EXAMINATION OF UNSTEADY HEAT CONDUCTION IN CASE OF VARYING TRANSPORT FACTORS

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Introduction

The investigations performed at our department on the unsteady period of drying, the unsteady processes of simultaneous heat and mas transfer directed our attention to the examination of heat conduction with varying boundary conditions.

Studies on heat transmission examine the involved processes by some particular solution of the general differential equation of heat conduction. One of the most important cases for the technical practice is the boundary condition of the third kind, when heat flux on the body surface is a linear function of temperature difference between the surface and the environment, assuming a constant heat transfer coefficient. This is however seldom the case.

In the most frequent case of convective heat transfer the coefficient is not constant but it is a function of the place.

Because of this, the solution of the differential equation of heat conduction was examined for "modified third kind boundary conditions" characterized by varying heat transfer coefficient as well as by varying simultaneous heat and mass transfer factors.

The two-dimensional parabolic partial differential equation of the phenomenon was solved by numerical methods, on a digital computer.

In addition measurements were made on heat transfer in a gypsum plate, and on drying of a wet gypsum plate and a wet cellulose plate.

Heat conduction characterized by varying boundary conditions

Differential equation of heat conduction in a space without source of heat, and having constant material characteristics is as follows:

$$\frac{\partial t}{\partial \tau} = a \nabla^2 t, \tag{1}$$

Its solution requires to know the boundary conditions describing the relation of the test body to the enviroment. The most common boundary condition without heat source:

$$\alpha(t_f - t_\infty) = -\lambda(\text{grad}_n t)_f \tag{2}$$

The heat transfer coefficient α is considered to be constant. In fact, in convection heat transfer α depends from the co-ordinates of the place, as illustrated by the following criterial equations.

In case of free convection along a smooth plane [1]:

$$Nu = C(Gr\,Pr)^n\,, \tag{3}$$

where

$$Nu = \frac{\alpha y}{\lambda} \quad \text{and} \quad Gr = \frac{g\beta \Delta t y^3}{\nu^2}\,.$$

In case of a heated vertical plate and in air this has the form [2]:

$$\alpha_y = 0.58 \frac{\lambda}{\sqrt[4]{y}} \, \frac{g(T_f - T_\infty)}{4\nu^2 T_\infty} \tag{4}$$

In case of forced laminar stream along a smooth plane [3]:

$$Nu_{\text{local}} = 0{,}332\, Pr^{1/3}\, Re_y^{1/2} \tag{5}$$

where

$$Re_y = \frac{v_\infty y}{\nu}\,.$$

In turbulent stream along a smooth plane [4]:

$$Nu_{\text{local}} = 0.0296\, Re_y^{0,8}\, Pr^{0,43} \left(\frac{Pr}{Re_f}\right)^{0,25}. \tag{6}$$

The heat transfer coefficient is seen from (3) to (6) to be a function of the distance from the beginning of the boundary layer.

The dependence, simplified:

$$\alpha_{\text{local}} = K_y^n$$

where

$$-0.2 > n > -0.5\,.$$

The place dependence is the highest for laminar forced flow.

Remind, the heat transfer coefficient depends on the surface temperature too [5]. Excessive complexity of the problem prevented the modifying effect of temperature on the transport coefficients from being investigated.

As inital condition for differential equation (1) uniform inital body temperature has been stipulated.

Heat transfer coefficient varying in flow dirention according to (7) produces in the body a temperature distribution varying in flow direction, therefore in (1):

$$\frac{\partial t}{\partial y} \neq 0 .$$

heat conduction cannot be regarded as linear.

The body can be assumed, however, to be a smooth plate of thickness 2X, and its third dimension (perpendicular to the flow) to be infinite. Thus

$$\frac{\partial t}{\partial z} = 0 .$$

Accordingly the phenomenon can be described by a two dimensiona differential equation:

$$\frac{\partial t}{\partial \tau} = a \left(\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} \right) .$$

For general validity, and mathematical simplicity, its dimensionless form will be applied:

$$\frac{\partial \vartheta}{\partial Fo} = \frac{\partial^2 \vartheta}{\partial \xi^2} + \frac{\partial^2 \vartheta}{\partial \psi^2} . \tag{8}$$

For the plane plate in Fig. 1 the dimensionless form of the initial condition is:

$$\vartheta_0 = 1 \quad \text{for} \quad Fo = 0 \ (\tau = 0).$$

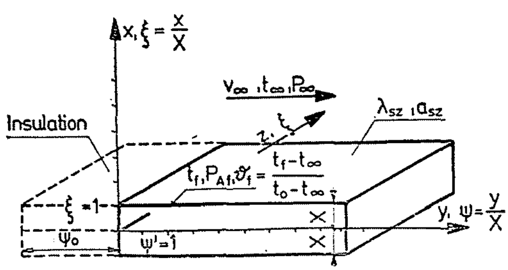

Fig. 1

The surface $\psi = 0$ of our specimen is perfectly heat insulated:

$$\left(\frac{\partial \vartheta}{\partial \psi}\right)_{\psi=0} = 0 \, .$$

Our specimen is symmetrical about the plane $\xi = 0$, consequently it is sufficient to accomplish the calculations in the range $0 \leq \xi \leq 1$ in arbitrary plane section $\xi =$ const. Owing to symmetry:

$$\left(\frac{\partial \vartheta}{\partial \xi}\right)_{\xi=0} = 0 \, .$$

The boundary condition for planes $\xi = 1$ and $\xi = -1$ of the specimen:

$$Bi(\psi)\, \vartheta_f = - \left(\frac{\partial \vartheta}{\partial \xi}\right)_{\xi=1} \tag{9}$$

Where, taking into consideration (5):

$$Bi(\psi) = 0.332 \, \frac{\lambda_g}{\lambda_{sz}} \, Pr^{1/3} \, \frac{v_\infty X}{\nu}^{1/2} \psi^{-1/2}. \tag{10}$$

The first part of the formula (10) is a function of the Reynolds number for plate half-thickness and it is constant for a given specimen and in given experimental circumstances:

$$k_a = k_a(Re_X) = 0.332 \, \frac{\lambda_g}{\lambda_{sz}} \, Pr^{1/3} \, Re_X^{1/2} \tag{11}$$

Taking this into consideration, simplified form of (10) is:

$$Bi(\psi) = k_a \psi^{-1/2} \, . \tag{12}$$

If momentum boundary layer begins by a distance before the thermal boundary layer, (5) and (12) modify [3]:

$$Nu' = 0.332 \, Pr^{1/3} Re^{1/2} \frac{1}{\sqrt[3]{1 - \left(\frac{y_0}{y}\right)^{3/4}}} \tag{5a}$$

$$Bi'(\psi) = \frac{k_a}{\sqrt[3]{1 - \left(\frac{\psi_0}{\psi}\right)^{3/4}}} \, \psi^{-1/2} \, . \tag{12a}$$

Variation of the boundary condition in case of simultaneous heat and mass transfer

In case of drying a wet solid, the heat flow is paralelled by a mass flow on the surface. The wetting agent evaporates, modifying the heat flux. The calculation of the inital temperature gradient in the drying material can however, be reduced to that of unsteady heat conduction [6]: the boundary condition of the third kind (2) will be of the form:

$$(\alpha + k_{AG} n M_A r_{An})(t_n - t_f) = -\lambda_{sz} \left(\frac{\partial t}{\partial x}\right)_f,$$

or in a dimensionless form:

$$Bi_n(\psi)\,\vartheta_{fn} = -\left(\frac{\partial \vartheta_n}{\partial \xi}\right)_{\xi=1}$$

where

$$Bi = \frac{\alpha + \alpha_M}{\lambda_{sz}} X .$$

$$\alpha_M = k_{AG} n M_A r_{An},$$

$$\vartheta_{fn} = \frac{t_f - t_n}{t_0 - t_n},$$

In the initial phase of the drying when evaporation of free humidity can be assumed, the temperature of the body tends to the limit temperature t_n (temperature of wet bulb thermometer).

Numerical value of Bi_n can be calculated by a formula similar to (10) by analogy to heat and mass transfer:

$$Bi_n(\psi) = \frac{0.332}{\lambda_{sz}} \sqrt{\frac{v_\infty X}{\nu}} \left(\lambda_g Pr^{1/3} + \frac{n M_A r_A}{RT} \frac{P}{P_{BM}} Sc^{1/3} D_{AG}\right) \psi^{-1/2} . \quad (13)$$

Solution of the differential equation of heat conduction by a numerical method

Analytical solutions of linear heat conduction are well-known and found in manuals. A solution for the two-dimensional heat conduction of the form of (8) is found in the literature only for particular cases, but no general one is known. Accordingly, a computerized solution has been attempted by numerical methods, for the more general, dimensionless formula.

For a numerical solution of partial differential equation (8) the test domain of a plane section ξ = constant was meshed and calculations applied

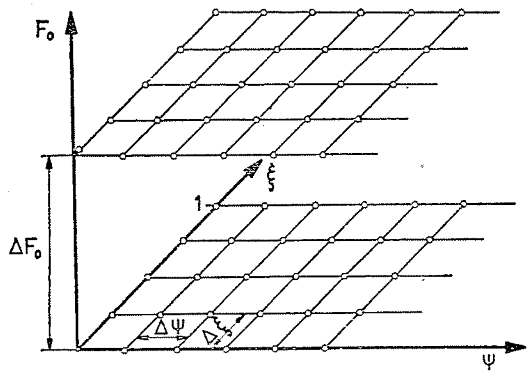

Fig. 2

the gridpoint values. A temperature distribution by ΔFo (time $\Delta\tau$ later) has a point mesh, shifted by ΔFo Fig. 2. Derivatives in (8) will be replaced by the difference quotients formed advancing differences:

$$\frac{\partial\vartheta}{\partial\xi} \simeq \frac{\Delta\vartheta_j}{\Delta\xi} = \frac{\vartheta_{i+1} - \vartheta_j}{g},$$

$$\frac{\partial\vartheta}{\partial Fo} \simeq \frac{\Delta\vartheta_i}{\Delta Fo} = \frac{\vartheta_{i+1} - \vartheta_i}{f},$$

$$\frac{\partial^2\vartheta}{\partial\xi^2} \simeq \frac{\Delta^2\vartheta_i}{\Delta\xi^2} = \frac{\Delta\vartheta_j - \Delta\vartheta_{j-1}}{\Delta\xi^2} = \frac{\vartheta_{j+1} - 2\vartheta_j + \vartheta_{j-1}}{g^2},$$

$$\frac{\partial^2\vartheta}{\partial\psi^2} \simeq \frac{\Delta^2\vartheta_k}{\Delta\psi^2} = \frac{\vartheta_{k+1} - 2\vartheta_k + \vartheta_{k-1}}{h^2}.$$

Suitable steps are:

$$g = h,$$

$$f = \frac{1}{4}g^2.$$

Transforming the differential equation (8) we get the following simple, explicit formula:

$$\vartheta_{i+1,j,k} = \frac{1}{4}(\vartheta_{i,j+1,k} + \vartheta_{i,j-1,k} + \vartheta_{i,j,k+1} + \vartheta_{i,j,k-1}) \tag{14}$$

The steps chosen to $f = 1/4\ g^2$ and $g = h$ provide stability to the calculation.

Derivate in boundary condition (9) is calculated by using three first terms in the Taylor series (15) derived with difference quotients, (16). This way the mutilation error arising from the boundary condition will equal the mutilation error in formula (14):

$$(y')_j = \frac{1}{\Delta x}\left(\Delta + \frac{1}{2}\Delta^2 - \frac{1}{6}\Delta^3 + - \dots\right) y_{j-1}\,, \tag{15}$$

$$\left(\frac{\partial \vartheta}{\partial \xi}\right)_j \cong \frac{1}{g}\left(\frac{4}{3}\vartheta_j - \frac{3}{2}\vartheta_{j-1} + \frac{1}{6}\vartheta_{j-3}\right). \tag{16}$$

Using (16) surface point values boundary condition (9) yields:

$$\vartheta_{i,m,k} = \frac{3/2\vartheta_{i,m-1,k} - 1/6\vartheta_{i,m-3,k}}{4/3 + g \cdot Bi_k}. \tag{17}$$

At $\xi = 0$

$$\left.\frac{\partial \vartheta}{\partial \xi} = 0\,.\right|_{\xi = 0}$$

Here ϑ values can be calculated by way of symmetry. Taking into consideration $\vartheta_{i,-1,k} = \vartheta_{i,1,k}$ and (14), yields $\vartheta_{i,0,k}$.

Just as for the symmetry points, the temperature gradient will be zero on the perfectly insulated butt ends:

$$\left(\frac{\partial \vartheta}{\partial \psi}\right)_{\psi=0} = 0 \quad \text{and} \quad \left(\frac{\partial \vartheta}{\partial \psi}\right)_{\psi=\text{final}} = 0\,.$$

These boundary conditions have been computed as the points in the symmetry: (14) has been computed from temperature values of symmetric points extrapolcted beyond the boundary.

The first surface point was obtained by parabolic extrapolation from adjacent surface values.

(14) and (17) provided a programable, speedy to run procedure. They have only one inconvenience, namely on account of the stability the stipulation $f = 1/4\ g^2$ requires to apply very little time steps.

As a remedy it was realized that change of temperature was rapid first both in time and in space later on it slowed down. Therefore a very dense mesh was applied from the onset: the calculation began with a grading dividing at the half thickness of the plate into 20 parts. As soon as the rate of change diminished, a sparser grading was sufficient. For this reason, gradually every second grid point was omitted to change: first to a division to 10 parts, then to 5 parts, with accordingly larger steps f.

Flow chart of the computer program for this mathematical model is shown in Fig. 3.

Long-term (Fo > 10) computations even by gradually increasing the steps run for hours on the computer ODRA 1204 of the Faculty of Mechanical

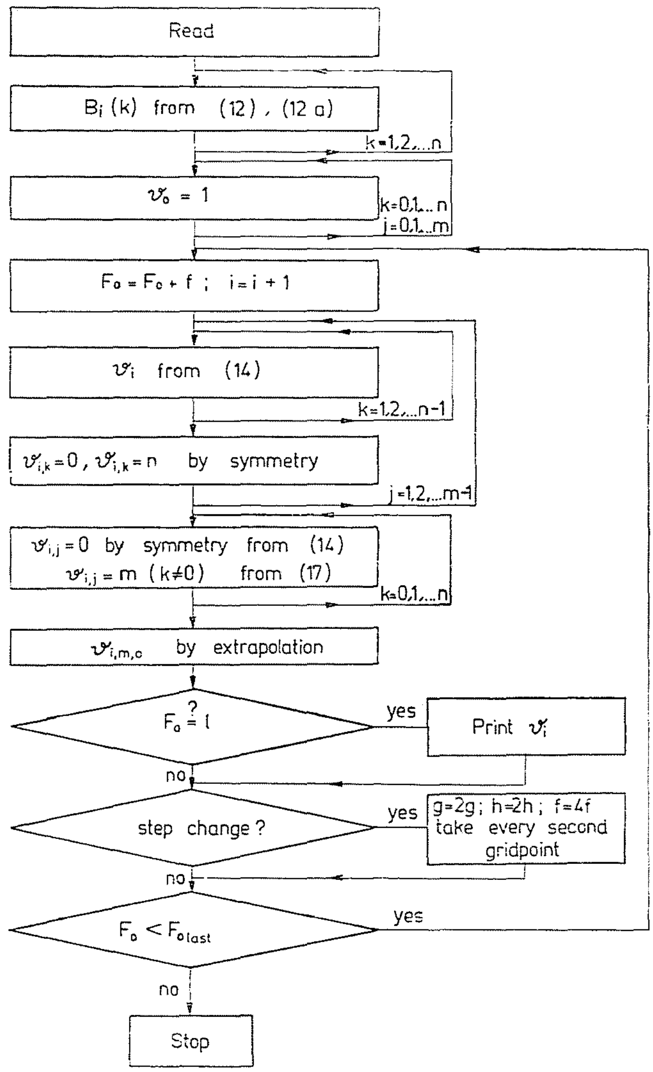

Fig. 3. Flow of the program based on explicit formula (14)

Engineering. In such cases it is more suitable to apply an implicit method, allowing larger steps f.

The "alternating direction implicit method" by PEACEMAN and RACHFORD [7] seemed to be very favourable to the solution of our parabolic partial differential equation.

Fig. 4 presents the flow chart of our program based on this method, involving Fo $>$ 10 calculations. (For details and formulae see [7] and [9]).

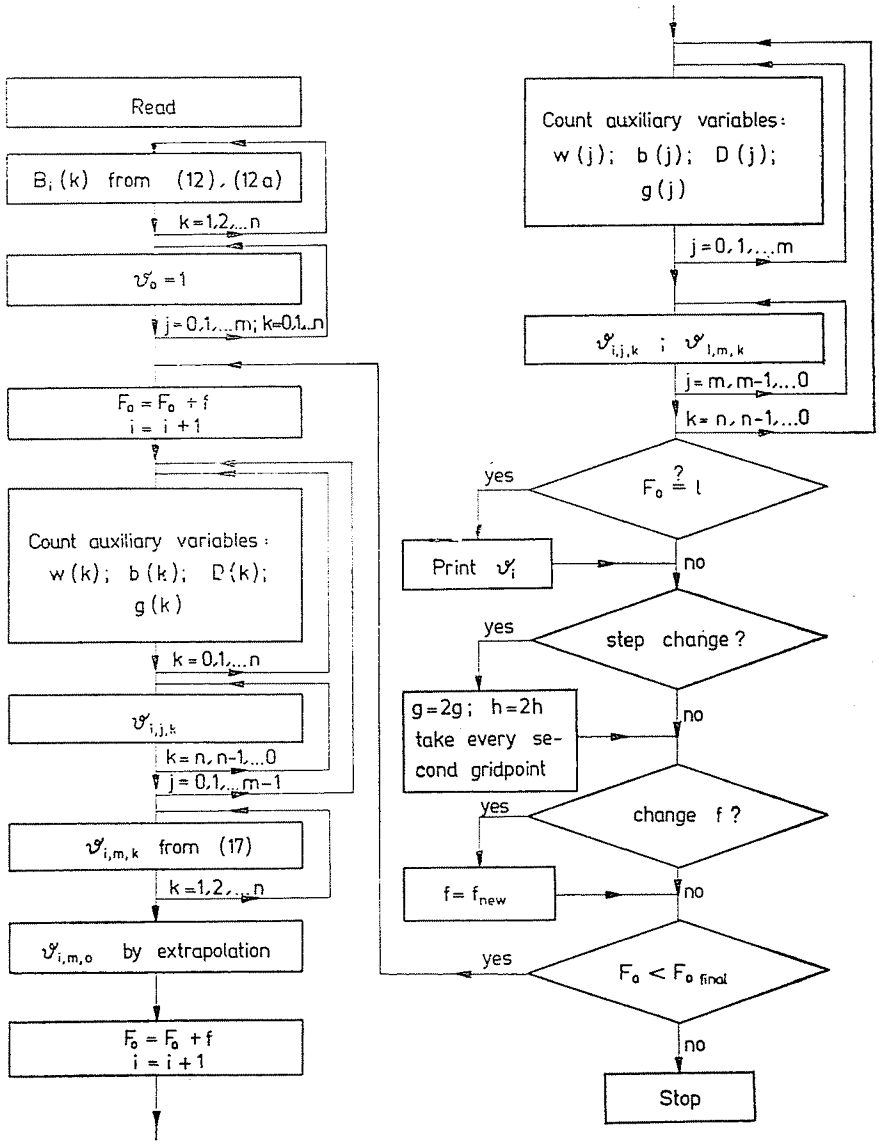

Fig. 4. Flow of the program based on alternating-direction implicit method

Outputs

Examination of boundary conditions (10) and (13) showed geometry and hydrodynamic characteristics as well as material data to be resumable in a place-independent k_a dimensionless number. Accordingly, the influence of plate thickness, flow speed and material characteristics on heat conduction, on time-dependent temperature distribution can be examined by computer with a series of k_a values. Publication of a set of computer runs would be voluminous but not necessary. Rather than this, some results for common physical problems will be presented as a good illustration.

Calculations were made for $2X=28$ mm thick plates by a medium temperature of 40 °C and drying limit temperature of $t_n = 20$ °C. Thermal characteristics of the materials were taken from [8].

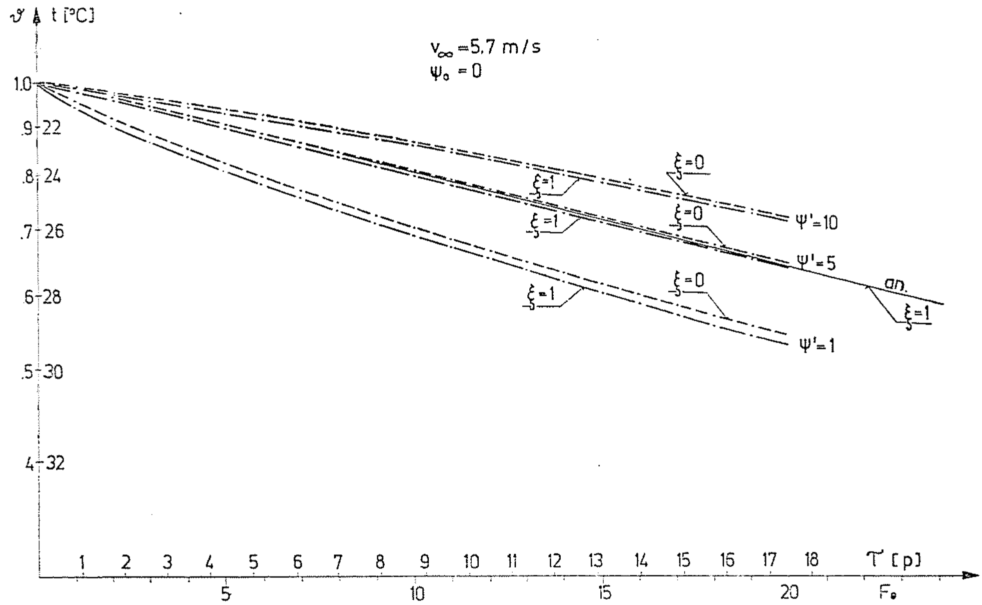

Fig. 5. Warming up of stainless steel plate: $k_a = 0{,}0364$

Our examples refer to warming up of stainless-steel plate and fibre board, as well as cooling down of wet loam and wet cellulose plate during drying. Tables composed from numerical results as well as further computer runs are found in [9].

Outputs have been plotted in Fig. 5 to 11. Reduced temperature distribution of surface and median plane points $\psi' = 1, 5, 10$ has been plotted with Fo number. (ψ' is the distance from the beginning of thermal boundary layer.) As a comparison, the analytic solution using an average heat coefficient for the given case has been superimposed as curves "an". Digrams show the heat transfer coefficient varying with place to result in different temperature distributions in a given cross section for medium and poor λ_{sz} values. Even then, analytic results nearly equal the computer outputs for mid-plate points. To better realize the results a time scale and a temperature scale were added to

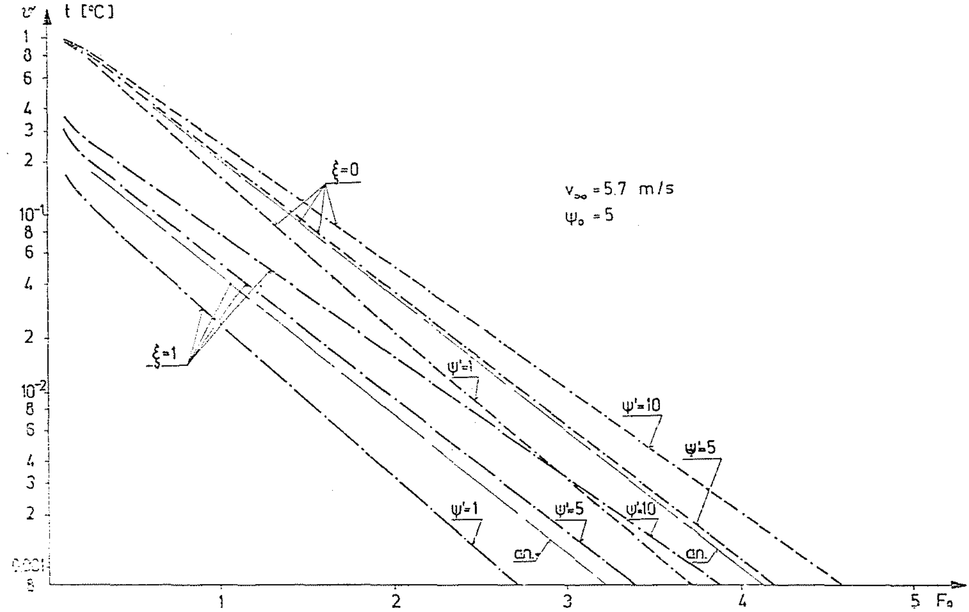

Fig. 6. Warming up of fibre-board: $k_a = 11,47$

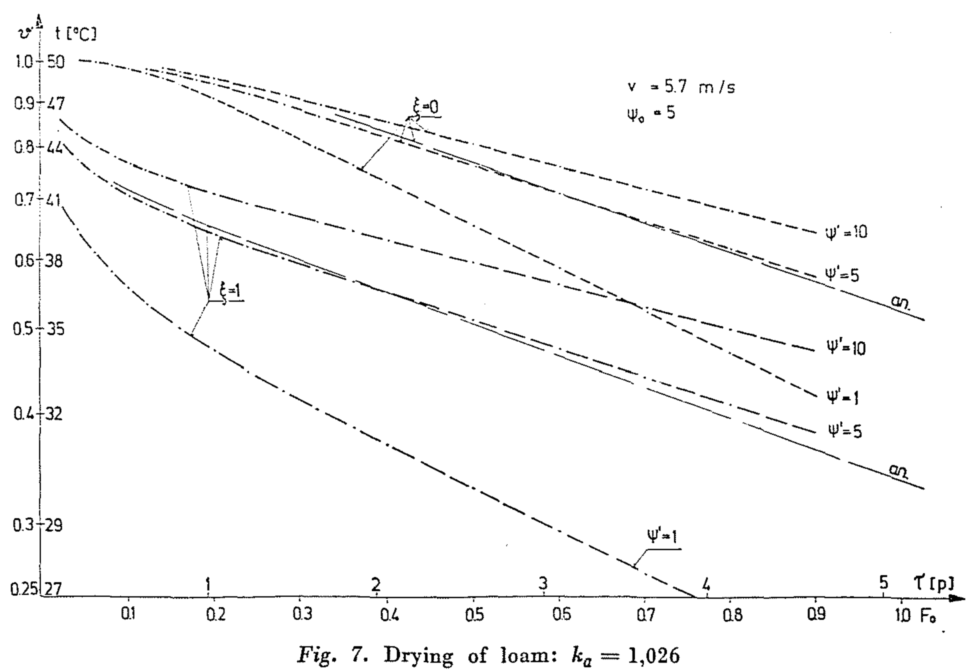

Fig. 7. Drying of loam: $k_a = 1,026$

the Fo axis and the ϑ axis, respectively. In these instances in case of heat transfer and of drying starting values were $t_0 = 20\ °C$ and $t_0 = 50\ °C$ respectively.

Figures 9 and 10 show the temperatures in cros sections of stainless steel plate and of drying loam plate at different times. In case of a high heat transfer coefficient, and low heat conduction, great temperature differences are seen to develop between the surface and median points. In case of a better heat

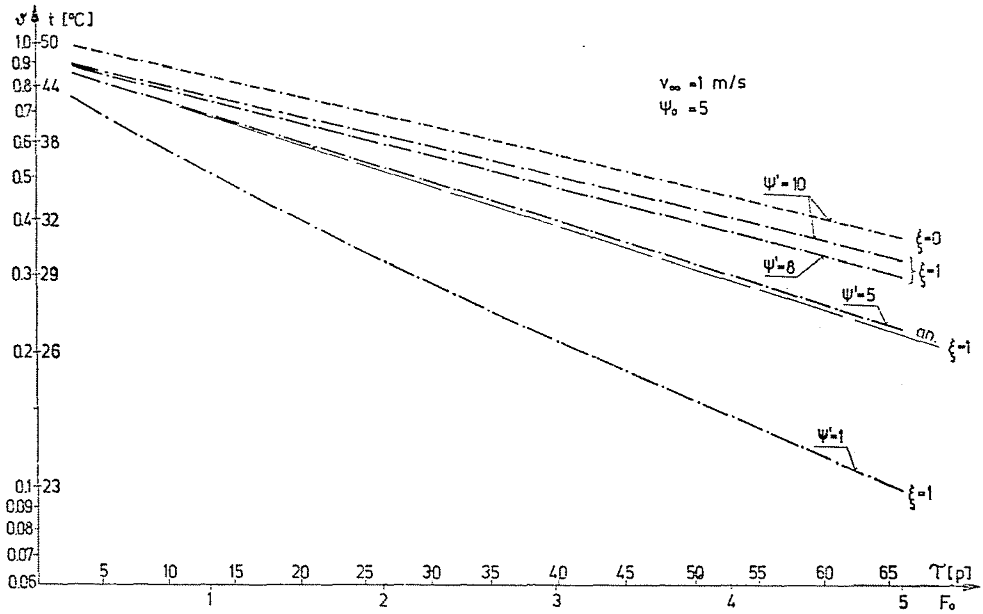

Fig. 8. Drying of cellulsoe: $k_a = 0{,}6305$

conduction coefficient, e. g. for a stainless steel plate small deviation will emerge in the given cross section, while there is a great temperature difference between distant points along the length of the plate.

Surface temperature differences between two ands of the plate are rapidly growing first during the process. After a maximum, they decrease and tend to complete heat equalization, Figure 11. Comparison of results for various k_a values shows for low heat conduction and high heat transfer coefficient e.g. in case of a board of $k_a = 11.5$ a maximum of difference to soon develop. For a high heat conduction coefficient (stainless steel: $k_a = 0.0364$), the maximum evolves later and has a higher value.

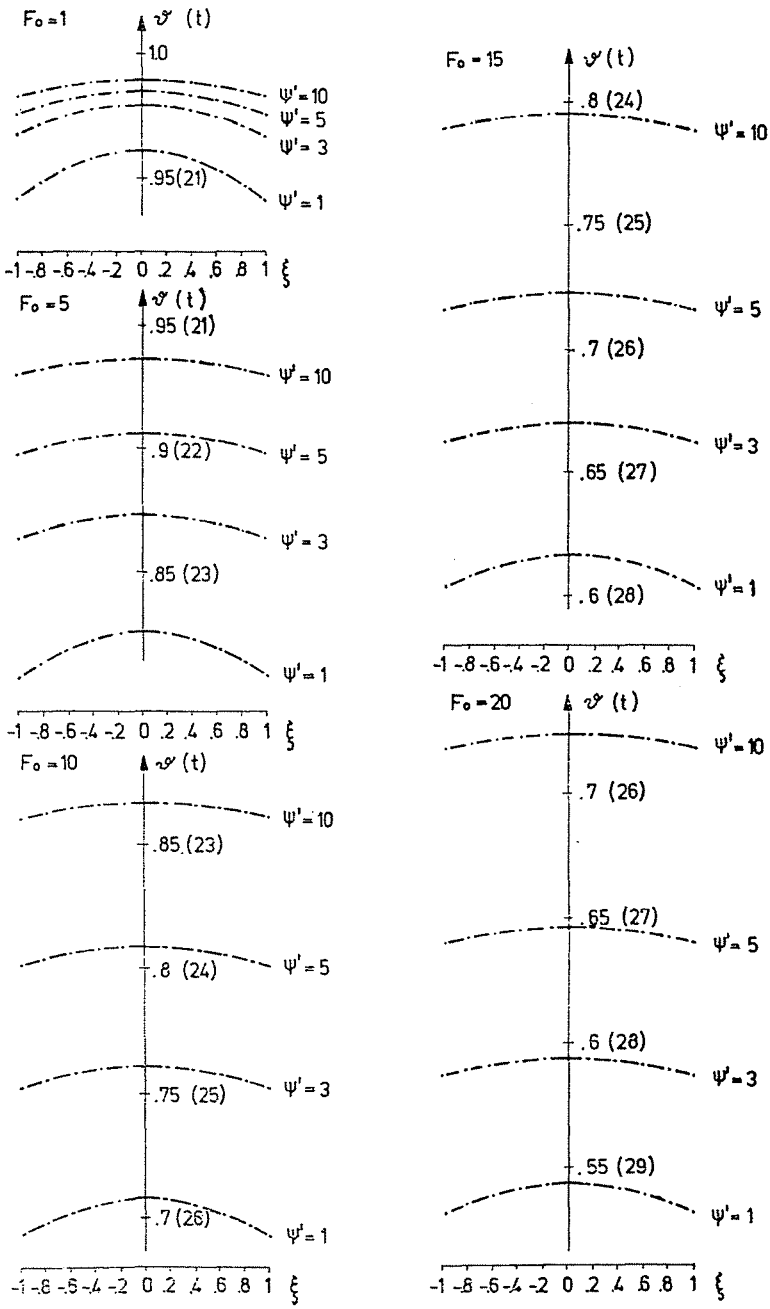

Fig. 9. Temperature gradient in stainless steel plate

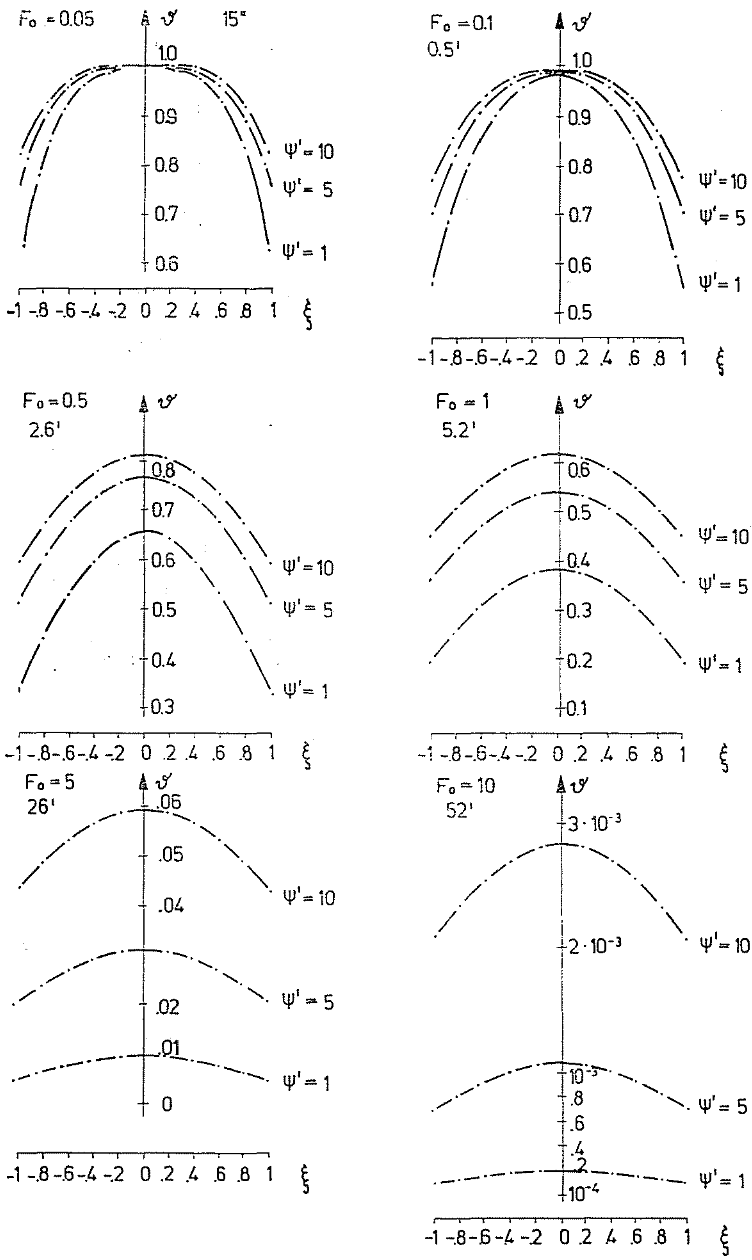

Fig. 10. Temperature gradient in drying loam

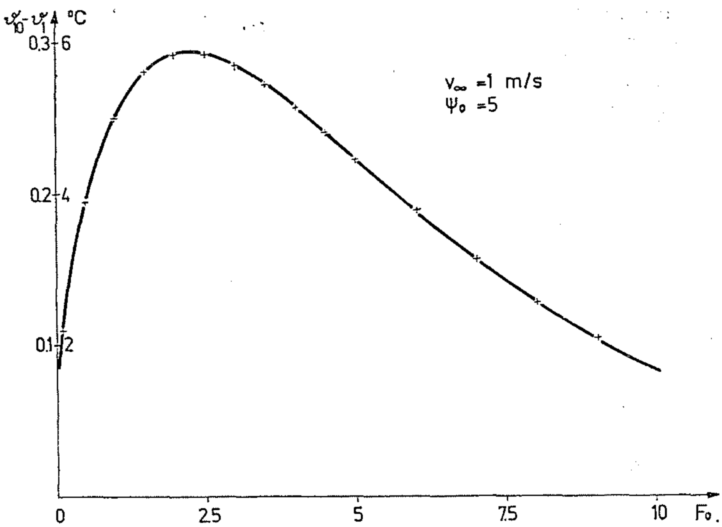

Fig. 11. Temperature differences between surface points of drying cellulose vs. Fo.

Test results

Specimens $2X = 28$ mm thick were prepared from gypsum and cellulose embedding thermocouples in the surface along the isothermal lines, at distances equal to the half thickness of plates. Thermocouples were also embedded in the plates at various depths.

Thermocouples were made of industrially varnished copper and constantan wires 0.1 mm thick, by butt welding with hardly perceivable welding points carefully calibrated and perfectly identical ones have been made in pairs.

Uniting the constantan wires gave one common cool point. Thus the selection of materials eliminated to apply compensation wires, a source of incertainty in dynamic tests. The output of thermocouples has been recorded by a 2 mV Kent multipoint compensograph.

The test plates were $11X$ in legth and width. Edges were plastic foam insulated which had a thickness of $5X$ at the entrance side and was wedge shaped. Tests were made in a wind-tunnel of a testing cross section of 300×340 mm and carefully controlled temperature. The specimen was on three fine plastic legs, little disturbing the gas flow. To provide for the temperature and moisture equalization of the plates, they were wrapped in a suitable insulation for hours before testing.

For evaluation and graphic representation of ϑ in Fo co-ordinates also the temperature conductivity coefficient of the tested material was needed, determined from our test dates by the method described in [6] there being no published data for gypsum or cellulose.

Figures 12 and 13 illustrate our results for heating gypsum board and drying cellulose sheet.

The test results justified the computer outputs: the specimen points in different positions related to the air flow are heated in different rates in the unsteady period. Besides, after an initial period of development both the tested

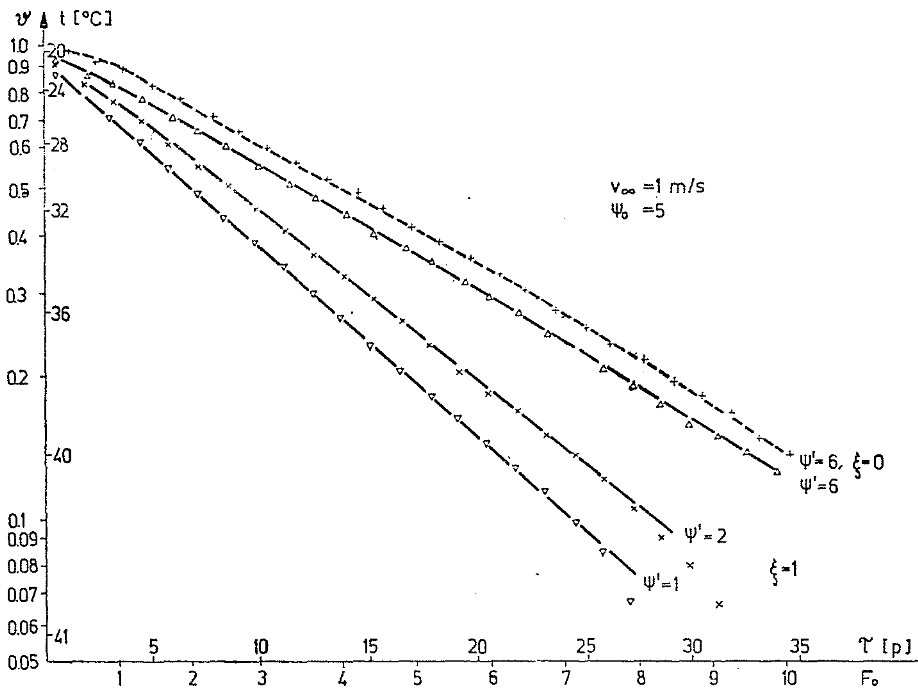

Fig. 12. Test results for heated gypsum plate

and calculated temperatures are aligned along a straight line plotted in logarithmic scale: the solution curves are exponential. Later on the test points showed an irregular nonlinearity: the testing uncertainty has a marked effect in the range of low relative temperatures.

The comparison of computer and test results shows the temperature differences between the points at both ends of the plate to be less in reality than the computer differences attributable to a deficiency of our mathematic model, namely the boundary conditions took only the place dependence of transport coefficients into consideration and ignored the complex change due to temperature dependence. At the same time, however this was a step towards reality.

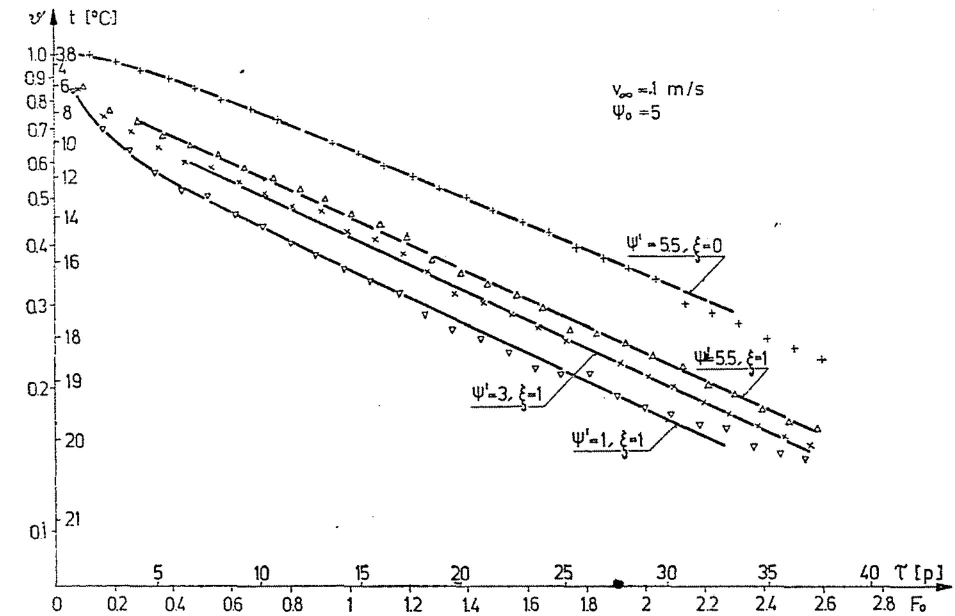

Fig. 13. Test results drying cellulose plate

Summary

In convective heat or mass transfer, transport coefficients depend on place co-ordinates in flow direction. Effect of varying transfer on unsteady heat conduction has been investigated. The parabolic partial differential equation describing the phenomenon was solved by a numerical method on a computer, and experimentally checked.

Transport coefficients varying with place were seen to create a place-dependent temperature gradient and a temperature difference of a few degrees was observed between two end points of the plate. Temperature change of mid-plate points, however, agreed with the analytic result for an average transfer coefficient. The linear method was seen to suit handling of convective, unsteady heat transfer but only for mid-plate points.

a	[m²/h]	temperature conductivity or thermal diffusion coefficient
Bi		Biot number
D_{AG}	[m²/h]	diffusion coefficient
Fo		Fourier number
f, g, h		steps in difference formulae
Gr		Grashof number
k_a		dimensionless constant for calculating the Biot number according to (11)
k_{AG}	[kmól/m²h at]	mass transfer coefficient
m	[at/grad]	slope of secant of tension curve
M_A	[kg/mól]	mol weight of wetting medium
Nu		Nusselt number
P	[at]	total pressure
p_{BM}	[at]	logarithmic mean partial pressure of non diffusible gas
Pr		Prandtl number

R	[at m³/mól grd]	universal gas constant
Re		Reynolds number
r_{An}	[kcal/kg]	latent heat
Sc		Smidt number
T	[grad K]	absolute temperature
t	[grad]	temperature
v	[m/]s	gas velocity
X	[m]	characteristic linear dimension: plate half thickness
x, y, z	[m]	co-ordinates
α	[kcal/m² h grad]	heat transfer coefficient
λ	[kcal/m h grad]	heat conduction coefficient
ν	[m²/s]	kinematic viscosity
ϑ		dimensionless temperature
ξ, ψ, ζ		dimensionless co-ordinates
τ	[s, h]	time

Subscripts

f	surface
g	relating to gas
i, j, k	step number
n	reference to wet-bulb temperature
0	reference to initial value
sz	relating to solid materials
∞	reference to main mass value

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