USE OF THE SUMMABLE SERIES (C,1) IN RESEARCH OF BAR VIBRATIONS

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A wheel, rolling on a bar, will be considered as the model of a car running on a bridge.

Bending vibration of a bar is expressed by the differential equation:

$$E\,\Theta\,\frac{\partial^4 u}{\partial x^4} + \mu\,\frac{\partial^2 u}{\partial t^2} = 0 \tag{1}$$

where E is the Young's modulus, Θ is the section modulus, and μ is the density of material.

Boundary conditions: A straight bar with supported ends, i.e. the ends can not change their position, but they can turn and the curvature is zero at the ends. This is expressed as follows:

$$u(0, t) = u(l, t) \tag{2}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{x=0} = \left(\frac{\partial^2 u}{\partial x^2}\right)_{x=l} = 0 \tag{3}$$

Instead of a rolling wheel, the force action of a frictionless sliding load should be considered. This problem is characterized by an outer force as a function of place and time, expressed by the following inhomogeneous differential equation:

$$E\,\Theta\,\frac{\partial^4 u}{\partial x^4} + \mu\,\frac{\partial^2 u}{\partial t^2} = P(x, t) \tag{4}$$

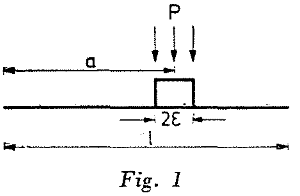

Fig. 1

The solution of the homogeneous differential equation (1) can be easily found by separation of variables:

$$u_h(x, t) = \sum_{n=1}^{\infty} \left[A_n \cos \frac{n^2\pi^2}{l^2} kt + B_n \sin \frac{n^2\pi^2}{l^2} kt\right] \cdot \sin \frac{n\pi}{l} x \tag{5}$$

where $k = \sqrt{\frac{E\Theta}{\mu}}$

satisfying both the differential equation (1) and the boundary conditions (2) and (3).

It will be assumed that the disturbing function $P(x, t)$ can be described as the load reduced to the unit length and sliding on the bar at a speed c. At a given time t_0, for the simplified function $P(x)$, we obtain:

$$P(x) = \frac{2P}{\pi\varepsilon} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{na\pi}{l} \sin \frac{n\pi\varepsilon}{l} \sin \frac{n\pi}{l} x \tag{6}$$

from the following consideration: the load P is considered as uniformly distributed along a length 2ε. Load density, i.e. the specific load on the unit length is $\frac{P}{2\varepsilon}$. Thus:

$$P(x) = \begin{cases} 0, & \text{if } 0 \leq x < a - \varepsilon \\ \frac{P}{2\varepsilon}, & \text{if } a - \varepsilon \leq x \leq a + \varepsilon \\ 0, & \text{if } a + \varepsilon < x \leq l \end{cases} \tag{7}$$

(See Fig. 2)

In expression (6), replace the quantity a, by product ct, to find the load that advances at a speed c:

$$P(x, t) = \frac{2P}{\pi\varepsilon} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi\varepsilon}{l} \sin \frac{n\pi c}{l} t \cdot \sin \frac{n\pi}{l} x \tag{8}$$

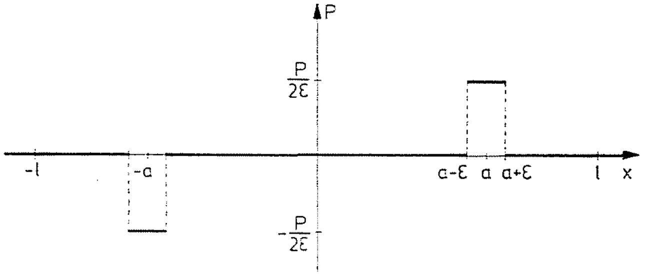

Fig. 2

Now, the function $P(x, t)$ represents the disturbing member of the differential equation (4). The solution of the inhomogeneous differential equation satisfying the relevant initial and boundary conditions, will be obtained as sum of two members. The first of these two members is the solution of the homogeneous differential equation that comprises any constants whatever and satisfies the boundary conditions; and the second one is a particular solution of the inhomogeneous differential equation that similarly satisfies the boundary conditions.

Solution of the homogeneous differential equation is given by function (5).

Particular solution of the inhomogeneous differential equation can be found in form of the eigenfunctions:

$$u(x, t) = \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi}{l} x \tag{9}$$

From this we obtain

$$a_n(t) = M\gamma_n \sin \frac{n\pi\varepsilon}{l} t \tag{10}$$

where

$$M = \frac{2Pl^4}{\pi^3\varepsilon}$$

$$\gamma_n = \frac{\sin \frac{n\pi\varepsilon}{l}}{n^3(E\Theta\, \pi^2 n^2 - l^2 \mu c^2)}$$

In this way, a particular solution of the inhomogeneous differential equation is:

$$u_i(x, t) = M \sum_{n=1}^{\infty} \gamma_n \sin \frac{n\pi c}{l} t \sin \frac{n\pi}{l} x \tag{11}$$

Consequently

$$u(x, t) = u_h(x, t) + u_i(x, t) =$$

$$= \sum_{n=1}^{\infty} \left(A_n \cos \frac{n^2\pi^2}{l^2} kt + B_n \sin \frac{n^2\pi^2}{l^2} kt + M\gamma_n \sin \frac{n\pi c}{l} t \right) \cdot \sin \frac{n\pi}{l} x \tag{12}$$

Considering the initial conditions:

$$u(x, 0) = 0, \qquad u_t(x, 0) = 0 \tag{13}$$

i.e. at the beginning of the process, the bar is motionless.

The unknown constants A_n and B_n can be found from initial conditions (13).

The first initial condition yields $A_n = 0$. Again, from the second initial condition:

$$B_n = -\frac{2Pl^5c}{\pi^4\varepsilon}\sqrt{\frac{\mu}{E\Theta}}\,\frac{\sin\dfrac{n\pi\varepsilon}{l}}{n^4(E\Theta\,\pi^2n^2 - l^2\,\mu c^2)} = -\frac{Mcl}{\pi k}\,\frac{\gamma_n}{n}$$

Thus, the solution of differential equation (4), satisfies both the initial and the boundary conditions:

$$u(x, t) = M\sum_{n=1}^{\infty}\gamma_n\left(\sin\frac{n\pi c}{l}t - \frac{lc}{\pi nk}\sin\frac{n^2\pi^2}{l^2}kt\right)\cdot\sin\frac{n\pi}{l}x \tag{14}$$

For the case of a fixed distance $\varepsilon > 0$, expression (14) represents a rather quickly converging series, as seen from the formula of γ_n. Again, in this latter formula, for certain values of c, the denominator is ZERO. For e.g. $n = 1$, the denominator in the first member of expression (14) equals ZERO, namely when

$$c = \frac{\pi k}{l} \tag{15}$$

This critical speed value is proportional to the first eigenvalue resulting in a phenomenon of resonance.

Now, let us consider the case of the rolling wheel. In this case, the value $\varepsilon \to 0$, i.e. the series (6) becomes divergent. Series (6) may take the following form:

$$P(x) = \frac{2P}{l}\sum_{n=1}^{\infty}\frac{\sin\dfrac{n\pi\varepsilon}{l}}{\dfrac{n\pi\varepsilon}{l}}\sin\frac{na\pi}{l}\sin\frac{n\pi}{l}x$$

which again can be simplified with $\varepsilon \to 0$ as follows:

$$P(x) = \frac{2P}{l}\sum_{n=1}^{\infty}\sin\frac{na\pi}{l}\sin\frac{n\pi}{l}x \tag{16}$$

This series (C,1) represents a summable Fourier—Stieltjes series; by its integration, we obtain a convergent Fourier series. In other words, the series (16) is, in the sense of distribution, convergent. In concreto, by using the averaging operation after Cesàro, series (16) is converging to the periodical distribution as shown in Fig. 3.

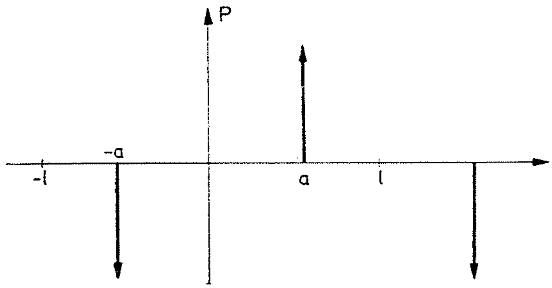

Fig. 3

In series (16), substituting a by ct, in analogy to series (8), the disturbing member of differential equation (4) is obtained as follows:

$$P(x, t) = \frac{2P}{l} \sum_{n=1}^{\infty} \sin \frac{n\pi c}{l} t \sin \frac{n\pi}{l} x \tag{17}$$

by which a certain distribution is determined, too.

In this case, the particular solution of the inhomogeneous differential equation (4) is found, again in form of expression (9). Now, we apply the rules of the distribution theory, viz, we proceed in an analogous way to the former case of dealing with common functions, and considering convergence always in the sense of distribution.

The solution meeting boundary conditions (2) and (3) is found as follows:

$$u(x, t) = \sum_{n=1}^{\infty} \left(E_n \cos \frac{n^2\pi^2 k}{l^2} t + F_n \sin \frac{n^2\pi^2 k}{l^2} t + \right.$$
$$\left. + \frac{2Pl^3}{\mu(n^4\pi^4k^2 - n^2\pi^2c^2l^2)} \sin \frac{n\pi c}{l} t \right) \sin \frac{n\pi}{l} x$$

In compliance with the initial conditions (13): $E_n = 0$ and

$$F_n = \frac{2Pl^4c}{\mu n^3\pi^3 k(c^2l^2 - n^2\pi^2k^2)} .$$

The obtained particular solution satisfies all auxiliary conditions in the following form:

$$u(x, t) = \frac{2Pl^3}{\mu\pi^2} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{l} x}{n^2(n^2\pi^2k^2 - c^2l^2)} \left(\sin \frac{n\pi c}{l} t - \frac{lc}{k\pi n} \sin \frac{n^2\pi^2}{l^2} kt \right) \tag{18}$$

Solution (14) tends to series (18) for $\varepsilon \to 0$, because of the following interrelation:

$$\lim_{\varepsilon \to 0} M\gamma_n = \frac{2Pl^3}{\mu\pi^2} \frac{1}{n^2(k^2\pi^2n^2 - l^2c^2)}$$

This result proves to be quite natural to calculate in the sense of distribution. Series (18) is as convergent as series (14). The possibility of resonance is the same.

Summary

In this paper, assuming a concentrated force action, solution of the differential equation of bar vibration is investigated. The disturbing function is established as a summable FOURIER—STIELTJES series (C,1) by which a periodical distribution is established.

References

1. SCHWANK, F.: Randwertprobleme. B. G. Teubner, Leipzig, 1951.
2. MIKUSIŃSKI, J.—SIKORSKI, R.: The elementary theory of distributions (I). Rozprawy Matematyczne XII. Warszawa, 1957.
3. GELFAND, T. M.—SCHILOW, G. E.: Verallgemeinerte Funktionen (Distributionen). VEB V. Berlin, 1967.
4. HOFFMANN, A.: Use of (C,1) summability series in the flow in water basins (in Hunga rian). Matematikai Lapok **23**, 3—4, p. 291—295 (1972).

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