METHOD FOR THE NUMERICAL CALCULATION OF VELOCITY DISTRIBUTION FOR A BLADE CASCADE ROTATING IN A PERFECT INCOMPRESSIBLE FLUID

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Introduction

Over the past three years, several papers have treated the determination of flow occurring around rotating cascades of air foils [1], [2]. The plane airfoil cascade is a mapping of a row of blade profiles [3] arranged on a surface with rotational symmetry, as is usual in an impeller (Fig 1). Theoretical considerations lead to a second type of the Fredholm integral equation which determines the absolute velocity along the profile contour [1]:

$$\frac{c_t(\zeta)}{2} - \frac{1}{t}\oint\limits_{(K)} c_t(\zeta')\, K_{\mathrm{I}}(\zeta,\zeta')\, |d\zeta'| = [c_{\infty x} + c_{Bx}(\zeta)] \cos\theta(\zeta) + \tag{1}$$

$$+ [c_{\infty y} + c_{By}(\zeta)] \sin\theta(\zeta) + \frac{1}{t}\oint\limits_{(K)} c_n(\zeta')\, K_{\mathrm{II}}(\zeta,\zeta')\, |d\zeta'|$$

where

$$K_{\mathrm{I}}(\zeta,\zeta') = -\Phi(\zeta,\zeta')\cos\theta(\zeta) - \Psi(\zeta,\zeta')\sin\theta(\zeta)$$

$$K_{\mathrm{II}}(\zeta,\zeta') = \Psi(\zeta,\zeta')\cos\theta(\zeta) - \Phi(\zeta,\zeta')\sin\theta(\zeta)$$

$$\Phi = \frac{-e^{2\pi\frac{\xi-\xi'}{t}} \sin 2\pi\frac{\eta-\eta'}{t}}{1 - 2e^{2\pi\frac{\xi-\xi'}{t}}\cos 2\pi\frac{\eta-\eta'}{t} + e^{4\pi\frac{\xi-\xi'}{t}}}$$

$$\Psi = \frac{e^{4\pi\frac{\xi-\xi'}{t}} - 1}{2\left[1 - 2e^{2\pi\frac{\xi-\xi'}{t}}\cos 2\pi\frac{\eta-\eta'}{t} + e^{4\pi\frac{\xi-\xi'}{t}}\right]}$$

and

$$c_{Bx}(\zeta) = -\frac{1}{t}\iint\limits_{(T)} \frac{1}{b}\,\frac{db}{dx'}\, c_x(z')\, \Phi(\zeta,z')\, dA(z')$$

$$c_{By}(\zeta) = \frac{1}{t} \iint\limits_{(T)} \frac{1}{b} \frac{db}{dx'} c_x(z') \Psi(\zeta, z') \, dA(z')$$

The kernel K_I is limited and continuous [2] but the function K_{II} is singular for $\zeta' = \zeta$.

Axial flow

Usually the flow is considered as proceeding in meridional streamlines of cylindrical form. In this case, the correspondent of the peripheral velocity in the straight cascade is a constant. In the same manner $b(x) = b_1 = b_2$ and consequently $C_{Bx} = C_{By} = 0$.

Substituting:

$$c_n(\zeta') = u_{yn}(\zeta') = u_y(\zeta') \cos \theta(\zeta')$$

and

$$c_t(\zeta') = w_t(\zeta') + u_t(\zeta') = w_t(\zeta') + u_y(\zeta') \sin \theta(\zeta')$$

Eq. (1) reduces to:

$$\frac{w_t(\xi)}{2} - \frac{1}{t} \oint\limits_{(K)} w_t(\zeta') \, K_I(\zeta,\zeta') \, |d\zeta'| = w_{\infty x} \cos \theta(\zeta) + w_{\infty y} \sin \theta(\zeta) + \\ + \frac{u_y \sin \theta(\zeta)}{2} + \frac{u_y}{t} \oint\limits_{(K)} [K_I \sin \theta(\zeta') + K_{II} \cos \theta(\zeta')] \, d\zeta' \tag{2}$$

Let us consider the integral

$$I = \oint\limits_{(K)} [K_I \sin \theta(\zeta') + K_{II} \cos \theta(\zeta')] \, |d\zeta'|$$

and putting $K = \Psi + i \, \Phi$ we have:

$$I = \oint\limits_{(K)} \{- I_m[e^{i\theta(\zeta)} K] \, I_m[e^{i\theta(\zeta')}] + Re[e^{i\theta(\zeta)} K] \, Re[e^{i\theta(\zeta')}]\} \, |d\zeta'| =$$

$$= Re \oint\limits_{(K)} e^{i\theta(\zeta)} K e^{i\theta(\zeta')} |d\zeta'| = Re \frac{e^{i\theta(\zeta)}}{2} \oint\limits_{(K)} \coth \frac{\pi}{t} (\zeta - \zeta') \, d\zeta' .$$

With the help of the residue theory, it is not difficult to see that:

$$I = - t \frac{\sin \theta(\zeta)}{2} .$$

Substituting the value of I, as determined above in Eq. (2), we have:

$$\frac{w_t(\zeta)}{2} - \frac{1}{t}\oint w_t(\zeta')\, K_{\mathrm{I}}(\zeta,\zeta')\, |d\zeta'| = w_{\infty x}\cos\theta(\zeta) + w_{\infty y}\sin\theta(\zeta) \tag{3}$$

Relation (3) is the well known equation for axial flow.

Mixed flow

In the following considerations, we shall begin by writing an integral equation applicable to determining the relative velocity, knowing that

$$c_{1y} - c_{2y} = \frac{1}{t}\oint_{(K)} c_t(\zeta')\, |d\zeta'|$$

$$c_{1x} = \frac{Q_n}{tb_1} = c_{2x}\frac{b_2}{b_1}$$

and using approximating values of C_{Bx} and C_{By} [1]

$$c_{Bx} \simeq \frac{Q_n}{b(x)\, t(x)} - c_{\infty x}\frac{t}{t(x)}$$

$$c_{By} \simeq 0$$

Eq. (1) takes the dimensionless form

$$\frac{c_t^*(\zeta)}{2} - \frac{1}{t}\oint_{(K)}\left[K_{\mathrm{I}} - \frac{\sin\theta(\zeta)}{2}\right] c_t^*(\zeta')\, |d\zeta'| = \mathrm{tg}\,\alpha_1 \sin\theta(\zeta) +$$

$$+ \frac{1}{t}\oint_{(K)} K_{\mathrm{II}}\, c_n^*(\zeta')\, |d\zeta'| + \left[\frac{1}{2}\left(1 + \frac{b_1}{b_2}\right)\left(1 - \frac{t}{t(x)}\right) + \frac{t\, b_1}{t(x)\, b(x)}\right]\cos\theta(\zeta)$$

where

$$c_t^*(\zeta) = \frac{c_t(\zeta)}{c_{1x}}; \quad c_n^*(\zeta) = \frac{c_n(\zeta)}{c_{1x}}.$$

Putting $\theta = \theta(\zeta)$ and $\theta' = \theta(\zeta')$ for convenience, we obtain

$$\frac{w_t^*(\zeta)}{2} - \frac{1}{t}\oint_{(K)}\left[K_{\mathrm{I}} - \frac{\sin\theta}{2}\right] w_t^*(\zeta')\, |d\zeta'| = -\frac{u_y^*(\zeta)}{2}\sin\theta -$$

$$
-\frac{\sin\theta}{2t}\oint\limits_{(K)} u_y^*(\zeta')\sin\theta'\,|d\zeta'| + \frac{1}{t}\oint\limits_{(K)} u_y^*(\zeta')\left[K_{\mathrm{I}}\sin\theta' + K_{\mathrm{II}}\cos\theta'\right)|d\zeta'| +
$$

$$
+\operatorname{tg}\alpha_1\sin\theta + \left[\frac{1}{2}\left(1+\frac{b_1}{b_2}\right)\left(1-\frac{t}{t(x)}\right) + \frac{tb_1}{t(x)\,b(x)}\right]\cos\theta \tag{4}
$$

for $\zeta' \to \zeta$ functions Φ and Ψ are known from reference [4]

$$
\Psi \approx \frac{t}{2\pi}\,\frac{\xi-\xi'}{(\xi-\xi')^2+(\eta-\eta')^2}
$$

$$
\Phi \approx -\frac{t}{2\pi}\,\frac{\eta-\eta'}{(\xi-\xi')^2+(\eta-\eta')^2}
$$

therefore the integral

$$
J = \frac{1}{t}\oint\limits_{(K)} u_y^*(\zeta')\,[K_{\mathrm{I}}\sin\theta' + K_{\mathrm{II}}\cos\theta']\,|d\zeta'| =
$$

$$
= \frac{1}{t}\oint\limits_{(K)} u_y^*(\zeta')\left[\psi(\zeta,\zeta')\cos(\theta+\theta') - \Phi(\zeta,\zeta')\sin(\theta+\theta')\right]|d\zeta'|
$$

will be transformed

$$
\begin{aligned}
J = \oint\limits_{(K)} \Bigg\{ & \left[\frac{\Psi}{t}\cos(\theta+\theta') - \frac{\Phi}{t}\sin(\theta+\theta')\right]u_y^*(\zeta') - \\
& -\frac{1}{2\pi}\left[\frac{\xi-\xi'}{(\xi-\xi')^2+(\eta-\eta')^2}\cos(\theta+\theta') + \right. \\
& \left. + \frac{\eta-\eta'}{(\xi-\xi')^2+(\eta-\eta')^2}\sin(\theta+\theta')\right]u_y^*(\zeta)\Bigg\}|d\zeta'| + \\
& + \frac{u_y^*(\zeta)}{2\pi}\oint\limits_{(K)}\left[\frac{\xi-\xi'}{(\xi-\xi')^2+(\eta-\eta')^2}\cos(\theta+\theta') + \right. \\
& \left. + \frac{\eta-\eta'}{(\xi-\xi')^2+(\eta-\eta')^2}\sin(\theta+\theta')\right]|d\zeta|\,.
\end{aligned}
$$

The first integral can be computed numerically. The computation of the latter can be made analytically with no difficulty.

$$M = \oint_{(K)} \left[\frac{\xi - \xi'}{(\xi - \xi')^2 + (\eta - \eta')^2} \cos(\theta + \theta') + \right.$$

$$\left. + \frac{\eta - \eta'}{(\xi - \xi')^2 + (\eta - \eta')^2} \sin(\theta + \theta') \right] |d\zeta'| =$$

$$= Re \oint_{(K)} \frac{e^{i(\theta + \theta')}}{\zeta - \zeta'} |d\zeta'| = Re\, e^{i\theta} \oint_{(K)} \frac{d\zeta'}{\zeta - \zeta'} = -\pi \sin\theta$$

Hence, Eq. (4) may be written as:

$$\frac{w_t^*(\zeta)}{2} - \frac{1}{t} \oint_{(K)} \left[K_1 - \frac{\sin\theta}{2} \right] w_t^*(\zeta') |d\zeta'| = - u_y^*(\zeta) \sin\theta -$$

$$- \frac{\sin\theta}{2t} \oint_{(K)} u_y^*(\zeta') \sin\theta' |d\zeta'| + \left[\frac{1}{2} \left(1 + \frac{b_1}{b_2}\right) \left(1 - \frac{t}{t(x)}\right) + \right.$$

$$\left. + \frac{t\, b_1}{t(x)\, b(x)} \right] \cos\theta + \oint_{(K)} \left\{ \left[\frac{\Psi}{t} \cos(\theta + \theta') - \frac{\Phi}{t} \sin(\theta + \theta') \right] u_y^*(\zeta') - \right.$$

$$- \frac{u_y^*(\zeta)}{2\pi} \left[\frac{\xi - \xi'}{(\xi - \xi')^2 + (\eta - \eta')^2} \cos(\theta + \theta') + \right.$$

$$\left. \left. + \frac{\eta - \eta'}{(\xi - \xi')^2 + (\eta - \eta')^2} \sin(\theta + \theta') \right] \right\} |d\zeta'| + \operatorname{tg} \alpha_1 \sin\theta$$

Results and conclusion

The solution of the integral equation can be reduced to the solution of a linear equation system, using Gaussian quadrature. The contour (K) is divided into two parts K_1 and K_2, the ends of which are the leading edge and the trailing edge. Since the abscissas and weights of Gaussian quadrature are symmetrical about the middle of the interval, such a division ensures a concentration of points in regions of great curvature.

The condition of a smooth flow at the trailing edge will be achieved by the restrictions $W_t^*(\zeta_1) = - w_t^*(\zeta_n)$ if n points are assumed on the contour.

Typical results are shown in Figs 2 and 3. Fig. 2 presents dimensionless circulations Γ^* for different numbers of points, with a view to increased accuracy. The convergency is good.

Fig. 3 illustrates the velocity distribution over the blade.

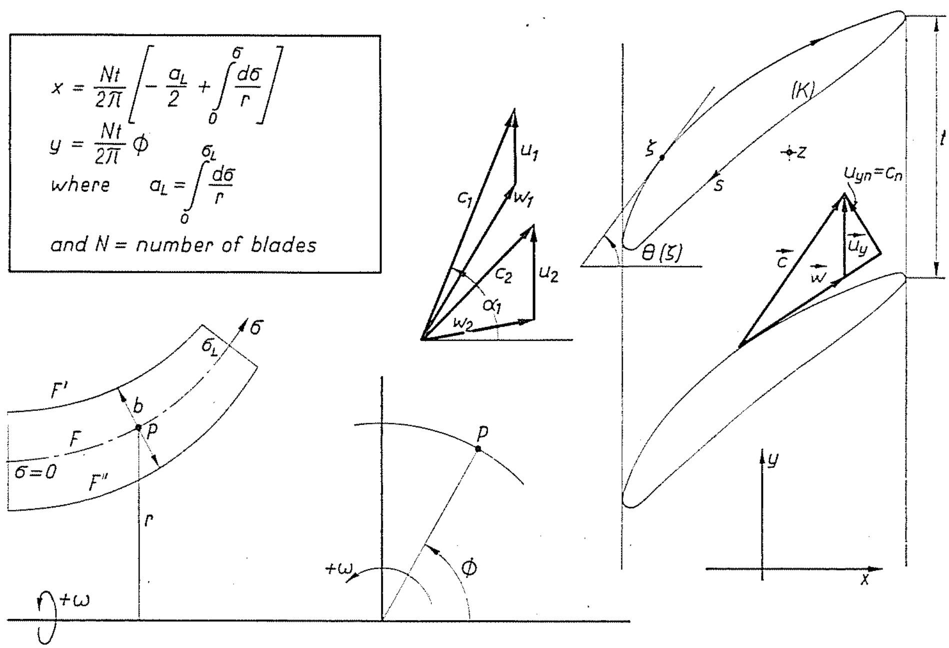
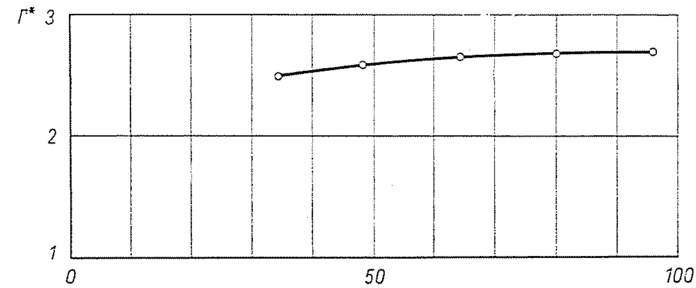


Fig. 1



points	Γ^*
34	2,5
48	2,583
64	2,657
80	2,674
96	2,686

Fig. 2

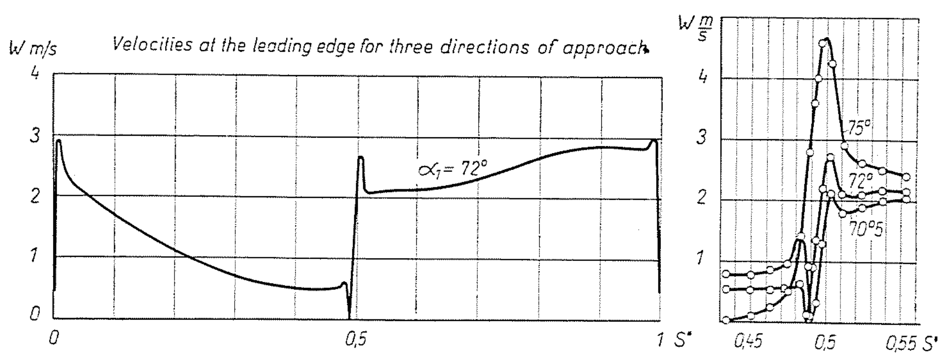


Fig. 3

An analysis is presented for finding out velocity distribution on a blade given by its geometry. The method outlined above seems to be one of the best for the numerical approach to the problem.

Notations in the straight cascade

x, y	co-ordinates in the Z plane;
$\zeta = \xi + i\eta$	point of the profile contour;
t	blade pitch;
b	width of the channel in the meridional section;
c	absolute velocity;
w	relative velocity;
C_∞	basic flow velocity;
C_B	velocity induced by the variation of the width of the partial channel;
U_y	the counterpart correspondent of the peripheral velocity in straight cascade;
Q_n	volumetric flow between two blades in the layer of thickness b;
Γ	blade circulation

References

1. Fűzy, O.—Thuma A. Calculating the velocity distribution of a plane airfoil cascade given by its geometry. Proceedings of the Third Conference on Fluid Mechanics and Fluid Machinery, Budapest 1969
2. Nyiri A.: Determination of the theoretical characteristics of hydraulic machines, based on potential theory. Acta Technica Hung. 69 (1971)
3. Czibere, T.: Über die Berechnung der Schaufelprofile von Strömungsmaschinen mit halbaxialer Durchströmung. Acta Technica Hung. 44 (1963)
4. Fűzy O. Design of Mixed Flow Impellers. Periodica Polytechnica, Vol. 6, 4 (1962)

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