GENERAL SOLUTION TO A CLASS OF UNSTEADY HEAT CONDUCTION PROBLEMS IN A SOLID CYLINDER WITH THE CONVECTIVE TYPE OF TIME-DEPENDENT BOUNDARY CONDITIONS

By

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Notations

$A_i \geq 0,\ B_i \geq 0$ $i = 1, 2, 3$	Constant boundary coefficients on surface.
A'_n	Coefficient defined by (7)
C_{kmn}	Coefficient defined by (10)
D_{km}	Coefficient defined by (11)
$f_i,\ i = 1, 2, 3$	Source function prescribed on surface
$I_{k(x)}$	Modified Bessel Function of the first kind of order k and argument x
$J_{k(x)}$	Bessel Function of the first kind of order k on argument $\varkappa$
k	$0, 1, 2, \ldots$
m, n	$1, 2, \ldots$
X, φ, Z	Cylindrical co-ordinate
$x = \frac{X}{R};\ Z z = \frac{Z}{R}$	Dimensionless co-ordinate
$Po(x, \varphi, z, Fo) = \frac{WR^2}{\lambda Ta}$	Pomerantesev criterion
$Fo = \frac{at}{R^2}$	Fourier number
$\theta(x, \varphi, z, Fo) = \theta = T \frac{Ta}{T}$	Dimensionless temperature distribution
$\theta o_j(x, \varphi, z, Fo) = \frac{To}{Ta} = \theta o_j$	Dimensionless temperature distribution
W	Volume-heat source
T	Temperature distribution
a	Thermal diffusivity
Ta	Ambient temperature
R	Radius
$\delta o_j,\ \delta ij$	Kronecker delta
λ_{kmn}	Eigenvalues in space x, φ, z
μ_{km}	defined by Equ. (7)
$\psi_{kmn}(x, \varnothing\ z)$	Eigenfunction defined in space x, φ, z
$(\bar{\ })$	(a) finite cosine transform of ( ) defined by Eq. (15)
$(\bar{\bar{\ }})$	(b) finite trigonometrical transform of $(\bar{\ })$ defined by Eq. (16)
$(\hat{\bar{\bar{\ }}})$	(c) finite Hankel transform of $(\bar{\bar{\ }})$ defined by Eq. (17)
$Z_n(z)$	defined by Eq. (10)
$\lambda_n = \frac{\alpha_n}{b^2}$	defined by Eq. (16).

Introduction

OLCER [1] studied convective heat transfer in finite region under the boundary conditions of third kind and later on he [2] extended his result for a three-dimensional rectangular region under the influence of an arbitrarily internal heat source and with arbitrary homogeneous boundary conditions of convective type.

In this paper we consider a more general and complete problem for a finite solid cylinder is subjected on its entire surface to boundary conditions of the third kind. The solution of the problem is obtained in quasi-steady and transient terms. A number of deductions have been obtained which are available in the literature.

Statement of the problem

Consider a three-dimensional problem of unsteady temperature distribution in a right circular cylinder of finite length 2b. The entire surface is subjected to boundary conditions of the third kind. Using cylindrical polar coordinates x, φ, z and choosing the z cordinate along the geometrical axis of the cylinder, the flow of heat conduction in dimensionless form can be written as

$$\frac{\partial^2 \Theta}{\partial x^2} + \frac{1}{x}\frac{\partial \Theta}{\partial x} + \frac{1}{x^2}\frac{\partial^2 \Theta}{\partial \Phi^2} + \frac{\partial^2 \Theta}{\partial z^2} + Po(x, \varphi, z, Fo) = \frac{\partial \Theta}{\partial Fo}$$

$$Fo > 0,\ 0 \leqslant x < 1,\ 0 \leqslant \varphi \leqslant 2x,\ |z| < b \tag{1}$$

where $\theta = \theta(x, \varphi, z, Fo)$ is the unsteady temperature distribution. Associated with (1), the boundary conditions are

$$A_1\frac{\partial \Theta}{\partial z} + B_1\Theta = f_1(x, \varphi, Fo) \quad 0 \leqslant x < 1,\ 0 \leqslant \varphi \leqslant 2x,\ z = -b,\ F_0 > 0 \tag{2a}$$

$$A_2\frac{\partial \Theta}{\partial z} + B_2\Theta = f_2(x, \varphi, Fo) \quad 0 \leqslant x < 1,\ 0 \leqslant \varphi \leqslant 2x,\ z = b,\ F_0 > 0 \tag{2b}$$

$$A_3\frac{\partial \Theta}{\partial x} + B_3\Theta = f_3(z, \varphi, Fo) \quad x = 1,\ 0 \leqslant \varphi \leqslant 2x,\ z = b,\ F_0 > 0 \tag{2c}$$

where $B_i (i = 1,2,3)$ are nonzero boundary coefficients and f_i $(i = 1, 2, 3)$ are integrable functions prescribed on the bases and curved surface. The statement of the problem is completed by introducing the initial condition

$$(\Theta)\ Fo = 0 = \Theta(x, \varphi, z, 0) = F(x, \varphi, z). \tag{2d}$$

Solution of the Problem

The eigenvalue problem corresponding to Eq. (1) is

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{1}{x}\frac{\partial \psi}{\partial x} + \frac{1}{x^2}\frac{\partial^2 \psi}{\partial \varphi^2} + \frac{\partial^2 \psi}{\partial z^2} + \lambda_{kmn}\psi = 0 \tag{3}$$

$$0 \leqslant \varphi \leqslant 2x,\ 0 \leqslant x < 1,\ |z| < b$$

and

$$A_1 \frac{\partial \psi}{\partial z} + B_1 \psi = 0, \quad 0 \leqslant x < 1, \; 0 \leqslant \varphi \leqslant 2x, \quad z = -b$$

$$A_2 \frac{\partial \psi}{\partial z} + B_2 \psi = 0 \quad 0 \leqslant x < 1, \; 0 \leqslant \varphi \leqslant 2x, \quad z = b \tag{4}$$

$$A_3 \frac{\partial \psi}{\partial z} + B_3 \psi = 0 \quad x = 1, \quad 0 \leqslant \varphi \leqslant 2x, \quad |z| < b$$

The eigenfunction of the differential equation (3) *well behaved* at the origin is

$$\psi_{kmn}(x, \varphi, z) = J_k(\mu_{km} x) \begin{Bmatrix} \cos k\varphi \\ \sin k\varphi \end{Bmatrix} Z_n(z)$$

$$(k = 0, 1, 2, \ldots, \infty \quad m, n = 1, 2, 3, \ldots, \infty) \tag{5}$$

where

$$Z_n(z) = \cos \alpha_n \left(1 + \frac{z}{b}\right) + \frac{M_1}{\alpha_n} \sin \alpha_n \left(1 + \frac{z}{b}\right) \tag{6}$$

The eigenvalue λ_{kmn} is given by

$$\lambda_{kmn}^2 = \mu_{km}^2 + \frac{\alpha_n}{b^2} \tag{7}$$

where μ_{km} is the root of

$$(k + \mu) J_k(\mu_{km}) = \mu_{km} J_{k+1}(\mu_{km})$$

$$\mu = \frac{B}{A} > 0 \tag{7a}$$

and α_n is the n^{th} root of

$$(M_1 + M_2)\alpha_n \cos 2\alpha_n = (\alpha_n^2 - M_1 M_2) \sin 2\alpha_n \tag{7b}$$

where

$$M_1 = \frac{B_1 b}{A_1} \quad \text{and} \quad M_2 = \frac{B_2 b}{A_2}$$

$$Z_n(-b) = 1, \; Z_n(b) = \frac{\alpha_n^2 + M_1^2}{\alpha_n^2 - M_1 M_2} \cos 2\alpha_n \tag{7c}$$

Taking $\psi(\lambda_{kmn} x)$ as kernel, a three-dimensional finite integral transform of $\theta(x, \varphi, z, Fo)$ is defined as

$$\Theta(k, m, n\ Fo) = \int_0^{2\lambda} \int_{-b}^{b} \int_0^{1} \psi(\lambda_{kmn} x)\, \Theta(x, \varphi, z, Fo) x\, dx d\varphi dz \tag{8}$$

with inversion

$$\Theta(x, \varphi, z, Fo) = \sum_k^\infty \sum_m^\infty \sum_n^\infty C_{kmn} \psi(\lambda\, kmn\, x) \Theta(k, m, n, Fo) \tag{9}$$

C_{kmn} is defined as

$$\frac{1}{C_{kmn}} = \frac{1}{A'_n}\,\frac{1}{D_{km}}$$

where $\frac{1}{A'_n} = \int_{-b}^{b} Z_n^2(z)dx$

$$= \frac{b(\alpha_n + M_1^2)(\alpha_n + M_2^2) + \frac{1}{2}b(M_1 + M_2)(\alpha_n^2 + M_1M_2)}{\alpha_n^2(\alpha_n^2 + M_2^2)} \quad n = 1, 2 \tag{10}$$

$$= 2b \qquad \alpha_0 = 0$$

and $\frac{1}{D_{km}} = \int_0^1\int_0^{2\lambda} J_k^2(\mu_{km}x)\begin{Bmatrix}\cos^2 k\varphi \\ \sin^2 k\varphi\end{Bmatrix} x\,dx\,d\varphi$

$$= \frac{1}{2}\frac{\pi}{\mu_{km}^2}(\mu_{km}^2 + \mu^2 - k^2)J_k^2(\mu_{km}) \qquad k = 1, 2, \ldots \infty \tag{11}$$

$$= \frac{\pi}{\mu_{om}^2}(\mu_{om}^2 + \mu^2)J_0^2(\mu_{om}) \qquad k = 0$$

It is difficult to show directly from (9) that it satisfies the boundary conditions (2). Such type of difficulty arises in dealing with the convergence of the series form of the solution. To get rid of this, it is essential to obtain from (9) an alternate form of solution composed of quasisteady state and transient parts. Thus the solution to the system of Eqs (1) and (2) can now be written down directly from the general expression (16) given in [1]. The result is

$$\begin{aligned}\Theta(x, \varphi, z, Fo) &= \sum_{j=0}^{3}\Theta o_j(x, \varphi, z, Fo) + \sum_{k}^{\infty}\sum_{m}^{\infty}\sum_{n}^{\infty}{}_{k=m=n\neq 0}\, C_{kmn}J_k(\mu_{km}x)\times Zn(z)e^{-\lambda_{kmn}^2 Fo}\\ &\times\Big[\int_0^{2\lambda}\int_{-b}^{b}\int_0^1 J_k(\mu_{km}x)Zn(z)\big[F(x, \varphi, z) - \sum_{j=0}^{3}\Theta 0_j(x, \varphi, z, Fo)\; xdxd\varphi dz\; -\\ &- \sum_{j=0}^{3}\int_0^{Fo} e^{\lambda_{kmn}^2 Fo}\int_0^{2\lambda}\int_{-b}^{b}\int_0^1 J_k(\mu_{km}x)Zn(z)\times\\ &\times\frac{\partial\Theta o_j}{\partial Fo}(x, \varphi, z, Fo)\,xdxd\varphi dzdFo'\Big]\end{aligned} \tag{12}$$

The functions $\theta o_j(x, \varphi, z, Fo)$ are defined by

$$\frac{\partial^2\Theta_{0j}}{\partial x^2} + \frac{1}{x}\frac{\partial\Theta_{0j}}{\partial x} + \frac{1}{x^2}\frac{\partial^2\Theta_{0j}}{\partial\varphi^2} + \frac{\partial\Theta_{0j}}{\partial z^2} + \delta_{0j}\,Po(x, \varphi, z, Fo) = 0 \tag{13}$$

$$0 \leqslant x < 1, |z| < b, 0 \leqslant \varphi \leqslant 2\lambda$$

and

$$-A_1 \frac{\partial \Theta o_j}{\partial z} + B_1 \Theta_{0j} = \delta_{1j} f_1(x, \varphi, F_0)$$
$$0 \leqslant x < 1, z = -b, 0 \leqslant \varphi \leqslant 2\pi$$

$$A_2 \frac{\partial \Theta o_j}{\partial z} + B_2 \Theta_{0j} = \delta_{2j} f_2(x, \varphi, F_0)$$
$$0 \leqslant x < 1, 0 \leqslant \varphi \leqslant 2\pi, z = b \qquad (14)$$

$$A_3 \frac{\partial \Theta o_j}{\partial z} + B_3 \Theta_{0j} = \delta_{3j} f_3(z, \varphi, F_0)$$
$$x = 1, 0 \leqslant \varphi \leqslant 2\pi \, | z | < b$$

Determination of functions: $\Theta 0_j(x, \varphi, z, Fo)$

The so-called pseudo-steady functions $\Theta o_j(x, \varphi, z, Fo), (j = 0, 1, 2, 3)$ are yet to be determined. We first define three finite transforms as follows:

(a) Finite cosine transform with respect to φ

$$\bar{\Theta} o_j (x, k, \varphi^1, z, Fo) = \int_0^{2\lambda} \Theta o_j (x, \varphi, z, Fo) \cos k(\varphi - \varphi')d\varphi \qquad (15)$$

with its inversion

$$\Theta o_j (x, \varphi, z, Fo) = \frac{1}{2\lambda} \bar{\Theta} o_j (x, 0, \varphi^1, z, Fo) + \frac{1}{\lambda} \sum_{k=p}^{\infty} \bar{\Theta} o_j (x, k, \varphi^1, z, Fo) \qquad (16)$$

(b) Finite trigonometrical transform with respect to z

$$\bar{\bar{\Theta}} o_j (x, n, k, \varphi', Fo) = \int_{-b}^{b} \Theta o_j (x, k, \varphi^1, z, Fo) \, Zn(z)dz \qquad (17)$$

with its inversion

$$\bar{\Theta} o_j (x, k, \varphi', z, Fo) = \sum_n^{\infty} A_n \bar{\bar{\Theta}} o_j (x, n, k, \varphi', Fo) \, Zn(z). \qquad (18)$$

c) Finite Hankel transform with respect to x

$$\hat{\bar{\Theta}} o_j (k, m, \varphi', z, Fo) = \int_0^1 \bar{\Theta} o_j(x, k, \varphi', z, Fo) \, x \, J_k(\mu_{km} x)dx \qquad (19)$$

with its inversion

$$\bar{\Theta} o_j (x, k, \varphi', z, Fo) = \sum_{m=1}^{\infty} D_{km} \, J_k(\mu_{km} x) \hat{\bar{\Theta}} o_j(m, k, \varphi', z, Fo) \qquad (20)$$

Determination of function: $\Theta_{00}(x, \varphi, z, Fo)$

From (13) with $j = 0$, the differential equation and the boundary conditions defining $\theta_{00}(x, 0, z, Fo)$ are

$$\frac{\partial^2\Theta_{00}}{\partial x^2} + \frac{1}{x}\frac{\partial\Theta_{00}}{\partial x} + \frac{1}{x^2}\frac{\partial^2\Theta_{00}}{\partial\varphi^2} + \frac{\partial^2\Theta_{00}}{\partial z^2} + Po(x, \varphi, z, Fo) = 0$$

$$0 \leqslant x < 1,\ 0 \leqslant \varphi \leqslant 2\lambda,\ |z| < b \qquad (21a)$$

and

$$-A_1\frac{\partial\Theta_{00}}{\partial z} + B_1\Theta_{00} = 0 \qquad 0 \leqslant x < 1,\ z = -b,\ 0 \leqslant \varphi \leqslant 2\pi$$

$$A_2\frac{\partial\Theta_{00}}{\partial z} + B_2\Theta_{00} = 0 \qquad 0 \leqslant x < 1,\ z = b,\ 0 \leqslant \varphi \leqslant 2\pi \qquad (21b)$$

$$A_3\frac{\partial\Theta_{00}}{\partial x} + B_3\Theta_{00} = 0 \qquad x = 1,\ |z| < b,\ 0 \leqslant \varphi \leqslant 2\pi$$

Under the transform (a) and (b) the set of Eqs (21) takes the form

$$\left[\frac{\partial^2}{\partial x^2} + \frac{1}{x} + \frac{\partial}{\partial x} - \left(\frac{k^2}{x^2} + \lambda_n^2\right)\right]\bar{\bar{\Theta}}o_0 + \bar{\bar{P}}o(x, k, n, \varphi', Fo) = 0$$

and (22)

$$A_3\frac{\partial\Theta o_0}{\partial x} + B_3\Theta o_0 = 0,$$

where

$$\overline{\overline{Po}}\,(x, k, n, \varphi', Fo) = \int_0^{2\pi}\int_{-b}^{b} Po(x, \varphi, z, Fo)\cos k(\varphi - \varphi')Z_n(z)d\varphi dz$$

The solution of the system (22) *well behaved* at $x = 0$ is

$$\bar{\bar{\Theta}}_{00}(x, k, n, \varphi', Fo) = \int_0^{x}\{G_{kn}(\varrho, x) + H_{kn}(\varrho, x)\}\,\overline{\overline{Po}}(x, k, n, \varphi', Fo)\cdot\varrho\, d\varrho +$$

$$+\int_1^{x}\{G_{kn}(x, \varrho) + H_{kn}(x, \varrho)\}\,\overline{\overline{Po}}(x, k, n, \varphi', Fo)\,\varrho\, d\varrho \qquad (23)$$

where

$$G_{kn}(\varrho, x) = \frac{\lambda_n I_k(\lambda_n x) A_3}{A_3\lambda_n I_k^1(\lambda_n) + B_3 I_k(\lambda_n)}\left[K_k(\lambda_n\varrho) I_k^1(\lambda_n) - I_k(\lambda_n\varrho) K_k^1(\lambda_n)\right] \qquad (24a)$$

and

$$H_{kn}(\varrho, x) = \frac{B_3 I_k(\lambda_n x)}{A_3\lambda_n I_k^1(\lambda_n) + B_3 I_k(\lambda_n)}\left[I_k(\lambda_n) K_k(\lambda_n\varrho) - K_k(\lambda_n) I_k(\lambda_n\varrho)\right] \qquad (24b)$$

The inverted transform is

$$\Theta o_0(x, \varphi, z, Fo) = \frac{1}{2\pi}\sum_{n=1}^{\infty} A_n^1 Z_n(z)\times$$

$$\int_0^{\varkappa}\int_0^{2\pi}\int_{-b}^{b}\{G_{0n}(\varrho, x) + H_{0n}(\varrho, x)\}\, Po(x, \varphi, z, Fo) Z_n(z)\varrho\, d\varrho\, d\varphi\, dz +$$

$$+\int_1^{\varkappa}\int_0^{2\pi}\int_{-b}^{b}\{G_{0n}(x, \varrho) + H_{0n}(x, \varrho)\}\, Po(x, \varphi, z, Fo) Z(z)\, \varrho d\, \varrho d\, \varphi\, dz +$$

$$+\frac{1}{\pi}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} A_n^1 Z_n(z)$$

$$\int_0^{\varkappa}\int_b^{2\pi}\int_{-b}^{b}\{G_{kn}(\varrho, x) + H_{kn}(\varrho, x)\}\, Po(x, \varphi, z, Fo) \cos k(\varphi - \varphi') Z_n(z)\varrho d\varrho d\varphi dz$$

$$+\int_1^{\varkappa}\int_0^{2\pi}\int_{-b}^{b}\{G_{kn}(x, \varrho) + H_{kn}(x, \varrho)\}\, Po(x, \varphi, z, Fo) \cos k(\varphi - \varphi') Z_n(z)\, \varrho d\varrho d\varphi dz$$

(25)

It is to be noted here that in determining an alternate expression for θo_0, a finite Hankel transform can be applied instead of the finite trigonometrical transform. Thus under the transform (a) and (c) Eq. (21) for $\hat{\Theta}_{00}$ reduces to

$$\frac{\partial^2 \hat{\hat{\Theta}}_{00}}{\partial z^2} - \mu_{km}^2 \Theta_{00} - - Po^*$$

$$(0 \leqslant x < 1,\ 0 \leqslant \varphi \leqslant 2\pi,\ |z| < b)$$

and

$$-A_1 \frac{\partial \hat{\hat{\Theta}}_{00}}{\partial z} + B_1 \hat{\hat{\Theta}}_{00} = 0$$

$$(0 \leqslant x < 1,\ 0 \leqslant \varphi \leqslant 2\pi,\ z = -b) \qquad (26)$$

$$A_2 \frac{\partial \hat{\hat{\Theta}}_{00}}{\partial z} + B_2 \hat{\hat{\Theta}}_{00} = 0$$

$$(0 \leqslant x < 1,\ 0 \leqslant \varphi \leqslant 2\pi,\ z = b)$$

where

$$Po^* = \int_0^{2\pi}\int_0^{1} Po(x, \varphi, z, Fo) \cos k(\varphi - \varphi') x\, J_k(\mu_{km} x)\, dx\, d\varphi$$

The solution of the system (26) is

$$\hat{\hat{\Theta}}_{00}(k, m, \varphi^1, z, Fo) =$$

$$[(A_1 \mu_{km} \cosh \mu_{km} b + B_1 \sinh \mu_{km} b)\int_0^{b} Po^*\{A_2 \mu_{km} \cosh \mu_{km}(z' - b) -$$

$$- B_2 \sinh \mu_{km}(z' - b)\}\, dz' + (A_2 \mu_{km} \cosh \mu_{km} b + B_2 \sinh \mu_{km} b)\cdot$$

$$\cdot \int_{-b}^{0} Po^*\{A_1 \mu_{km} \cosh \mu_{km}(z - b) + B_1 \sinh \mu_{km}(z' - b)\}\, dz'] \cosh \mu_{km} z$$

$$+ [-(A_1\mu_{km}\sinh\mu_{km}b + B_1\cosh\mu_{km}b)\int_0^b Po^*\{A_2\mu_{km}\cosh\mu_{km}(z'-b) -$$

$$- B_2\sinh\mu_{km}(z'-b)\}\,dz' + (A_2\mu_{km}\sinh\mu_{km}b + B_2\cosh\mu_{km}b)\cdot$$

$$\int_{-b}^{0} Po^*\{A_1\mu\cosh\mu_{km}(z'-b) + B_1\sinh\mu_{km}(z'-b)\}\,dz']\,\sinh\mu_{km}z)$$

$$[\mu_{km}(B_1B_2 + \mu_{km}^2A_1A_2)\sinh 2\mu_{km}b + \mu_{km}^2(A_1B_2 + A_2B_1)\cosh 2\mu_{km}b]^{-1} -$$

$$-\int^{Z}\frac{Po^*}{\mu_{km}}\sinh\mu_{km}(z'-z)\,dz' \qquad (27)$$

With its inversion

$$\frac{1}{2\pi}\sum_{m=1}^{\infty}\frac{D_{0m}J_0(\mu_{0m}x)}{\mu_{0m}(B_1B_2+\mu_{0m}^2A_1A_2)\sinh 2\mu_{0m}b + \mu_{0m}^2(A_1B_2+B_2A_1)\cosh 2\mu_{0m}b}\times$$

$$([-(A_1\mu_{0m}\cosh\mu_{0m}b + B_1\sinh\mu_{0m}b)\cdot$$

$$\int_0^b\int_0^{2\pi}\int_0^1 Po(x,\varphi,z,Fo)\,x\,J_0(\mu_{0m}x)\{A_2\mu_{0m}\cosh\mu_{0m}(z'-B) -$$

$$- B_2\sinh\mu_{0m}(z'-b)\}\,dx\,d\varphi\,dz'$$

$$-(A_2\mu_{0m}\cosh\mu_{0m}b + B_2\sinh\mu_{0m}b)$$

$$\int_{-b}^{0}\int_0^{2\pi}\int_0^1 Po(x,\varphi,z,Fo)\,xJ_0(\mu_{0m}x)\{A_1\mu_{0m}\cosh\mu_{0m}(z'-b) +$$

$$+ B_1\sinh\mu_{0m}(z'-b)\,dx\,d\varphi\,dz']\cosh\mu_{0m}z$$

$$+[-(A_1\mu_{0m}\sinh\mu_{0m}b + B_1\cosh\mu_{0m}b)$$

$$\int_0^b\int_0^{2\pi}\int_0^1 Po(x,\varphi,z,Fo)\,x\,J_0(\mu_{0m}x)\{A_2\mu_{0m}\cosh\mu_{0m}(z'-b) -$$

$$- B_2\sinh\mu_{0m}(z'-b)\}\,dx\,d\varphi\,dz'$$

$$+(A_2\mu_{0m}\sinh\mu_{0m}b + B_2\cosh\mu_{0m}b)$$

$$\int_{-b}^{0}\int_0^{2\pi}\int_0^1 Po(x,\varphi,z,Fo)\,x\,J_0(\mu_{0m}x)\{A_1\mu_{0m}\cosh\mu_{0m}(z'-b) +$$

$$+ B_1\sinh\mu_{0m}(z'-b)\}\,dx\,d\varphi\,dz']\,\sinh\mu_{0m}z)$$

$$-\frac{1}{2\pi}\sum_{m=1}^{\infty}\int^{Z}\int_0^{2\pi}\int_0^1\frac{Po}{\mu_{0m}}(x,\varphi,z',Fo)\sinh\mu_{0m}(z'-z)\,D_{0m}J_0(\mu_{0m}x)\,xdxd\varphi dz'\,.$$

$$+\frac{1}{\pi}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{D_{km}J_k(\mu_{km}x)}{\mu_{km}(B_1B_2+\mu_{km}A_1A_2)\sinh 2\mu_{km}b + \mu_{km}(A_1B_2+B_2A_1)\cosh 2\mu_{km}b}$$

$$\times\Big(\Big[-(A_1\mu_{km}\cosh\mu_{km}b+B_1\sinh\mu_{km}b)$$

$$\int_0^b\int_0^{2\pi}\int_0^1 Po(x,\varphi,z',Fo)\cos k(\varphi-\varphi')J_k(\mu_{km}x)\{A_2\mu_{km}\cosh\mu_{km}(z'-b)-$$

$$-B_2\sinh\mu_{km}(z'-b)xdxd\varphi dz'$$

$$-(A_2\mu_{km}\cosh\mu_{km}b+B_2\sinh\mu_{km}b)\,.$$

$$\cdot\int_{-b}^0\int_0^{2\pi}\int_0^1 Po(x,\varphi,z,Fo)\cos k(\varphi-\varphi')J_k(\mu_{km}x)\{A_1\mu_{km}\cosh\mu_{km}(z'-b)+$$

$$+B_1\sinh\mu_{km}(z'-b)\}\,xdxd\varphi dz'\Big]\cosh\mu_{km}z$$

$$+\Big[-(A_1\mu_{km}\sinh\mu_{km}b+B_1\cosh\mu_{km}b)$$

$$\int_0^b\int_0^{2\pi}\int_0^1 Po(x,\varphi,z',Fo)\cos k(\varphi-\varphi')J_k(\mu_{km}x)\{A_2\mu_{km}\cosh\mu_{km}(z'-b)+$$

$$+B_2\sinh\mu_{km}(z'-b)\}\,xdxd\varphi dz'$$

$$+(A_2\mu_{km}\sinh\mu_{km}+B_2\cosh\mu_{km}b)$$

$$\int_b^0\int_0^{2\pi}\int_0^1 Po(x,\varphi,z,Fo)\cosh(\varphi-\varphi')J_k(\mu_{km}x)\{A_1\mu_{km}\cosh\mu_{km}(z'-b)+$$

$$+B_1\sinh\mu_{km}(z'-b)\}\,xdxd\varphi dz'\Big]$$

$$\sinh\mu_{km}z\Big)$$

$$+\frac{1}{\pi}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}D_{km}\int^{Z}\int_0^{2\pi}\int_0^1\frac{Po}{\mu_{km}}(x,\varphi,z',Fo)\sinh\mu_{km}(z'-z)xJ_k(\mu_{km}x)\,\cdot$$

$$\cdot\cos k(\varphi-\varphi')dxd\varphi dz'\Big) \tag{28}$$

Determination of function $\Theta_1(x,\varphi,z,Fo)$

From systems (13) and (14) with $j=1$ the differential equation and boundary conditions defining $\Theta_{01}(x,\varphi,z,Fo)$ are

$$\left(\frac{\partial^2}{\partial x^2}+\frac{1}{x}\frac{\partial}{\partial x}+\frac{1}{x^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}\right)\Theta_{01}(x,\varphi,z,Fo)=0$$

$$0\leqslant x<1,\quad 0\leqslant\varphi\leqslant 2\pi,\ |z|<b \tag{29}$$

and

$$-A_1\frac{\partial\Theta_{01}}{\partial z}+B_1\Theta_{01}=f_1(x,\varphi,Fo)$$

$$0\leqslant x<1,\quad 0\leqslant\varphi\leqslant 2\pi,\ z=-b$$

$$A_2 \frac{\partial \Theta_{01}}{\partial z} + B_2\, \Theta_{01} = 0$$

$$0 \leqslant x < 1, \quad 0 \leqslant \varphi \leqslant 2\pi \quad z = b \tag{30}$$

$$A_3 \frac{\partial \Theta_{01}}{\partial x} + B_3\, \Theta_{01} = 0$$

$$x = 1, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad |z| < b$$

Under the transform (a) and (b) the sets of equations (29) and (30) yield for the expression $\bar{\bar{\Theta}}_{01}(x, k, \varphi_1, n, Fo)$ as

$$\bar{\bar{\Theta}}_{01}(x, k, n, \varphi', Fo)$$

$$= -\frac{1}{A_1}\left[\int_0^{\varkappa} \{G_{kn}(\varrho, x) + H_{kn}(\varrho, x)\}\, \bar{f}_1\, \varrho d\varrho + \int_{\varkappa}^{1} \{G_{kn}(x, \varrho) + H_{kn}(x, \varrho)\}\, \bar{f}_1\, \varrho d\varrho\right]$$

where (31)

$$\bar{f}_1 = \int_0^{2\pi} f_1(x, \varphi, Fo) \cosh k(\varphi - \varphi') d\varphi'$$

with its inversion

$$-\frac{1}{2\pi A_1} \sum_{n=1}^{\infty} A_n^1 Z_n(z) \left[\int_0^{2\pi}\int_0^{\varkappa} \{G_{on}(\varrho, x) + H_{on}(\varrho, x)\} f_1(x, \varphi, Fo)\, \varrho d\varrho d\varphi + \right.$$

$$\left. + \int_0^{2\pi}\int_{\varkappa}^{1} \{G_{on}(x, p) + H_{on}(x, \varrho)\} f_1(x, \varphi, Fo)\, \varrho d\varrho d\varphi\right]$$

$$-\frac{1}{\pi A_1} \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} A_n^1 Z_n(z) \left[\begin{array}{l} \int_0^{2\pi}\int_0^{\varkappa} \{G_{kn}(\varrho, x) + H_{kn}(\varrho, x)\} f_1(x, \varphi, Fo) \cos k(\varphi - \varphi')\, \varrho d\varrho d\varphi \\ \int_0^{2\pi}\int_{\varkappa}^{1} \{G_{kn}(x, \varrho) + H_{kn}(x, \varrho)\} f_1(x, \varphi, Fo) \cos k(\varphi - \varphi')\, \varrho d\varrho d\varphi \end{array}\right] \tag{32}$$

An alternate expression for $\Theta_{01}(x, \varphi, z, F_0)$ under the transform (a) and (c) is

$$\hat{\bar{\Theta}}_{01}(k, m, \varphi', z, Fo) = \frac{A_2 \mu_{km} \cosh \mu_{km}(b + z) + B_2 \sinh \mu_{km}(b + z)}{(\mu_{km}^2 A_1 A_2 + B_1 B_2) \sinh 2\mu_{km} b + \mu_{km}(A_1 B_2 + B_2 A_1) \cosh 2\mu_{km} b} \tag{33}$$

with its inversion

$$\Theta_{01}(x, \varphi, z, Fo) = \frac{1}{2\pi} \sum_{m=1}^{\infty} \frac{D_{om} J_o(\mu_{om} x) \{A_2 \mu_{om} \cosh \mu_{om}(b + z) + B_2 \sinh \mu_{om}(b + z)}{(\mu_{om}^2 A_1 A_2 + B_1 B_2) \sinh 2\mu_{om} b + \mu_{om}(A_1 B_2 + B_2 A_1) \cosh 2\mu_{om} b}$$

$$\int_0^1 \int_0^{2\pi} f_1(x, \varphi, Fo) J_0(\mu_{om} x) x dx d\varphi$$

$$+ \frac{1}{\pi} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{D_{km} J_k(\mu_{km} x) \{A_2 \mu_{km} \cosh \mu_{km}(b+z) + B_2 \sinh \mu_{km}(b+z)\}}{(\mu_{km}^2 A_1 A_2 + B_1 B_2) \sinh 2\mu_{km} b + \mu_{km}(A_1 B_2 + B_2 A_1) \cosh 2\mu_{km} b}$$

$$\int_0^1 \int_0^{2\pi} f_1(x, \varphi, Fo) J_1(\mu_{km} x) \cosh(\varphi - \varphi') x dx d\varphi \tag{34}$$

Determination for function $\Theta_{02}(x, \Theta, z, Fo)$:

The expression for $\Theta_{02}(x, \varphi, z, Fo)$ under the transform (a) and (b) is

$$\bar{\bar{\Theta}}_{o2}(x, k, n, \varphi', Fo) = \frac{1}{A_2} \cosh 2\mu_n + \frac{M_1}{\alpha_n} \sinh 2\alpha_n) \cdot$$

$$\times \left[\begin{array}{l} \int_0^{2\pi} \int^{\varkappa} \{G_{kn}(\varrho, x) + H_{kn}(\varrho, x)\} \varrho f_2(x, \varphi, Fo) \cos k(\varphi - \varphi') d\varrho d\varphi \\ + \int_0^{2\pi} \int_0^{\varkappa} \{G_{kn}(x, \varrho) + H_{kn}(x, \varrho)\} f_2(x, \varphi, Fo) \cos k(\varphi - \varphi) d\varrho d\varphi \end{array} \right] \tag{35}$$

with its inversion

$$\Theta_{o2}(x, \varphi, z, Fo) = \frac{1}{2\pi A_2} \sum_{n=1}^{\infty} (\cosh 2\alpha_n + \frac{M_1}{\alpha_n} \sinh 2\alpha_n)$$

$$\times \left(A_n \cos \alpha_n \left(1 + \frac{z}{b}\right) + \frac{M_1}{\alpha_n} \sin \alpha_n \left(1 + \frac{z}{b}\right)\right.$$

$$\times \int_0^{2\pi} \int_0^{\varkappa} \{G_{on}(\varrho, x) + H_{on}(\varrho, x)\} f_2(x, \varphi, Fo) \varrho d\varrho d\varphi +$$

$$\int_0^{2\pi} \int_1^{\varkappa} \{G_{on}(x, \varrho) + H_{on}(x, \varrho)\} f_2(x, \varphi, Fo) +$$

$$+ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{A_2} \left(\cos 2\alpha_n + \frac{M_1}{\alpha_n} \sinh 2\alpha_n\right) A_n \left[\cos k \alpha_n \left(1 + \frac{z}{b} + \frac{M_1}{\alpha_n} + \left(1 + \frac{z}{b}\right)\right)\right]$$

$$\left[\int_0^{2\pi} \int_1^{\varkappa} \{G_{kn}(x, \varrho) + H_{kn}(x, \varrho)\} f_2(x, \varphi, Fo) \cos k(\varphi - \varphi) \varrho d\varrho d\varphi\right]$$

$$+ \int_0^{2\pi} \int_0^{\varkappa} \{G_{kn}(\varrho, x) + H_{kn}(\varrho, x)\} f_2(x, \varphi, Fo) \cos k(\varphi - \varphi) \varrho d\varrho d\varphi \sin \alpha_n \tag{36}$$

An alternate expression for $\Theta_{02}(x,\varphi,z,Fo)$ under the transform (a) and (c) is obtained as

$$\Theta_{02}(k, m, \varphi', z, Fo) = \frac{[\mu_{km}A_1 \cosh \mu_{km}(b+z) + B_1 \sinh \mu_{km}(b+z)]}{(B_1B_2 + \mu_{km}^2 A_1A_2) \sinh 2\mu_{km}b + \mu_{km}(A_1B_2 + B_1A_2) \cosh 2_{km}b} \int_0^{2\pi}\int_{-b}^{b} f_2(x, \varphi, Fo) \cos k(\varphi - \varphi) J_k(\mu_{km}x) x dx d\varphi \tag{37}$$

with its inversion

$$\Theta_{02}(x, \varphi, z, Fo) = \frac{1}{2\pi}\sum_{m=1}^{\infty} D_{om} J_o(\mu_{om}x)$$

$$\frac{[\mu_{om}A_1 \cosh \mu_{om}(b+z) + B_1 \sinh\mu_{om}(b+z)] \int_0^{2\pi}\int_{-b}^{b} f_2(x, \varphi, Fo) J_0(\mu_{om}x) x dx d\varphi}{(B_1B_2 + \mu_{om}^2 A_1A_2) \sinh 2\mu_{om}b + \mu_{om}(A_1B_2 + B_1A_2) \cosh 2\mu_{om}b}$$

$$+ \frac{1}{\pi}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} D_{km} J_k(\mu_{km}x)$$

$$\frac{[\mu_{km}A_1 \cosh \mu_{km}(b+z) + B_1 \sinh \mu_{km}(b+z)] \int_0^{2\pi}\int_{-b}^{b} f_2(x, \varphi, Fo) \cos k(\varphi - \varphi) x J_k(\mu_{km}x) dx d\varphi}{(B_1B_2 + \mu_{km} A_1A_2) \sinh 2\mu_{km} b + \mu_{km} (A_1B_2 + B_1A_2) \cosh 2\mu_{km}b} \tag{38}$$

The expression for $\Theta_{03}(x,\varphi,z,Fo)$ under the transform (a) and (b) is

$$\bar{\bar{\Theta}}_{03}(x, k, n, \varphi, Fo) = \frac{I_k(\lambda_n x)}{A_3\lambda\, I_k'(\lambda_n) + B_3 I_k(\lambda_n)} \int_{-b}^{b}\int_0^{2\pi} f_3(z, \varphi, Fo) \cos k(\varphi - \varphi') Z_n(z)\, dz d\varphi \tag{39}$$

with its inversion

$$\frac{1}{2\pi}\sum_{n=1}^{\infty} \frac{A_n' Z_n(z)}{A_3\lambda_n I_o'(\lambda_n) + B_3 I_0(\lambda_n)} I_0(\lambda_n x) \int_0^{2\pi}\int_{-b}^{b} f_3(z, \varphi, Fo) Z_n(z) dz d\varphi$$

$$+ \frac{1}{\pi}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{A_n Z_n(z)}{A_3\lambda_n I_o'(\lambda_n) + B_3 I_0(\lambda_n)} I_k(\lambda_n x) \int_0^{2\pi}\int_{-b}^{b} f_3(z, \varphi, Fo) Z_n(z) \cos k(\varphi - \varphi')\, d\varphi dz \tag{40}$$

An alternate expression for $\Theta_{03}(x, \varphi, z, Fo)$ function is

$\hat{\hat{\Theta}}_{o3}(k, m, z, Fo) =$

$$\frac{1}{\mu_{km}(B_1B_2 + \mu_{km}A_1A_2)\sinh 2\mu_{km}b + \mu_{km}^2(A_1B_2 + B_1A_2)\cosh 2\mu_{km}b}$$

$$\Big(-\Big[(A_1\mu_{km}\cosh\mu_{km}b + B_1\sinh\mu_{km}b)\int_0^b\int_0^{2\pi} f_3(z,\varphi,Fo)J_k(\mu_{km})\cos k(\varphi-\varphi')\cdot$$

$$\cdot\{A_2\mu_{km}\cosh\mu_{km}(z'-b) - B_2\sinh\mu_{km}(z'-b)\}\,dz'd\varphi$$

$$(A_2\mu_{km}\cosh\mu_{km}b + B_2\sinh\mu_{km}b)\int_{-b}^0\int_0^{2\pi} f_3(z,\varphi,Fo)J_k(\mu_{km})\cos k(\varphi-\varphi')$$

$$\{A_1\mu_{km}\cosh\mu_{km}(z'-b) + B_1\sinh\mu_{km}(z'-b)\}\,dz'd\varphi$$

$$\cosh\mu_{km}z$$

$$\Big[-(A_1\mu_{km}\sinh\mu_{km}b + B_1\cosh\mu_{km}b)\int_0^b\int_0^{2\pi} f_3(z,\varphi,Fo)J_k(\mu_{km})\cos k(\varphi-\varphi')$$

$$\{A_2\mu_{km}\cosh\mu_{km}(z'-b) - B_2\sinh\mu_{km}(z'-b)\}\,dz'd\varphi$$

$$+ (A_2\mu_{km}\sinh\mu_{km}b + B_2\cosh\mu_{km}b)\int_{-b}^0\int_0^{2\pi} f_3(z,\varphi,Fo)J_k(\mu_{km})\cos k(\varphi-\varphi')$$

$$\{A_1\mu_{km}\cosh\mu_{km}(z-b) + B_1\sinh\mu_{km}(z'-b)\}\,dz'd\varphi\Big]$$

$$\sinh\mu_{km}z\Big)$$

$$-\int^Z \frac{f_3}{\mu_{km}}(z'\varphi, Fo)J_k(\mu_{km})\sinh\mu_{km}(z'-z)\cos k(\varphi-\varphi')\,dz'\,d\varphi \qquad (41)$$

with its inversion

$\Theta_{o3}(x, \varphi, z, Fo) =$

$$\frac{1}{2\pi}\sum_{m=1}^{\infty}\frac{D_{om}J_o(\mu_{om}x)}{\mu_{om}(B_1B_2 + \mu_{om}A_1A_2)\sinh 2\mu_{om}b + \mu_{om}^2(A_1B_2 + B_1B_2)\cosh 2\mu_{om}b}$$

$$\Big(-\Big[(A_1\mu_{om}\cosh\mu_{om}b + B_1\sinh\mu_{om}b)\int_0^b\int_0^{2\pi} f_3(z,\varphi,Fo)J_o(\mu_{om})$$

$$\{A_2\mu_{om}\cosh\mu_{om}(z'-b) - B_2\sinh\mu_{om}(z'-b)\}\,dz'd\varphi$$

$$+ (A_2\mu_{om}\cosh\mu_{om}b + B_2\sinh\mu_{om}b)\int_{-b}^0\int_0^{2\pi} f_3(z,\varphi,Fo)J_o(\mu_{om})$$

$$\{A_1\mu_{om}\cosh\mu_{om}(z'-b) + B_1\sinh\mu_{om}(z'-b)\}\,dz'd\varphi\Big]$$

$$\cosh\mu_{om}z\Big)$$

$$+\Big[-(A_1\mu_{om}\sinh\mu_{om}b+B_1\cosh\mu_{om}b)\int_0^b\int_0^{2\pi} f_3(z,\varphi,Fo)\,J_o(\mu_{om})$$

$$A_2\,\mu_{om}\cosh\mu_{om}(z'-b)-B_2\sinh\mu_{om}\{(z'-b)\}\,dz'd\varphi$$

$$+(A_2\,\mu_{om}\sinh\mu_{om}\,b+B_2\cosh\mu_{om}\,b)\int_{-b}^{0}\int_0^{2\pi} f_3(z,\varphi,Fo)J_o(\mu_{om})$$

$$\{A_1\mu_{om}\cosh\mu_{om}(z'-b)+B_1\sinh\mu_{om}(z'-b)\}\,dz'd\varphi\Big]$$

$$\sinh\mu_{om}z)$$

$$-\frac{1}{2\pi}\sum_{m=1}^{\infty}D_{om}\,J_o(\mu_{om}\,x)\int_0^{2\pi}\int_0^{Z}\frac{f_3}{\mu_{om}}(z',\varphi,Fo)J_o(\mu_{om})\sinh\mu_{om}(z'-z)\,dz'd\varphi$$

$$+\frac{1}{\pi}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{D_{km}\,J_k(\mu_{km}\,x)}{\mu_{km}\,(B_1B_2+\mu_{km}^2A_1A_2)\sinh 2\mu_{km}\,b+\mu_{km}(A_1B_2+B_1A_2)\cosh 2\mu_{km}\,b}$$

$$\Big(-\Big[(A_1\,\mu_{km}\cosh\mu_{km}\,b+B_1\sinh\mu_{km}\,b)\int_0^b\int_0^{\pi} f_3(\varphi,z,Fo)J_k(\mu_{km})\cos k\,(\varphi-\varphi)$$

$$\{A_2\,\mu_{km}\cosh\mu_{km}\,(z'-b)-B_2\sinh\mu_{km}\,(z'-b)\}\,dz'd\varphi$$

$$+(A_2\,\mu_{km}\cosh\mu_{km}\,b+B_2\sinh\mu_{km}\,b)\int_{-b}^{0}\int_0^{2\pi} f_3(\varphi,z,Fo)J_k(\mu_{km})\cos k\,(\varphi-\varphi')$$

$$\{A_1\,\mu_{km}\cosh\mu_{km}\,(z'-b)+B_1\sinh\mu_{km}\,(z'-b)\}\,dz'd\varphi\Big]$$

$$\cosh\mu_{km}\,z$$

$$(A_1\,\mu_{km}\sinh\mu_{km}\,b+B_1\cosh\mu_{km}\,b)\int_{-b}^{b}\int_0^{2\pi} f_3(\varphi,z,Fo)J_k(\mu_{km})\cos k\,(\varphi-\varphi')$$

$$\{A_2\,\mu_{km}\cosh\mu_{km}\,(z'-b)-B_2\sinh\mu_{km}\,(z'-b)\}\,dz'd\varphi$$

$$+(A_2\,\mu_{km}\sinh\mu_{km}\,b+B_2\cosh\mu_{km}b)\int_{-b}^{0}\int_0^{2\pi} f_3(\varphi,z,Fo)J_k(\mu_{km})\cos k\,(\varphi-\varphi')$$

$$\{A_1\,\mu_{km}\cosh\mu_{km}\,(z'-b)+B_1\sinh\mu_{km}\,(z'-b)\}\,dz'd\varphi\Big]$$

$$\sinh\mu_{km}\,z\Big)$$

$$-\frac{1}{\pi}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}D_{km}\,J_k(\mu_{km}\,x)\int_0^{Z}\int_0^{2\pi} f_3(z,\varphi,Fo)\,J_k(\mu_{km})\sinh\mu_{km}\,(z'-z)$$

$$\cos k\,(\varphi-\varphi')\,dz'd\varphi \qquad (42)$$

The boundary conditions given in (2) cover a wide variety of cases occurring in technological applications. Specially, conditions of prescribed surface temperature can be obtained by putting $B_i = 0$, $i = 1, 2, 3$) and Newtonian boundary conditions by putting $f_i = B_i a_i$ or any combination of these can be obtained by assigning appropriate values to A_i, B_i and $f_i (i = 1, 2, 3)$. The general problem treated by OJALVO [3] happens to be a particular case of the problem treated here.

Summary

General expressions have been derived for the unsteady heating of a finite solid cylinder under the influence of an arbitrary volume heat source and an arbitrary initial temperature distribution when convective type of time dependent boundary conditions are prescribed on the bases and curved surface of the cylinder. By using finite integral transform techniques, expressions for temperature distributions are obtained in various forms and contain valuable results of technological importance.

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