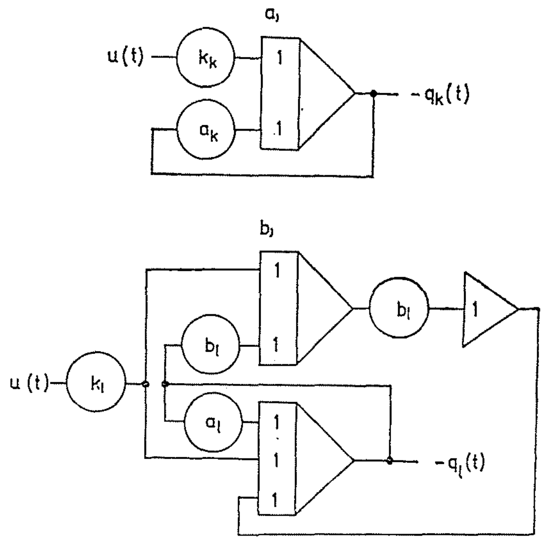

Fig. 2. Filter circuits to be adjusted on the analogue computer

The transfer function realized by the analogue program of the differential equation is:

$$H_k^*(i\omega) = \frac{Q_k(i\omega)}{U(i\omega)} = \frac{k_k}{a_k + i\omega}. \tag{12}$$

Namely:

$$|H_k^*(i\omega)|^2 = H_k^*(i\omega) \cdot H_k^*(-i\omega) = \frac{k_k^2}{a_k^2 + \omega^2} \tag{13}$$

For

$$a_k = \alpha_k \quad \text{and} \quad k_k = \frac{2}{\tilde{u}} \sqrt{DA_k\alpha_k \frac{\omega_f - \omega_a}{2\pi}},$$

(13) is exactly identical with relationship (9). The analogue program of differential equation (10) is shown in Fig. 2/a. As many connections are needed on the analogue computer as there are such members in autocorrelation function (2), missing β. If in the l-th member $\beta_l \neq 0$, then

$$[R_{qq}(\tau)]_l = DA_l e^{-\alpha_l|\tau|} \cdot \cos \beta_l \tau .$$

The deducible power density function is:

$$[\Phi_{qq}(\omega)]_l = 2DA_l \frac{\alpha_l(\alpha_l^2 + \beta_l^2 + \omega^2)}{(\alpha_l^2 + \beta_l^2)^2 + 2(\alpha_l^2 - \beta_l^2)\,\omega^2 + \omega^4} \tag{14}$$

Similarly as in preceding calculations, on the basis of the Laplace transform of the second-order differential equation

$$\ddot{q}_l + a_l\dot{q}_l + b_l^2 q_l = k_l(\dot{u} + c_l u) \tag{15}$$

the transfer function is:

$$H_l^*(i\omega) = \frac{Q_l(i\omega)}{U(i\omega)} = \frac{k_l(c_l + i\omega)}{b_l^2 - \omega^2 + i\omega a_l} .$$

Hence,

$$|H_l^*(i\omega)|^2 = \frac{k_l^2(c_l^2 + \omega^2)}{b_l^4 + (a_l^2 - 2b_l^2)\,\omega^2 + \omega^4} . \tag{16}$$

With coefficients in differential equation (15):

$$a_l = 2\alpha_l ; \quad b_l^2 = \alpha_l^2 + \beta_l^2 ; \quad c_l = \sqrt{\alpha_l^2 + \beta_l^2} = b_l$$

and

$$k_l = \frac{2}{\tilde{u}} \sqrt{DA_l\alpha_l \frac{\omega_f - \omega_a}{2\pi}} ,$$

the l-th member of excitation can be produced by computer connection shown in Fig. 2/b.

2. Analogue simulation of the road profile specified by statistical characteristics

The applicability of the process described in the preceding is demontsrated on an example. The motion of the non-linear vibrating system of an autobus was examined. Oscillations of the vehicle travelling with velocity v are generated by road irregularities of random distribution.

Public roads are built in lengths with the same technology. Thus the road can be considered as steady with respect to random irregularities. Road quality is practically time-independent, that is, a road of this kind can be taken as ergodical [14].

Results of measurements performed along different roads are approximated in general by the autocorrelation function distance displacement λ.

As an illustration cases were chosen, where this was an expression of two members at the maximum:

$$R_{qq}(\lambda) = D[A_1 e^{-\alpha_1|\lambda|} + A_2 e^{-\alpha_2|\lambda|} \cos \beta_2 \lambda] . \tag{17}$$

Values of constants in the formula are given in Table 1 with reference to sources.

Table 1

Coefficients in autocorrelation functions characterising the quality of modelled roads

$$R_{qq}(\lambda) = D[A_1 e^{-\alpha_1|\lambda|} + A_2 e^{-\alpha_2|\lambda|} \cos \beta_2 \lambda]$$

Serial number	Road surface	D cm²	A_1	α_1 1/cm	A_2	α_2 1/cm	β_2 1/cm	Source
1	Good quality concrete	0.25	1	0.0015	—	—	—	[14], [15]
2	Bad quality concrete	1.54	1	0.0015	—	—	—	[15]
3	Good quality asphalt	0.639	0.85	0.002	0.15	0.0005	0.0060	[14], [15]
4	Bad quality asphalt	1.44	—	—	1	0.0002	0.0044	[15]

Autocorrelation function (2) contains displacement time τ while (17) depends on distance vs. displacement λ. Relationship $\lambda = v\tau$, for a vehicle travelling with a velocity v, transforms (17) to:

$$R_{qq}(\tau) = D[A_1 e^{-\alpha_1 v|\tau|} + A_2 e^{-\alpha_2 v|\tau|} \cos \beta_2 v\tau] . \tag{18}$$

The output of our noise generator is, type NRG-201 made by RFT according to specifications, a white noise of effective voltage $\tilde{u} = 1$ V in the frequency range 5 Hz to 20 kHz, and deviation from the ideal spectrum is less than ± 0.5 dB.

In analogue modelling, a time scale corresponding to the character of the problem is selected. In examining the vibrating system of the autobus it is advisable to choose a time scale $M_t = 0.1$, representing a tenfold faster run. Frequency limits of the above mentioned noise generator, converted to our problem, are 0.5 Hz and 2000 Hz. This range contains natural frequencies of the vibrating system of the examined autobus.

In determining coefficients of differential equations (10) and (15), bus velocity v, the chosen time scale, further the fact that the time constant of integrators in the "rapid" mode of operation of the electronic analogue computer Type MEDA 81 is 0.005 s, have to be taken into consideration. Displacement scale 1 V/cm chosen in the example does not affect numerical values of the coefficients.

Coefficients calculated in accordance with the aforesaid are given in Table 2.

Table 2

Coefficients for the program of modelled roads

Serial number	Road surface	v km/h	Fig. 2a k_k	Fig. 2a a_k	Fig. 2b k_l	Fig. 2b a_l	Fig. 2b b_l
1	Good quality concrete	10	4.564	0.417	—	—	—
		25	7.216	1.042	—	—	—
		40	9.127	1.667	—	—	—
		55	10.703	2.292	—	—	—
2	Bad quality concrete	10	11.328	0.417	—	—	—
		25	17.910	1.042	—	—	—
		40	22.653	1.667	—	—	—
		55	26.564	2.292	—	—	—
3	Good quality asphalt	10	7.768	0.556	1.632	0.278	1.672
		25	12.282	1.389	2.580	0.694	4.181
		40	15.535	2.222	3.263	1.111	6.690
		55	18.216	3.056	3.826	1.528	9.198
4	Bad quality asphalt	10	—	—	4.000	0.111	1.223
		25	—	—	6.324	0.278	3.059
		40	—	—	8.000	0.444	4.894
		55	—	—	9.380	0.611	6.729

3. Analogue model of the vibration system of the examined bus

The aim of the present paper is to present the random excitation of non-linear vibration systems, thus, the model of the bus in the example, its motion equations and the analogue program are described only in their main lines.

In [11] and [16] it is proved in detail that the body of the local traffic bus shown in Fig. 3/a and considered to be a rigid body can be represented by three rigidly coupled point-like masses, arranged in the plane of motion (Fig. 3/c). In the case of road vehicles mass M_s reduced to the centre of gravity represents only 5 to 7% of the total mass of the body, thus the error caused by omitting M_s representing the coupling of the front and rear vibration systems can be practically neglected.

Error can be further reduced by distributing spring mounted mass in inverse proportion with the distances between centres of gravity (Fig. 3/d). Accordingly the front and rear parts of the bus are represented by two systems of two degrees of freedom, differing only by data. Subscript "e" refers to the front carriage, "h" to the rear one. Omitting these subscripts, motion equations of the two systems are:

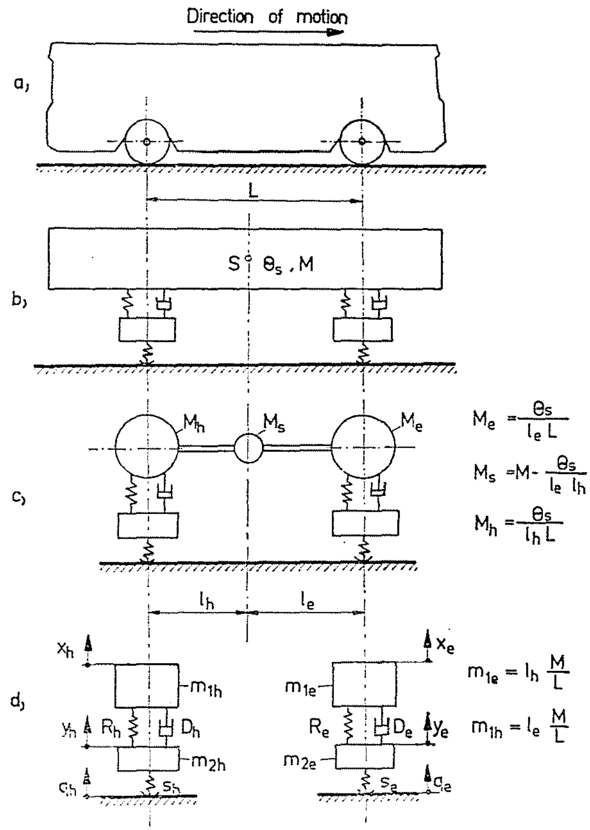

Fig. 3. Deduction of the vibration model for the bus

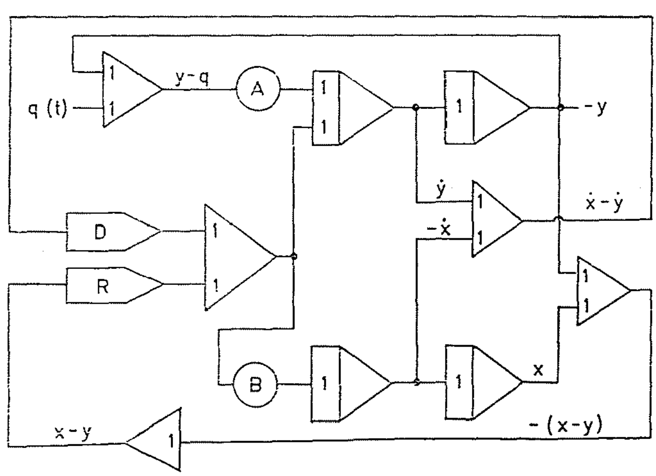

Fig. 4. Analogue program of the vibration system of the bus

$$\left.\begin{aligned} m_1\ddot{x} + D[\dot{x} - \dot{y}] + R[x - y] &= 0 \\ m_2\ddot{y} - D[\dot{x} - \dot{y}] - R[x - y] + sy &= sq(t). \end{aligned}\right\} \tag{19}$$

Here m_1 suitably reduced mass of the body;
m_2 unsprung mass of the carriage;
x and y vertical displacement of masses with respect to static equilibrium position;
$D[\dot{x} - \dot{y}]$ damping force, a non-linear function of relative velocity;
$R[x - y]$ variabale component of spring force (on account of air spring, non-linear function of elongation);
s resultant spring stiffness of tires considered as linear springs;
$q(t)$ road excitation.

The analogue program of motion equations (19) is shown in Fig. 4. Values for coefficients A and B depend on data of the system, further on the chosen time scale. Non-linear characteristics of the damper and of springs were adjusted on function generators D and R.

The results of a simulation are shown in Fig. 5, plotted by an UV Recorder. Statistical processing of records of this type is very time consuming and inaccurate. It is more advisable to process electric signals directly, as described in the next item.

4. Statistical processing of the outputs of computer simulation

The great advantage of modelling by an analogue computer is that output data of the system, i.e. electric voltages proportional to motion characteristics are available continuously. In the case of a random excitation, signals are also changing at random.

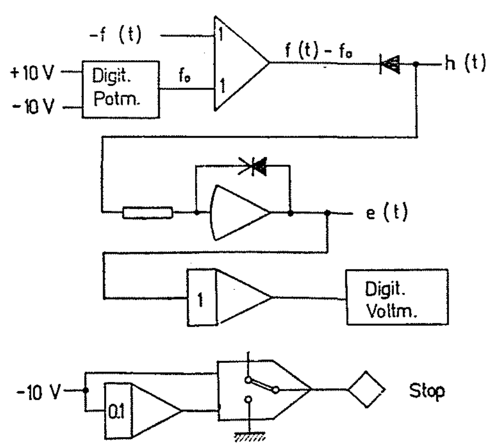

Fig. 6. Program of the statistical processing of the analogue signal

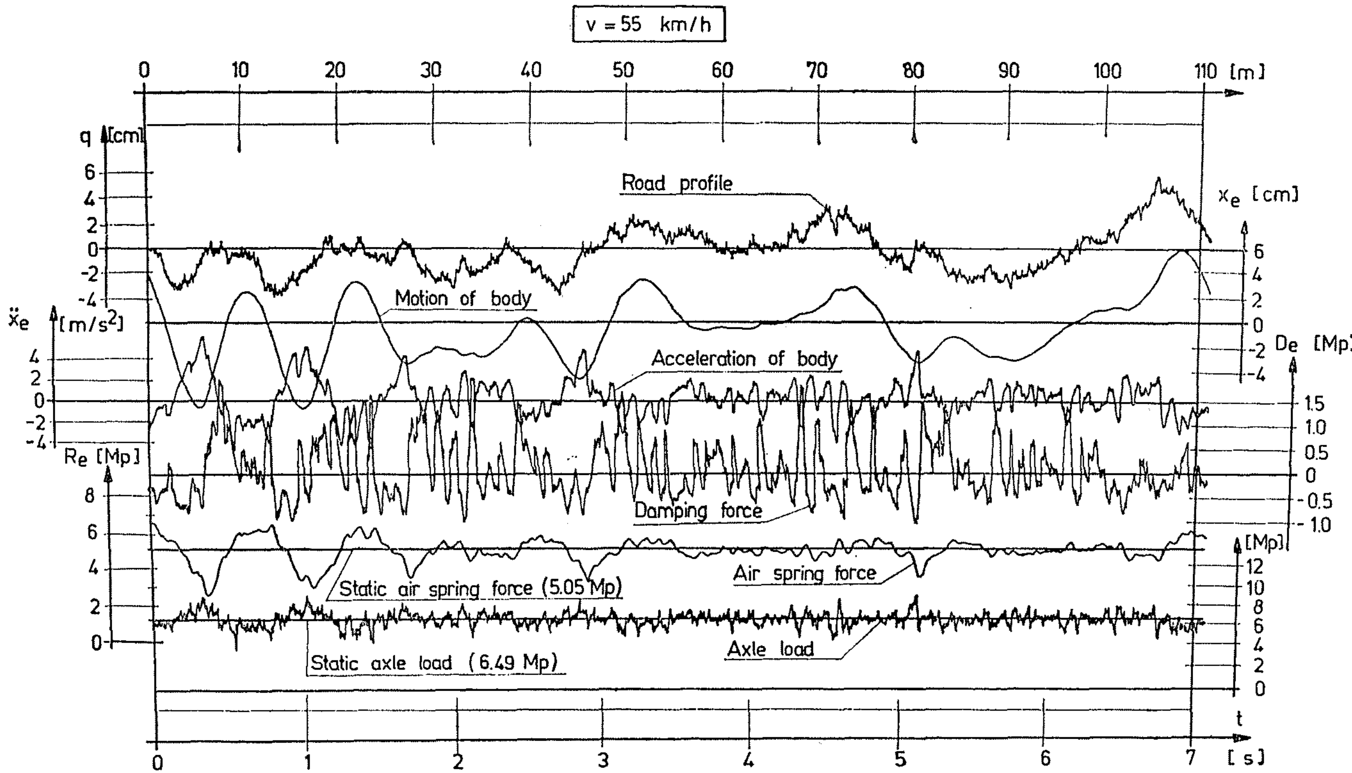

Fig. 5. Motion characteristics fo the front part of a loaded bus travelling on a good quality asphalt road

Let $f(t)$ designate the probability variable, that is the time function of the signal to be processed. Determine its probability distribution i.e., the probability of $f(t)$ assume values not higher than f_0. For this purpose the analogue circuit shown in Fig. 6 was elaborated.

The first adder forms the difference of $f(t)$ and of voltage f_0 adjusted on the digital potentiometer. The operational amplifier connected after the diode and fed back by Zener diode produces rectangular pulses of voltage -1 V. By integrating this, we obtain the probability $P[f(t) < f_0]$. The digital

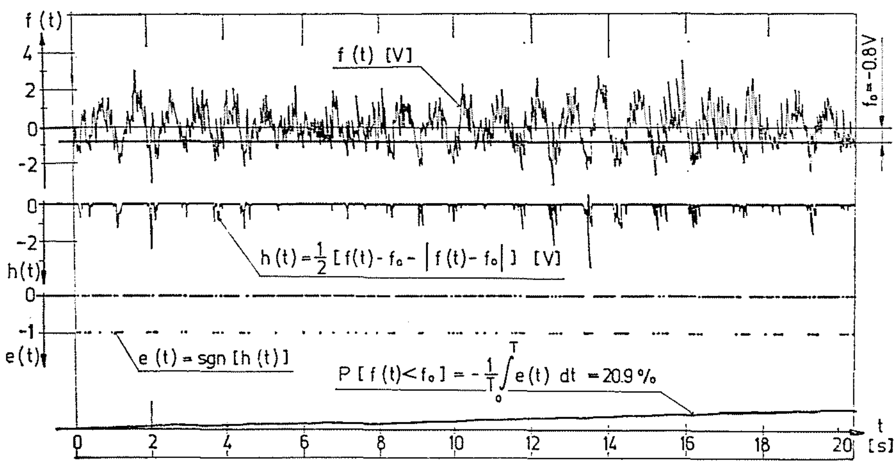

Fig. 7. Formation of signals in the course of statistical processing

voltmeter indicates its percentages, since the auxiliary circuit shown at the lower part of Fig. 6 interrupts integration after a time of 10 s. In accordance with the chosen time scale, this represents a period of 100 s, permitting a vehicle travelling e.g. at a speed of 55 km/h to do 1528 m.

The change vs. time of the individual signals of the circuit described above is shown in Fig. 7.

According to measurement experience [17], vibration characteristics follow the normal distribution. Accordingly it is advisable to construct examination results on a Gaussian plane. In the case presented in Fig. 8, the loaded bus was travelling on a good quality asphalt road. Random distribution functions pertaining to the vertical acceleration of the spring supported mass of the front oscillating system can be well approximated by straight lines. Separate examination points may deviate from the theoretical straight line, nevertheless a statistical hypothesis test [18] demonstrated that the distribution of oscillation accelerations pertaining to the given velocities can be considered as normal even in this case.

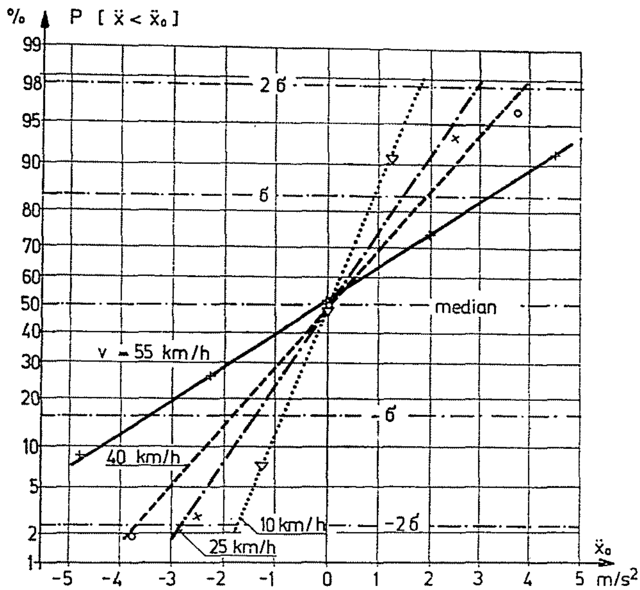

Fig. 8. Distribution functions for accelerations at the front part of a loaded bus traveling on a good quality asphalt road

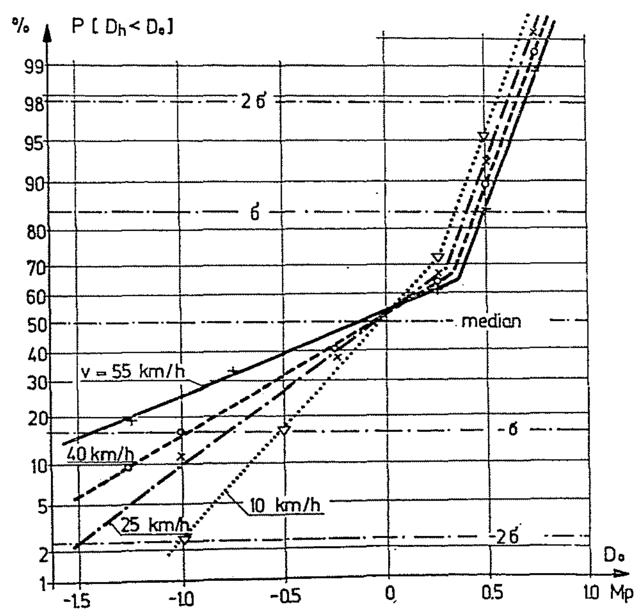

Fig. 9. Distribution functions for damping force, if the loaded bus travels on a good quality asphalt road

As mentioned, the system of the examined bus is non-linear. Especially force-velocity characteristics of vibration dampers built into the bus are deviating from the linear. Distribution function pertaining to damping force is shown in Fig. 9. Directional tangents of the straight lines of normal distributions corresponding to tension and compression strokes, respectively, differ from each other. A detailed analysis of the problem goes beyond the scope of the present paper. Results of the complete examination are given in [19].

It can be stated in summary that non-linear systems excited by random processes can truly be modelled by electronic analogue computers. To this end the autocorrelation function of the exciting effect has to be known in a form approximated by an exponential or harmonic function. Results of such examinations can be successfully used in design work.

Summary

Vibrating systems are excited in many practical cases by random effects. For the examination of non-linear systems the method of statistical linearization is generally used. Reality can much better be approximated by modelling the phenomenon on an analogue computer.

The paper presents the way of the analogue realization of the steady random signal given by its autocorrelation function. The authors elaborated a method also for the statistical processing of the resulting analogue signal.

Applicability of the developed methods is demonstrated on the example of the non-linear system of a bus travelling on a road given by its statistical characteristics.

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