ON PERTURBATIONS OF A CLASS OF SELF-EXCITED OSCILLATORS

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1. It is well known that, for ε small, both Van der Pol's equation

$$\ddot{u} - \varepsilon(1 - u^2)\dot{u} + u = 0 \tag{1}$$

and Rayleigh's equation

$$\ddot{u} - \varepsilon(1 - \dot{u}^2)\dot{u} + u = 0 \tag{2}$$

have a unique non-constant periodic solution. It is interesting to note that if Van der Pol's equation is modified by the addition of the term $\varepsilon\dot{u}^3$

$$\ddot{u} - \varepsilon(1 - u^2 - \dot{u}^2)\dot{u} + u = 0 \tag{3}$$

the resulting oscillator (3) has a unique periodic solution for all positive values of ε, no matter how large (cf. [2]).

Here we shall consider a class of self-excited oscillators, i.e.

$$\ddot{u} - \varepsilon[1 - \varphi(H)]\dot{u} + \psi(u) = 0 \tag{4}$$

where $\varphi(H)$ and $\psi(u)$ will be defined later.

Since the class of oscillators (4) is response to stochastic excitation, and since under certain conditions to be specified later (cf. [2]), it possesses a unique periodic solution $u_0(t)$ of period τ, so the perturbed class of self-excited oscillators

$$\ddot{u} - \varepsilon(1 - \varphi(H))\dot{u} + \psi(u) = \mu\,\gamma\left(\frac{t}{\tau}, u, \dot{u}, \mu, \tau\right) \tag{5}$$

is considered, where $\varepsilon > 0$, μ is a small parameter and the perturbation $\gamma \varepsilon C^1$, is periodic in t with period τ.

It will also be assumed that the period of the perturbation is controllable.

The results of this paper are mainly based on those of papers [2], [6], [7].

Let $\psi \varepsilon C^1$ be an odd function, $u\psi(u) > 0$ for $u \neq 0$ and $\Psi(u) = \int\limits_0^u \psi(t)dt$ a positive, strictly increasing function. Assume further that $H(u,\dot{u}) = \frac{1}{2}\dot{u}^2 + \Psi(u)$ and $\varphi \varepsilon C^1$ is a positive strictly increasing function, such that

$$\varphi(c_0) = 1, \quad 0 < c_0 < \infty, \text{ and } \lim_{H\to\infty} \varphi^{-2} \cdot d\varphi/dH = 0.$$

Under these conditions Caughey and Payne [2] proved that the class of self-excited oscillators (4) has a unique periodic solution $u_0(t)$. Its least (positive) period will be denoted by τ_0.

Eq. (3) is a special case of Eq. (4) with $\varphi(H) = 2H$ and $\psi(u) = u$. It is assumed without loss of generality that

$$u_0(0) = 0, \quad \dot{u}_0(0) = a > 0, \quad a \neq 1. \tag{6}$$

Introducing the notations

$$x_1 = u, \qquad x_2 = \dot{u}, \tag{7}$$

Eq. (4) is reduced to the system

$$\dot{x} = f(x), \tag{8}$$

where

$$x = \text{col}\,[x_1, x_2], \text{ and } f(x) = \text{col}\,[x_2, \ \varepsilon[1 - \varphi(H)]\,x_2 - \psi(x_1)].$$

The periodic solution of (8) corresponding to the solution $u_0(t)$ of (4) is

$$p(t) = \text{col}\,[u_0(t), \dot{u}_0(t)]. \tag{9}$$

The first variational system of (8) corresponding to $p(t)$ is

$$y = f'_x(p(t))\,y \quad \text{with} \quad f'_x(p(t)) = \begin{bmatrix} 0 & 1 \\ \alpha_1(t) & \alpha_2(t) \end{bmatrix}, \tag{10}$$

where

$$\alpha_1(t) = -\varepsilon\,\dot{u}_0(t)\,\varphi'_H \cdot \psi(u_0(t)) - \psi'(u_0(t)),$$
$$\alpha_2(t) = \varepsilon[1 - \varphi - \dot{u}_0^2(t)\,\varphi'_H]. \tag{11}$$

The system (10) is a linear system with τ_0-periodic coefficients and has the periodic solution

$$\dot{p}(t) = \text{col}[\dot{u}_0(t), \alpha_3(t)], \tag{12}$$

of period τ_0, where $\alpha_3(t) = \varepsilon\,[1 - \varphi(H)]\dot{u}_0(t) - \psi(u_0(t))$.
The scalar form of (10) is

$$\dot{y}_1 = y_2,$$
$$\dot{y}_2 = \alpha_1(t)y_1 + \alpha_2(t)y_2.$$

By eliminating y_2 we obtain

$$\ddot{y}_1 - \alpha_2(t)\dot{y}_1 - \alpha_1(t)y_1 = 0. \tag{13}$$

One solution of (13) is clearly $\dot{u}_0(t)$. A second solution of (13) is found by setting $y_1 = \dot{u}_0(t)v(t)$ in (13). The result is

$$y_1 = v_0(t) = \dot{u}_0(t)\int_0^t [\dot{u}_0(s)]^{-2}\left[\exp\int_0^s \alpha_2(\tau)d\tau\right]ds. \tag{14}$$

However, $v_0(t)$ must be properly defined at zeros of $u_0(t)$, since at such points (14) is meaningless. It is clear from (14) that

$$v_0(0) = 0, \qquad \dot{v}_0(0) = \frac{1}{\ddot{u}_0(0)} = \frac{1}{a}. \tag{15}$$

Let $Y_0(t)$ denote the fundamental matrix solution of the system (10) for which $Y_0(0) = U$,(U is the unit matrix,) holds,

$$Y_0(t) = \begin{bmatrix} \frac{1}{a}\dot{u}_0(t) & a\,v_0(t) \\ \frac{1}{a}\alpha_3(t) & a\,\dot{v}_0(t) \end{bmatrix}. \tag{16}$$

Also the principal matrix of (10) is

$$C_0 = Y_0(\tau_0) = \begin{bmatrix} 1 & a\,v_0(\tau_0) \\ 0 & a\,v_0(\tau_0) \end{bmatrix} \tag{17}$$

According to Liouville's formula, the Wronksian determinant $W(t) = \det[Y_0(t)]$, for which $W(0) = 1$, is

$$W(t) = \dot{u}_0(t)\dot{v}_0(t) - v_0(t)\alpha_3(t) = \exp\left[\int_0^t \alpha_2(s)ds\right]. \tag{18}$$

Thus the characteristic multipliers of (10) (i.e. the eigenvalues of the principal matrix C_0) are 1 and the following expression:

$$W(\tau_0) = \exp\left[\int_0^{\tau_0} \alpha_2(t)dt\right]. \tag{19}$$

Now let us consider the perturbed class of oscillators (5) reduced by the substitution (7) to the system

$$\dot{x} = f(x) + \mu g\left(\frac{t}{\tau}, x, \mu, \tau,\right) \tag{20}$$

where x and $f(x)$ are as defined before and

$$g\left(\frac{t}{\tau}, x, \mu, \tau\right) = \mathrm{col}\left[0, \gamma\left(\frac{t}{\tau}, x, \mu, \tau\right)\right], \tag{21}$$

Here we want to emphasize that the perturbed system (20) can be treated, provided that we know (nothing else but):

I) a unique periodic solution $p(t)$ of the unperturbed system (8),

II) the fundamental matrix solution of the first variational system of the unperturbed system (8) corresponding to the solution $p(t)$.

We shall assume that 1 is a simple characteristic multiplier of (10)

$$\left(\text{i. e.} \int_0^{\tau_0} \alpha_2(t)dt \neq 0\right).$$

2. Consider the perturbed system

$$\dot{x} = f(x) + \mu g\left(\frac{t}{\tau}, x, \mu, \tau\right) \tag{20}$$

where the right-hand side is periodic in t with period τ, and analytic [although much less would suffice for the discussion here] in the region $I_t \times \Omega \times I_\mu \times I_\tau$ where: $I_t = \{t : -\omega < t < \omega\}$, Ω is an open connected region of the two-dimensional plane, $I_\mu = \{\mu : |\mu| < \alpha\}$, for some $\alpha > 0$, and $I_\tau = \{\tau : |\tau - \tau_0| < \beta\}$ for some β, $0 < \beta < \tau_0$.

According to the above conditions and M. Farkas's general theorem 2 (cf. [6]), for each small value of $|\mu|$ and the parameter $|\vartheta|$ (defined there), the system (20) has a unique periodic solution $x(t; \mu, \vartheta)$ with period $\tau(\mu, \vartheta)$ (provided that $\tau(\mu, \vartheta)$ is substituted into (20) for τ). Let

$$u_p(t; \mu, \vartheta) = u_p(t; \vartheta, \mu, \tau(\mu, \vartheta)) \tag{22}$$

denote the corresponding unique periodic solution of (5), for which $u_p(\vartheta; \mu, \vartheta) = 0$ holds. The functions $\tau(\mu, \vartheta)$ and $u_p(t; \mu, \vartheta)$ are analytic in the neighbourhood of $\mu = \vartheta = 0$, and $\tau(0, 0) = \tau_0$ $u_p(t; 0, 0) = u_0(t)$.

Now Poincaré's method will be worked out for the approximate determination of the periodic solution $x(t; \mu, \vartheta)$ of the system (20) up to the first approximation.

THEOREM 1

Let the conditions stated in this section hold, then the perturbed equation (5) has a unique period

$$\tau(\mu, \vartheta) = \tau_0 + \mu\tau_1(\vartheta) + o(\mu),$$

where

$$\tau_1(\vartheta) = -\int_0^{\tau_0} \frac{\gamma(r)}{W(r)}\left[v_0(r) - \frac{v_0(\tau_0)}{a[1 - W(\tau_0)]}\,\dot{u}_0(r)\right]dr, \tag{23}$$

and the unique periodic solution

$$u_p(t; \mu, \vartheta) = u_0(t - \vartheta) + \mu\left\{\frac{v_0(\tau_0)W(\tau_0)}{1 - W(\tau_0)}\int_0^{\tau_0}\frac{\gamma(r)}{W(r)}\,\dot{u}_0(r)dr + \right.$$

$$\left. + \int_0^{\frac{t-\vartheta}{\tau(\mu,\vartheta)}\tau_0} \frac{\gamma(r)}{W(r)}\,[\dot{u}(r)v_0(\varepsilon - \vartheta) - v_0(r)\dot{u}_0(t - \vartheta)]\,dr\right\} + o(\mu), \tag{24}$$

where

$$\gamma(r) = \gamma\left(\frac{r + \vartheta}{\tau_0},\; u_0(r),\; \dot{u}_0(r),\; 0,\; \tau_0\right).$$

Proof: Using the substitution

$$t = \vartheta + s\tau(\mu, \vartheta) \tag{25}$$

the system (20) and its periodic solution $x(t; \mu, \vartheta)$ assume the forms:

$$\frac{dx}{ds} = \tau(\mu, \vartheta)\left[f(x) + \mu g\left(s + \frac{\vartheta}{\tau(\mu, \vartheta)},\, x,\, \mu,\, \tau(\mu, \vartheta)\right)\right] \tag{26}$$

and

$$\psi(s; \mu, \vartheta) = x(\vartheta + s\tau(\mu, \vartheta); \mu, \vartheta) \tag{27}$$

respectively. Expand the solution $\psi(s; \mu, \vartheta)$ and the function $\tau(\mu, \vartheta)$ for fixed ϑ by powers of μ up to the first approximation, i.e.

$$\psi(s; \mu, \vartheta) = \psi^0(s, \vartheta) + \mu\psi^1(s, \vartheta) + o(\mu), \tag{28}$$

$$\tau(\mu, \vartheta) = \tau_0(\vartheta) + \mu\tau_1(\vartheta) + o(\mu). \tag{29}$$

It is clear that $x(t; 0, \vartheta) = p(t - \vartheta)$, $\tau_0(\vartheta) = \tau(0, \vartheta) = \tau_0$, and

$$\psi^0(s; \vartheta) = \psi(s; 0, \vartheta) = x(\vartheta + s\tau_0; 0, \vartheta) = p(s\,\tau_0). \tag{30}$$

The function ψ is obviously periodic in s with period one and such is, as a consequence, $\psi^0(s; \vartheta)$ and $\psi^1(s, \vartheta)$.

Substituting the expansions (28) and (29) into (26) and equating the corresponding coefficients of μ^i on both sides, we have:

$$\mu^0: \frac{d\psi^0(s,\vartheta)}{ds} = \tau_0 f(p(s\,\tau_0)),$$

$$\mu^1: \frac{d\psi^1(s,\vartheta)}{ds} = \tau_0 f'_x(p(s\,\tau_0))\,\psi^1(s,\vartheta) + \tau_0 g\left(s + \frac{\vartheta}{\tau_0} p(s\tau_0),\, 0,\, \tau_0\right) + \tau_1(\vartheta) f(p(s\,\tau_0)).$$

It is easy to prove that the system of equations given above determine ψ^1 and τ_1 uniquely when subject to the conditions that $\psi^1_1(0,\vartheta) = 0$ and $\psi^1(s,\vartheta)$ is periodic in s with period one.

After long but easy calculations we obtain the expression (23) for $\tau_1(\vartheta)$ and the two components of $\psi^1(s;\vartheta)$, i.e.

$$\psi^1_1(s,\vartheta) = \frac{v_0(s\tau_0)W(\tau_0)}{1 - W(\tau_0)} \int_0^{\tau_0} \frac{\gamma(r)}{W(r)} \dot{u}_0(r)dr + \tau_1 s\, \dot{u}_0(s\tau_0) +$$

$$+ \int_0^{s\tau_0} \frac{\gamma(r)}{W(r)} [-\dot{u}_0(r)v_0(s\tau_0) + v_0(r)\dot{u}_0(s\tau_0)]dr \tag{31}$$

and

$$\psi^1_2(s,\vartheta) = \frac{\dot{v}_0(s\tau_0)W(\tau_0)}{1 - W(\tau_0)} \int_0^{\tau_0} \frac{\gamma(r)}{W(r)} \dot{u}_0(r)dr + \tau_1 s\, \alpha_3(s\tau_0) +$$

$$+ \int_0^{s\tau_0} \frac{\gamma(r)}{W(r)} [\dot{u}_0(r)\dot{v}_0(s\tau_0) - v_0(r)\alpha_3(s\tau_0)]dr. \tag{32}$$

According to (7) and by using (30) we have:

$$u_p(s;\mu,\vartheta) = p_1(s\,\tau_0) + \mu\psi^1_1(s,\vartheta) + o(\mu).$$

Thus

$$u_p(t;\mu,\vartheta) = p_1\left(\frac{t-\vartheta}{\tau(\mu,\vartheta)}\tau_0\right) + \mu\psi^1_1\left(\frac{t-\vartheta}{\tau(\mu,\vartheta)},\, \vartheta\right) + o(\mu). \tag{33}$$

Expanding again the first term on the right-hand side of the last equation into power series in μ, we obtain the expression (24), and by that the theorem is proved.

3. Now we are going to study the stability of the solution $x(t;\mu,\vartheta)$ of the perturbed system (20). The first variational system of (20) corresponding to $x(t;\mu,\vartheta)$ is

$$\dot{y} = \left[\frac{df}{dx} + \mu \frac{\partial g\left(\frac{t}{\tau}, x, \mu, \tau\right)}{\partial x}\right]_{x = x(t,\mu,\vartheta)} y. \tag{34}$$

Let $Y(t;\mu,\vartheta)$ be the fundamental matrix solution of (34) for which $Y(0;0,0) = U$ (U is the unit matrix) holds. The principal matrix $C(\mu,\vartheta)$ of (34) corresponding to $Y(t;\mu,\vartheta)$ is then

$$C(\mu,\vartheta) = Y(\tau(\mu,\vartheta);\mu,\vartheta). \tag{35}$$

Also $C(0,0) = C_0(0) = Y(\tau_0;\ 0,0) = Y_0(\tau_0)$ is the characteristic matrix of system (10). Let also $\lambda(\mu,\vartheta)$ be the characteristic multiplier of system (34) for which $\lambda(0,0) = 1$ holds. $\lambda'_\mu(0,0)$ denotes the partial derivative of $\lambda(\mu,\vartheta)$ with respect to μ at $\mu = \vartheta = 0$.

THEOREM 2.

Let the hypotheses of theorem 1 be satisfied, there are $\varrho_1 > 0$ and $\varrho_2 > 0$ such that in the region

$$|\mu| < \varrho_1, \qquad |\vartheta| < \varrho_2 \tag{36}$$

$\lambda(\mu,\vartheta)$ is a real valued analytic function of its arguments μ and ϑ, and if $\int_0^{\tau_0} \alpha_2(t)\,dt < 0$, then the periodic solution $x(t;\mu,\vartheta)$ of the perturbed system (20) with period $\tau(\mu,\vartheta)$ is asymptotically stable for μ and ϑ that are in the region (36) and satisfy the condition

$$\mu\lambda'_\mu(0,0) < 0$$

where

$$\lambda'_\mu(0,0) = -\int_0^{\tau_0} \frac{\gamma'_t}{W(t)}\left(\frac{t}{\tau_0},\dot u_0(t),u_0(t),0,\tau_0\right)\left[v_0(t) - \frac{v_0(\tau_0)}{a[1-W(\tau_0)]}\,\dot u_0(t)\right]dt. \tag{37}$$

PROOF: The first part of the theorem is a consequence of M. Farkas's theorem 3 (cf. [7]) and of (19). Only the formula (37) is left to be proved. Since the matrix $C(\mu,\vartheta)$ is analytic in its argument, so it can be expanded in the form

$$C(\mu,\vartheta) = C^\circ(\vartheta) + \mu C^1(\vartheta) + \mu^2 R(\mu,\vartheta),$$

$R(\mu,\vartheta)$ is analytic.
Let c_i^0, c_i^1 and u_i be the i^{th} row of the matrices of $C^\circ(0)$, $C^1(0)$ and u and $(-1)^n\, d(\lambda;\mu,\vartheta) = \det\,[C(\mu,\vartheta) - \lambda U]$ be the characteristic polynomial of $C(\mu,\vartheta)$. According to M. Farkas's theorem: (cf. [7]) and to the notations defined there, we have

$$d(\lambda;0,0) = \begin{vmatrix} 1-\lambda & av_0(\tau_0) \\ 0 & av_0(\tau_0)-\lambda \end{vmatrix},$$

$$d'_\lambda(1;0,0) = 1 - W(\tau_0),$$

$$C^1(0) = \tau_1(0) f_x'(p(\tau_0)) Y_0(\tau_0) + Y_0(\tau_0) \int_0^{\tau_0} Y_0^{-1}(t) B(t, 0) dt , \tag{38}$$

where

$$B(t, 0) = \left[f_{xx}''(p(t)) x_\mu'(t; 0, 0) + g_x' \left(\frac{t}{\tau_0}, p(t), 0, \tau_0 \right) \right] Y_0(t), \tag{39}$$

with

$$x_\mu'' (t, 0, 0) = \psi^1 \left(\frac{t}{\tau_0}, 0 \right) - \dot{p}(t)\, t \frac{\tau_1(0)}{\tau_0} , \tag{40}$$

$$f_{xx}''(p(t)) x_\mu''(t; 0, 0) = \begin{bmatrix} 0 & 0 \\ \delta_1 x_{1\mu}' + \delta_2 x_{2\mu}' & \delta_2 x_{1\mu}' + \delta_3 x_{2\mu}' \end{bmatrix}, \tag{41}$$

where
$$\delta_1 = -\varepsilon \dot{u}_0(t) [\psi'(u_0(t)) \varphi_H' + \varphi_{HH}'' \psi^2(u_0(t))] - \psi''(u_0(t)) ,$$
$$\delta_2 = -\varepsilon \psi(u_0(t)) [\varphi_H' + \dot{u}_0^2(t) \varphi_{HH}''] ,$$
$$\delta_3 = -\varepsilon [3\, \dot{u}_0(t)\, \varphi_H' + \dot{u}_0^3(t)\, \varphi_{HH}''] ,$$

and

$$g_x' \left(\frac{t}{\tau_0}, p(t), 0, \tau_0 \right) = \begin{bmatrix} 0 & 0 \\ \gamma_{x_1}' & \gamma_{x_2}' \end{bmatrix}, \tag{42}$$

where $\gamma_{x_1}' \left(\frac{t}{\tau_0}, p(t), 0, \tau_0 \right)$ and $\gamma_{x_2}' \left(\frac{t}{\tau_0} p(t), 0\ \tau_0 \right)$ are the partial derivatives of the function $\gamma \left(\frac{t}{\tau}, x, \mu, \tau \right)$ with respect to x_1 and x_2 evaluated at $\mu = \vartheta = 0$ and $x = p(t)$.

Thus by the quoted theorem of [7], we have

$$\lambda_\mu'(0, 0) = \frac{1}{W(\tau_0) - 1} \left\{ \det \begin{vmatrix} C_{11}^1 & C_{12}^1 \\ C_{21}^0 & C_{22}^0 \end{vmatrix} + \det \begin{vmatrix} C_{11}^0 - 1 & C_{12}^0 \\ C_{21}^1 & C_{22}^1 \end{vmatrix} \right\} =$$
$$= C_{11}^1(0) + \frac{a v_0(\tau)}{1 - W(\tau_0)} C_{21}^1 (0) . \tag{43}$$

Consider the periodic vector

$$z(t) = \frac{1}{W(t)} \left[\dot{v}_0(t) - \frac{v_0(\tau_0)}{a[1 - W(\tau_0)]} \alpha_3(t), - v_0(t) + \frac{v_0(\tau_0)}{a[1 - W(\tau_0)]} \dot{u}_0(t) \right] \tag{44}$$

of period τ_0 in t. It is clear that the row vector $z(t)$ satisfies the equation

$$\dot{z} = - z f_x'(p(t)) .$$

Also it is easy to prove that $x'_\mu(t; 0, 0)$ satisfies the system

$$\dot{x}'_\mu = f'_x(p(t))\, x'_\mu(t; 0, 0) + g\left(\frac{t}{\tau_0}, p(t); 0, \tau_0\right).$$

After long but simple calculations, substituting the expressions (39), (40), (41) and (42) into (38) and using (43) and (44) we get:

$$\lambda'_\mu(0, 0) = -\frac{av_0(\tau_0)}{1 - W(\tau_0)}\tau_1(0) + \int_0^{\tau_0} z(t)\Big[f''_{xx}(p(t))x'_\mu(t; 0, 0) + + g'_x\left(\frac{t}{\tau_0}, p(t), 0, \tau_0\right)\Big]\dot{p}(t)dt \tag{45}$$

Taking into account the periodicity of $f'_x(p(t))$ and $g\left(\frac{t}{\tau_0}, p(t), 0, \tau_0\right)$, it is easy to prove that

$$\int_0^{\tau_0} z(t)\left[f''_{xx}(p(t))\, x'_\mu(t; 0, 0) + g'_x\left(\frac{t}{\tau_0}\right) p(t), 0, \tau_0\right] dt = = \frac{av_0(\tau_0)\, \tau_1(0)}{1 - W(\tau_0)} - \int_0^{\tau_0} z(t)\, g'_t\left(\frac{t}{\tau_0}, p(t), 0, \tau_0\right) dt.$$

Substituting this into (45) we get expression (37) and this completes the proof of the theorem.

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Summary

The perturbation of a class of self-excited oscillators is considered. Conditions for existence, uniqueness and stability of a periodic solution are given. Poincaré's method is used to obtain expressions for the unique periodic solution, its period and for the stability criteria.

References

1. CAUGHEY, T. K.: Response of Van der Pol's oscillator to random excitation. J. Appl. Mech. 26 (1959), pp. 245
2. CAUGHEY, T. K. and PAYNE, H. J.: On response of a class of self-excited oscillators to stochastic excitation. Int. J. Nonlinear Mec. Vol. 2, pp. 125—151 (1967)

3. CODDINGTON E. and LEVINSON, N.: Theory of ordinary differential equations, McGraw-Hill, 1955, New York, London
4. EL OWAIDY, H.: Further stability conditions for controllably periodic perturbed solutions (to appear)
5. EL OWAIDY, H.: On perturbations of Liénard's equation (to appear)
6. FARKAS, M.: Controllably periodic perturbations of autonomous system, Acta Math. Acad. Sci. Hungar. 22 (1971), pp. 337—348.
7. FARKAS, M.: Determination of controllably periodic perturbed solutions by Poincaré's method, Studia Sci. Math. Hungary 7 (1972), 259—268.
8. FARKAS J. and FARKAS, M.: On perturbations of Van der Pol's equation Annales Univ. Sci. Budapest, Math. 15 (1972), 155—164.
9 HALE J. K.: Oscillations in nonlinear system. McGraw Hill, Advanced Math. with application (1963), New York, London.
10. HALE, J. K.: Ordinary differential equations, Wiley-Interscience, (1969)

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