PROBABILISTIC APPROACH TO DESIGN FOR MEASURING INSTRUMENT MECHANISMS

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The members of actually produced, so-called real mechanisms are burdened with numerous errors due to production inaccuracies. In designing instrument mechanisms, dimensions, shape, and positional relations are indicated not only by a single value (nominal dimension), but also by tolerances. In assembly, all the mechanism components classified as good will evidently be within prescribed tolerance limits, dimensions of individual components, however, may be differently sited within the tolerance range [1]. Consequently, deterministic methods supposing the occurrence of minimum or maximum errors for each component when calculating the combined (resultant) error of the mechanism (as if component dimensions were exactly at the tolerance limit) result in unjustified low or high values for the resultant error.

Experience has shown that components sited differently within the tolerance range are incorporated into the mechanism. Therefore, in calculating resultant errors of mechanisms, the combined effect of individual errors has to be determined by probability calculus.

On the basis of the aforesaid, the question may arise, how near the results obtained by using methods of accuracy synthesis and analysis employing deterministic functions are approximating reality, i. e. which are the conditions for them to suit description of motion conditions in real mechanisms.

The above considerations give rise to various problems (connected to the essential characteristics of mechanisms), such as:

— How do the various functions describing the mechanism modify the deterministic, or probabilistic characteristics of the input signal?

— How the probabilistic characteristics of the output signal are influenced by the various (practically frequent) distributions of design parameters interpreted as random variables?

— How the correct tolerance ranges (or variances) of design parameters of various distributions can be determined so that the tolerance range of the output signal does not exceed a value prescribed or permitted at a confidence level determined from functional aspects?

— What is the correlation between the characteristic function of the theoretical mechanism and the output signal interpreted as random variable?

Deterministic methods of accuracy synthesis and analysis are not suited to answer the above-mentioned questions. To solve the raised and similar problems, the probabilistic model of mechanisms is set up and the accuracy examination method concerning the model is elaborated.

Mathematical formulation of the problem

At the time of designing the instrument or the instrument mechanism, some information is already available, on the basis of practical experience of many years, concerning input signal (or signal to be transmitted) and design parameters. What we now need is information on the output signal being in a known functional relationship with the input signal and with design parameters.

In probability calculus this problem can be formulated in the following way. Density and distribution functions of e. g. the input signal and the design parameters (as random variables) are known and the density and distribution functions of the output signal have to be found.

The characteristic function $y = h(x)$ of mechanisms applied in measuring instruments can generally be an arbitrary, not mutually unambiguous, transformation, defined for all possible values of random variable x.

Let us examine how in this case the density function of y (as output signal) can be determined in the knowledge of the density function of variable x. Under the examined circumstances it is advisable to divide the range of interpretation of x into sections where function $y = h(x)$ is thoroughout monotonously neither increasing, nor decreasing. In practical cases it may be assumed that function $y = h(x)$ has a single-valued inverse in each of the above intervals, it can be differentiated, and its derivative differs from zero. In this case the density function of the output signal of the mechanism (i. e. of random variable y) can be calculated by the following relationship [2]:

$$g(y) = \sum_i f(x_i) \frac{1}{|h'(x_i)|} \tag{1}$$

where $f(x)$ is the density function of random variable x, and summation should be extended to all x_i values, where $h(x_i) = y$.

Be $f(x)$ the density function of x, if in the interval including its values, the function $y = h(x)$ is monotonously increasing or decreasing, but is always differentiable, and its inverse function is $x = h^{-1}(y)$, then the density function of output signal $y = h(x)$ as random variable, is found to be [3]

$$g(y)=\begin{cases}0, & \text{if } y \leq h(-\infty);\\ \dfrac{f(x)}{\left|\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|}, & \text{if } h(-\infty)<y<h(+\infty);\\ 0, & \text{if } y \geq h(+\infty).\end{cases} \tag{2}$$

Distribution function $G(y)$ of the new random variable $y = h(x)$ is, according to [4]:

$$G(y)=\begin{cases}P[h(x)\leq y]=1, & \text{if } y \geq h(+\infty);\\ P[h(x)\leq y]=F[h^{-1}(y)], & \text{if } h(-\infty)<y<h(+\infty);\\ 0, & \text{if } y \leq h(-\infty);\end{cases} \tag{3}$$

where $h^{-1}(y)$ denotes the inverse function of $h(y)$ and $F(x)$ is the distribution function of random variable x.

By using Eqs (1), (2) and (3), further the relationship for calculating density functions for the composition, multiplication, and quotient of continuous distributions, the density or distribution functions of the output signals of tested mechanisms can be generated. As density and distribution functions contain all the information relating to the random variable (in our case to the output signal of the mechanism), they can be used to determine parameters characterizing accuracy (e. g. mean value, standard deviation, etc.), if these exist, and the values of tolerance limits for a prescribed level of probability.

Model and examination method

The probabilistic model of measuring instrument mechanisms is understood to interpret design parameters of mechanisms (such as dimensions, position and shape relationships, etc.), and also input and output signals as random variables, which seem us to better approach reality, and to apply methods of probability calculus for examining the model.

In calculating the accuracy of measuring instrument mechanisms, our examinations will be limited to the summation of random stationary errors inaffected by displacements of the mechanism.

The algorithm of the proposed method for the accuracy examination of the probabilistic model of real measuring instrument mechanisms is the following.

a) Determine the mechanism characteristic for the theoretical mechanism

$$y = h(x, k, l, r, \ldots).$$

b) Consider parameters x, k, l, r, in the mechanism characteristics as random variables, and determine the distributions of these random variables on the basis of empirical (or measured) data.

c) On the basis of poitems a and b, determine the density or distribution function of the output signal of the real mechanism, by means of the aforementioned probabilistic model.

d) In possession of the density, or distribution function of the output signal, calculate, on the one hand, the parameters characterizing the output signal (or its accuracy) (e. g. mean value, standard deviation, etc.), on the other hand the deviations (tolerance ranges) pertaining to the prescribed tolerance level, as a function of design parameters.

e) From accuracy requirements prescribed for the output signal, taken as preset conditions, determine the correct parameter values for real mechanisms by deduction on the basis of item point d.

The probabilistic model for measuring instrument mechanisms and the proposed examination method permit to solve the set problems and to clarify, or to examine further essential characteristics of mechanisms.

The application of the proposed examination method is illustrated on two examples.

The mechanism illustrated in Fig. 1 is frequently applied in measuring instruments.

In general, only the restoring force of the spring and gravity force is acting and the pin is producing the rectilinear displacement or rotation. On account of the existence of friction forces, the above mentioned, relatively small forces are insufficient to safely unambiguously position the pin in its guide. The position of the pin in the guide is therefore regarded as a random variable of uniform distribution [5].

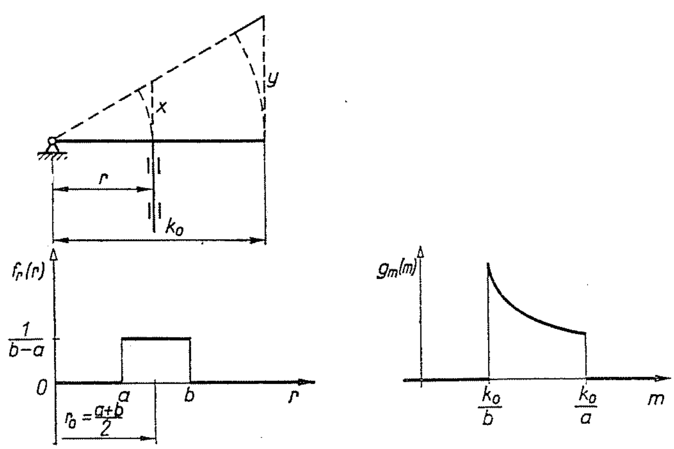

Fig. 1. Simple measuring instrument mechanisms

Let dimension r in Fig. 1 be uniformly distributed in the interval $[a, b]$.

On the one hand, bearings of the mechanism are supposed to be produced very carefully, thus their effect on the precise functioning of the mechanism is negligible, and, on the other hand, lever length k_0 is supposed to be measured by an instrument of appropriate accuracy. Under such conditions k_0 can be regarded as deterministic and only the uncertainty of dimension r may affect the theoretical modification

$$m_e = \frac{k_0}{r_0},$$

Let us examine, whether the uniform distribution of dimension r affects the theoretical modification of the mechanism, and if so, in what manner.

According to the proposed examination method first the density function of the modification $m = k_0/r$ of the real mechanism, as of a random variable has to be determined, taking into consideration that dimension r has a uniform distribution in the interval $[a, b]$ (Fig. 1).

This problem can be solved by relationship (2) for determining the distribution of the random variable. In our case, the density function of transmission ratio m, considered as a random variable, is found to be

$$g_m(m) = \frac{f_r(r)}{\left|\dfrac{\mathrm{d}\,m}{\mathrm{d}\,r}\right|} \tag{4}$$

Form the derivative of modification $m = k_0/r$ with respect to r:

$$\frac{\mathrm{d}\,m}{\mathrm{d}\,r} = -\frac{m^2}{k_0}$$

Upon considering the distribution of dimension r,

$$f_r(r) = \begin{cases} \dfrac{1}{b-a}, & \text{for } a < r < b \\ 0 & \text{otherwise}. \end{cases}$$

The density function of modification m can already be calculated on the basis of relationship (4) (see Fig. 1).

$$g_m(m) = \begin{cases} \dfrac{k_0}{(b-a)\,m^2}, & \text{for } \dfrac{k_0}{b} < m < \dfrac{k_0}{a} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

In the knowledge of the density function, let us determine the expected (most probable) mean value of the modification of the real mechanism, under the examined conditions.

$$M(m) = \int_{-\infty}^{\infty} m\, g_m(m)\, dm = \int_{\frac{k_0}{b}}^{\frac{k_0}{a}} m \frac{k_0}{(b-a)\, m^2} dm = \frac{k_0}{b-a} \ln \frac{b}{a} \qquad (6)$$

In our case the $b-a$ value is actually the dimensional difference between the inside diameter of the guide bush and the pin diameter, i. e. a quantity proportional to the tolerance range characterizing the fit.

From the result obtained for $M(m)$, the modification m of the real mechanism is seen to equal the modification m_e of the theoretical mechanism only under the condition that $b-a = 0$, i. e. when there is no clearance between the pin and the bush.

$$\lim_{\substack{b \to r_0+0 \\ a \to r_0-0}} M(m) = \lim_{\substack{b \to r_0+0 \\ a \to r_0-0}} \frac{k_0}{b-a} \ln \frac{b}{a} = \frac{k_0}{r_0}$$

In practice, however, correct functioning of the mechanism is preconditioned by $b-a > 0$, i. e. a clearance between the dimensions of pin and bush.

Under the condition that $b-a > 0$, the inequality

$$\frac{\ln \frac{b}{a}}{b-a} \neq \frac{1}{r_0}$$

is always valid, it can therefore be stated that under the examined conditions the mean value of the real mechanism, i. e. the most probable modification, will always differ from the modification of the theoretical mechanism:

$$M(m) \neq m_e.$$

This is an essential finding since it points to an error component Δm not detectable by deterministic methods, originating from the difference of the respective modifications of theoretical and real mechanisms, considered as a systematic error:

$$\Delta m = m_e - M(m) = k_0 \left(\frac{1}{r_0} - \frac{\ln \frac{b}{a}}{b-a} \right) \neq 0$$

to be taken into consideration in designing mechanisms functioning under the given conditions.

After having determined the mean value of modification, let us determine another accuracy parameter namely the standard deviation of modification.

The standard deviation of modification can be calculated in the knowledge of density function $g_m(m)$.

$$\sigma(m) = \left[\int_{-\infty}^{\infty} [m - M(m)]^2 \, g_m(m) \, \mathrm{d}\, m\right]^{1/2} = \left[\frac{k_0^2}{ab} - \left(\frac{k_0}{b-a} \ln \frac{b}{a}\right)^2\right]^{1/2} \tag{7}$$

As a solution of the problem, the two parameters characterizing the accuracy of modification of a real mechanism functioning under the examined conditions have been determined, namely the mean $M(m)$ of the modification (Eq. 6) and its standard deviation $\sigma(m)$ (Eq. 7), vs. design parameters k_0, r_0, a, and b.

Let us consider the case of sinusoidal mechanisms (Fig. 2) frequently applied in measuring instruments. Examine, how dimension r of normal distribution with parameters $N_r[r_0, \sigma_r]$ is influencing or disturbing the correct transmission of an input signal of normal distribution with parameters $N_x[x_0, \sigma_x]$ (e. g. the dimension of a workpiece).

According to the suggested examination method, let us determine the density function of the output signal of a sinusoidal mechanism (angular displacement).

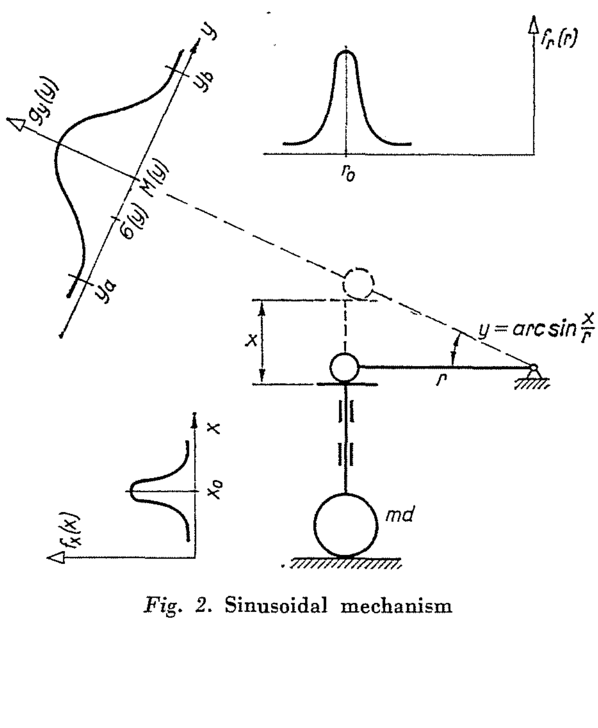

Fig. 2. Sinusoidal mechanism

The mechanism characteristics are given by

$$y = \text{arc}\ \sin \frac{x}{r}.$$

Applying notation $m^* = \frac{x}{r}$, the inverse function becomes

$$m^* = \sin y.$$

The derivative of the inverse function is

$$\frac{\mathrm{d}\, m^*}{\mathrm{d}\, y} = \cos y. \tag{8}$$

The density function of $m^* = \frac{x}{r}$, i. e. of the quotient of two normally distributed random variables can be approximated according to [6] by the relationship

$$h^*(m^*) = \frac{\sigma_x^2 \cdot r_0 + \sigma_r^2 \cdot m^* \cdot x_0}{\sqrt{2\pi(\sigma_x^2 + \sigma_r^2 \cdot m^{*2})^3}} \exp\left[-\frac{\left(m^* - \frac{x_0}{r_0}\right)^2}{2\left(\frac{1}{r_0}\sqrt{\sigma_r^2 m^{*2} + \sigma_x^2}\right)^2}\right]. \tag{9}$$

Relationships (8) and (9) field the density function of the output signal (angular displacement) of the sinusoidal mechanism operating under the examined conditions

$$g_y(y) = h^*(m^*)\left|\frac{\mathrm{d}\, m^*}{\mathrm{d}\, y}\right| =$$

$$= \cos y \cdot \frac{\sigma_x^2 \cdot r_0 + \sigma_r^2 m^* \cdot x_0}{\sqrt{2\pi(\sigma_x^2 + \sigma_r^2 m^{*2})^3}} \exp\left[\frac{\left(m^* - \frac{x_0}{r_0}\right)^2}{2\left(\frac{1}{r_0}\sqrt{\sigma_r^2 m^{*2} + \sigma_x^2}\right)^2}\right]. \tag{10}$$

Relationship (10) concerning the output signal of the sinusoidal mechanism delivers all the information on the output signal as a random variable and thus it can be applied for calculating the parameters characterizing the output signal.

From the aspect of accuracy, the calculation of the mean value, of the standard deviation, and of the tolerance limits pertaining to the prescribed confidence levels as a function of design parameters are the most important.

Determination of the mean value

The mean value can be determined in the knowledge of the density function, by using the following well-known relationship:

$$M(y) = \int_{-\pi/2}^{\pi/2} y\, g_y(y)\, \mathrm{d}\, y =$$

$$= \int_{-\pi/2}^{\pi/2} y \cos y \frac{\sigma_x^2 \cdot r_0 + \sigma_r^2 \cdot \sin y \cdot x_0}{\sqrt{2\pi (\sigma_x^2 + \sigma_r^2 \cdot \sin^2 y)^3}} \exp \left[- \frac{\left(\sin y - \frac{x_0}{r_0} \right)^2}{2 \left(\frac{1}{r_0} \sqrt{\sigma_r^2 \sin^2 y + \sigma_x^2} \right)^2} \right] \mathrm{d}\, y. \tag{11}$$

The mean determined by relationship (11) supplies the most probable value of the output signal of the sinusoidal mechanism operating under the examined conditions, as a function of output signal and of design parameters.

This formula helps to establish the relationship between the mechanism characteristics (output signal) of the theoretical sinusoidal mechanism and the most probable output signal of the real sinusoidal mechanism. Eventual differences between the characteristics of the theoretical sinusoidal mechanism and relationship (11) obtained for the mean of the real sinusoidal mechanism can be considered as systematic errors, offering in turn a possibility of calculating Δy and of making corrections.

$$\Delta y = M(y) - y.$$

Determination of the standard deviation

The second significant characteristic of the output signal of the mechanism is its standard deviation.

$$\sigma(y) = \left[\int_{-\pi/2}^{\pi/2} [y - M(y)]^2 g(y)\, dy \right]^{1/2} =$$

$$= \left[\int_{-\pi/2}^{\pi/2} y^2 \cos y \frac{\sigma_x^2 r_0 + \sigma_r^2 \sin y\, x_0}{\sqrt{2\pi (\sigma_x^2 + \sigma_r^2 \sin^2 y)^3}} \exp \left[- \frac{\left(\sin y - \frac{x_0}{r_0} \right)^2}{2 \left(\frac{1}{r_0} \sqrt{\sigma_r^2 \sin^2 y + \sigma_x^2} \right)^2} \right] dy - \right. \tag{12}$$

$$\left. - \left(\int_{-\pi/2}^{\pi/2} y \cos y \frac{\sigma_x^2 r_0 + \sigma_r^2 \sin y\, x_0}{\sqrt{2\pi (\sigma_x^2 + \sigma_r^2 \sin^2 y)^3}} \exp \left[- \frac{\left(\sin y - \frac{x_0}{r_0} \right)^2}{2 \left(\frac{1}{r_0} \sqrt{\sigma_r^2 \sin^2 y + \sigma_x^2} \right)^2} \right] \right)^2 \right]^{1/2}$$

Standard deviation is characteristic for the variation of the output signal, interpreted as a random variable, with respect to the mean.

Determination of tolerance limits for the prescribed confidence level

By means of density function (10) calculated for the ouput signal of the sinusoidal mechanism operating under the examined conditions, limits y_a and y_b of the interval can be determined, according to accuracy considerations, for various confidence levels:

$$P_i = \int_{y_a}^{y_b} g_y(y)\,dy, \qquad i = 1, 2, \ldots n; \tag{13}$$

On account of the symmetry of density function (10), it is sufficient to determine the integral for the following interval:

$$P_i^* = \int_{M(y)}^{y_b} g_y(y)\,dy =$$

$$= \int_{M(y)}^{y_b} \frac{\sigma_x^2 r_0 + \sigma_r^2 \sin y\, x_0}{\sqrt{2\pi(\sigma_x^2 + \sigma_r^2 \sin^2 y)^3}} \exp\left[-\frac{\left(\sin y - \frac{x_0}{r_0}\right)^2}{2\left(\frac{1}{r_0}\sqrt{\sigma_r^2 \sin^2 y + \sigma_x^2}\right)^2}\right] dy$$

where

$$P_i^* = \frac{1}{2} P_i .$$

It is advisable to determine the mean, standard deviation, and tolerance limits pertaining to prescribed confidence levels, for the output signal of the sinusoidal mechanism operating under the examined conditions in a computer, taking into consideration the character of relationships (11), (12), and (13), and the high number of variations of the parameters.

As an example, Fig. 3 presents the variation of the tolerance range $\pm T$ of a sinusoidal mechanism operating under the examined conditions, for the confidence level $P = 99.73\%$, in the following range of parameters x_0, r_0, σ_x, and σ_r: $x_0/r_0 = 0.34907$ $(y = 20°)$; $r_0 = 10$ mm; $\sigma_r = \sigma_x = 2, 4, 6, 8, 10 \cdot 10^{-3}$ mm.

On the basis of the performed examinations, the proposed examination method can be stated in to permit answering the questions raised in this paper.

Accordingly, the suggested examination method can be of help in determining for the output signal interpreted as a random variable, the mean, the

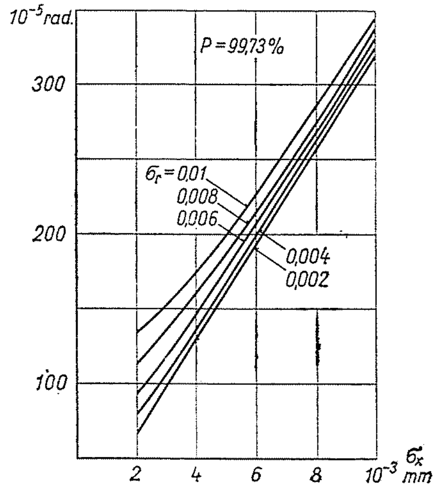

Fig. 3. Variation of the tolerance ranges of the output signal of a sinusoidal mechanism vs. various parameters

standard deviation, and the tolerance limits to various confidence levels, vs. design parameters interpreted similarly as random variables.

The relationships developed in this way greatly facilitate the work of the designer by constricting the range of trial-and-error techniques.

Summary

The probabilistic model of mechanisms is set up with the aim of detecting further significant characteristics of instrument mechanisms. The examination method connected with the model is exposed.

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