ON PERTURBATIONS OF SECOND ORDER AND THIRD ORDER EQUATIONS

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Presented by Prof. M. FARKAS

M. FARKAS [1, 2] considered the perturbed system

$$\dot{x} = f(x) + \mu r\left(\frac{t}{\tau}, x, \mu, \tau\right), \tag{1}$$

where x, f and r are n-dimensional vectors, t, μ and τ are real scalars. He obtained general results pertaining to the existence and stability of the D-periodic solution of (1). Then I. FARKAS, M. FARKAS [3] and H. M. EL OWAIDY [4] have considered perturbed van der Pol's and Liénard's equation in detail and obtained the actual determination of D-periodic solution and an effective form of the stability criterion.

In the present paper the above results are extended to general perturbed second order (section 1) and third order equations (section 2).

1. Consider the scalar differential equation

$$\ddot{u} + g(u, \dot{u}) = \mu\gamma\left(\frac{t}{\tau}, u, \dot{x}, \mu, \tau\right), \tag{2}$$

where μ is a small parameter. We shall suppose that (2) satisfies the following conditions:

i) The functions g and γ are analytic in the region

$$I_t \times \Omega \times I_\mu \times I_\tau,$$

where

$$I_t = \{t : -\infty < t + \infty\},$$

Ω an open and connected region in the $(u, \dot{u})$ — plane,

$$I_\mu = \{\mu : |\mu| < \alpha\}, \qquad \alpha > 0,$$

$$I_\tau = \{\tau : |\tau - \tau_0| < \beta\}, \quad 0 < \beta < \tau_0.$$

ii) The function γ is periodic in t with period $\tau \in I_\tau$, g and γ are also periodic in u with period $a\tau = a_0\tau_0$; where a_0 is a constant. In the case $a_0 \neq 0$,

Ω is supposed to have the following property: if $(u, \dot{u}) \in \Omega$, then $(u + a_0 t, \dot{u}) \in \in \Omega$, $-\infty < t < +\infty$.

iii) The unperturbed equation

$$\ddot{u} + g(u, \dot{u}) = 0$$

has a non-constant D-periodic solution $u_0(t)$ with period $\tau_0 > 0$ and with coefficient a_0.

We replace the equation (2) by the equivalent system

$$\dot{x} = f(x) + \mu r\left(\frac{t}{\tau}, x, \mu, \tau\right), \tag{3}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ f(x) = \begin{bmatrix} x_2 \\ -g(x_1, x_2) \end{bmatrix},$$

$$r\left(\frac{t}{\tau}, x, \mu, \tau\right) = \begin{bmatrix} 0 \\ \gamma\left(\frac{t}{\tau}, x_1, x_2, \mu, \tau\right) \end{bmatrix}.$$

Obviously, for $\mu = 0$ the first variational system of (3) corresponding to the solution $p(t) = \text{col}\,(u_0(t), \dot{u}_0(t))$,

$$\dot{y} = f'_x(p(t))y \tag{4}$$

has a non-trivial periodic solution $\dot{p}(t)$ and thus 1 is a characteristic multiplier of (4). Using Liouville's formula the other characteristic multiplier is given by

$$\varrho_2 = \exp\left(-\int_0^{\tau_0} g'_{x_2}(u_0(t), \dot{u}_0(t))dt\right)$$

We can assume without loss of generality that

$$\dot{p}(0) = \text{col}\,(\dot{u}_0(0), 0), \qquad \dot{u}_0(0) \neq 0.$$

M. Farkas [1] has showed that for $|t_0|$, $\|x^0 - p^0\|$, $|\mu|$ and $|\tau - \tau_0|$ sufficiently small the solution $x(t; t_0, x^0, \mu, \tau)$ can be written in the form

$$x = x(t; \vartheta, p^0 + h, \mu, \tau).$$

Here

$$x(\vartheta; \vartheta, p^0 + h, \mu, \tau) = p^0 + h,$$
$$p^0 = p(0) = \text{col}\,(u_0^0, \dot{u}_0^0) = \text{col}\,(u_0(0), \dot{u}_0(0)),$$
$$h = \text{col}\,(0, h_2).$$

As a consequence of theorem 1 of [1], we have the following statement:

Theorem 1. If the condition

$$\int_0^{\tau_0} g'_{\dot u}(u_0(t), \dot u_0(t))dt \neq 0$$

is satisfied, then to all μ, ϑ, which are in modulus sufficiently small there belongs a unique period $\tau = \tau(\mu, \vartheta)$ and a unique $h_2 = h_2(\mu, \vartheta)$ such that

$$\omega(t; \mu, \vartheta) = u(t; \vartheta, u_0^0, \dot u_0^0 + h_2(\mu, \vartheta), \mu, \tau(\mu, \vartheta)) \tag{5}$$

is a D-periodic solution with period $\tau(\mu, \vartheta)$ and coefficient $a(\mu, \vartheta) = \dfrac{a_0 \tau_0}{\tau(\mu, \vartheta)}$ of the equation (2). The functions $\tau(\mu, \vartheta)$, $h_2(\mu, \vartheta)$ are analytic in a neighbourhood of $\{\mu = \vartheta = 0\}$, the solution (5) is analytic in a neighbourhood of $\{\mu = \vartheta = 0\}$ for all t, $\tau(0, 0) = 0$, $h_2(0, 0) = 0$ and $\omega(t; 0, 0) = u_0(t)$.

By using Poincaré's method we shall give expressions up to first approximations for $\tau(\mu, \vartheta)$, $a(\mu, \vartheta)$ and $\omega(t; \mu, \vartheta)$. We have already mentioned that the system (4) has the periodic solution

$$p_1(t) = \operatorname{col}\left(p_{11}(t), p_{21}(t)\right) = \frac{\dot p(t)}{\dot u_0^0}$$

for which $p_1(0) = \operatorname{col}(1, 0)$. A second linearly independent solution of (4) is denoted by $p_2(t) = \operatorname{col}(p_{12}(t), p_{22}(t))$ with $p_2(0) = \operatorname{col}(0, 1)$. The Wronskian $W(t)$ (for which $W(0) = 1$) according to Liouville-formula is given by

$$W(t) = \exp\left(-\int_0^t g'_{\dot u}\left(u_0(s), \dot u_0(s)\right) ds\right)$$

Theorem 2. Under the conditions of theorem 1, the period $\tau(\mu, \vartheta)$, the coefficient $a(\mu, \vartheta)$ and the D-periodic solution $\omega(t; \mu, \vartheta)$ of the perturbed equation (2) are given by

$$\tau(\mu, \vartheta) = \tau_0 + \mu\tau_1(\vartheta) + o(\mu),$$

$$a(\mu, \vartheta) = a_0 - \mu \frac{a_0\tau_1(\vartheta)}{\tau_0} + o(\mu)$$

and

$$\omega(t; \mu, \vartheta) = u_0(t - \vartheta) + \mu \frac{1}{\dot u_0^0}\left[\frac{p_{12}(t - \vartheta)W(\tau_0)}{1 - W(\tau_0)} \int_0^{\tau_0} \frac{\gamma^*(q)}{W(q)} \dot u_0(q)dq + \right.$$

$$\left. + p_{12}(t - \vartheta)\int_0^{t-\vartheta} \frac{\gamma^*(q)}{W(q)} \dot u_0(q)dq - \dot u_0(t - \vartheta)\int_0^{t-\vartheta} \frac{\gamma^*(q)}{W(q)} p_{12}(q)dq\right] + o(\mu),$$

where

$$\tau_1(\vartheta) = \frac{1}{\dot{u}_0^0}\int_0^{\tau_0} \frac{\gamma^*(q)}{W(q)}\left[p_{12}(q) - \frac{p_{12}(\tau_0)}{\dot{u}_0^0(1 - W(\tau_0))}\,\dot{u}_0(q)\right] dq\,,$$

$$\gamma^*(q) = \gamma\left(\frac{q+\vartheta}{\tau_0}\,,\ u_0(q),\, \dot{u}_0(q),\, 0,\, \tau_0\right).$$

The proof is analogous to that of theorem 2 of [3] and will therefore be omitted here.

A stability condition is given also for the D-periodic solution $\omega(t; \mu, \vartheta)$.

Theorem 3. Under the conditions of theorem 1 if

$$\int_0^{\tau_0} g'_{\dot{u}}(u_0(t),\, \dot{u}_0(t))dt > 0,$$

then there exist $\alpha_1 > 0, \alpha_2 > 0$ such that the D-periodic solution $\omega(t; \mu, \vartheta)$ of (2) with period $\tau(\mu, \vartheta)$ is asymptotically stable for all (μ, ϑ) satisfying the conditions

$$|\,\mu\,| < \alpha_1, \qquad |\,\vartheta\,| < \alpha_2$$

and

$$\mu\varrho'_\mu(0, 0) < 0.$$

Here

$$\varrho'_\mu(0, 0) = \frac{1}{\dot{u}_0^0}\int_0^{\tau_0} \frac{\gamma'_t\left(\frac{t}{\tau_0}\,,\ u_0(t),\, \dot{u}_0(t),\, 0,\, \tau_0\right)}{W(t)}\left[p_{12}(t) - \frac{p_{12}(\tau_0)\,u_0(t)}{\dot{u}_0^0(1-W(\tau_0))}\right] dt$$

where γ'_t is the partial derivative of γ with respect to t.

The proof, which we omit here, is analogous to that of theorem 3 of [3].

In the applications of theorem 2 and 3 we need to find the first component $p_{12}(t)$ of the solution $p_2(t)$ of (4). Generally the determination of $p_{12}(t)$ is difficult, but if $\dot{u}_0(t) \neq 0$ for all t, using Liouville formula the function $p_{12}(t)$ is given by

$$p_{12}(t) = \dot{u}_0^0\,\dot{u}_0(t)\int_0^t \frac{\exp \int_0^s g'_{\dot{u}}\left(u_0(s_1),\, \dot{u}_0(s_1)\right) ds_1}{(\dot{u}_0(s))^2}\, ds\,.$$

In the case $a_0 \neq 0$ we can show that $\dot{u}_0(t) \neq 0$ for all t.

Theorem 4. If $a_0 > 0$ $(a_0 < 0)$ then $\dot{u}_0(t) > 0$ $(\dot{u}_0(t) < 0)$ for all t.

Proof. In the (x_1, x_2) — plane the solution $p(t) = \mathrm{col}\,(u_0(t), \dot{u}_0(t))$ of system $\dot{x} = f(x)$ defines a trajectory K. First we show that if the trajectory reaches the x_1 axis at a point then it is crossing at this point from the upper (lower) half plane into the lower (upper) one. Suppose that t_0 is a finite value of t such that $\dot{u}_0(t_0) = 0$. Since the trajectory K can not reach a critical point in a finite time, we have $g(u_0(t_0), 0) \neq 0$ and hence K must cross the x_1 axis. Obviously for $x_2 > 0$ ($x_2 < 0$) the function $u_0(t)$ is increasing (decreasing).

In the case $a_0 > 0$ it is easy to see that $\max \dot{u}_0(t) = \dot{u}_0(t_1) > 0$, $0 \leqq t_1 \leqq \tau_0$. Now let us consider the behaviour of K in the interval $t_1 \leqq t \leqq t_1 + \tau_0$. If the theorem is incorrect, then K is crossing the x_1 axis at some time t_2, $t_1 < t_2 < t_1 + \tau_0$. Since

$$(u_0(t_1 + \tau_0), \quad \dot{u}_0(t_1 + \tau_0)) = (u_0(t_1) + a_0\tau_0, \quad \dot{u}_0(t_1))$$

and

$$\dot{u}_0(t) \leqq \dot{u}_0(t_1),$$

the trajectory K must cross itself and we obtained a contradiction.

The case $a_0 < 0$ can be discussed analogously.

The theorem may be considered as extension of F. Tricomi's results [5] (p. 290) to the general equation

$$\ddot{u} + g(u, \dot{u}) = 0.$$

2. We shall consider now the perturbed equation

$$\dddot{u} + g(u, \dot{u}, \ddot{u}) = \mu\gamma\left(\frac{t}{\tau}, u, \dot{u}, \ddot{u}, \mu, \tau\right), \tag{6}$$

which satisfies hypotheses analogous to i), ii) and iii) of section 1. We shall discuss also the existence and the stability of a D-periodic solution of (6). The equation (6) is equivalent to the system

$$\dot{x} = f(x) + \mu r\left(\frac{t}{\tau}, x, \mu, \tau\right), \tag{7}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \; f(x) = \begin{bmatrix} x_2 \\ x_3 \\ -g(x_1, x_2, x_3) \end{bmatrix}$$

and

$$r\left(\frac{t}{\tau}, x, \mu, \tau\right) = \begin{bmatrix} 0 \\ 0 \\ \gamma\left(\frac{t}{\tau}, x_1, x_2, x_3, \mu, \tau\right) \end{bmatrix}.$$

Obviously, the unperturbed system of (7),

$$\dot{x} = f(x) \tag{8}$$

has a D-periodic solution $p(t) = \text{col}\,(u_0(t), \dot{u}_0(t), \ddot{u}_0(t))$. Since (8) is autonomous, the first variational system of (8) corresponding to the solution $p(t)$, i.e.

$$\dot{y} = f'_x(p(t))y \tag{9}$$

has a periodic solution $p_1(t) = \dot{p}(t) = \text{col}\,(p_{11}(t), p_{21}(t), p_{31}(t))$. Without loss of generality we may assume that

$$p_1(0) = \text{col}\,(1, 0, 0)$$

and the first component of $p_1(t)$ is a solution of the unperturbed equation of (6). (This can be achieved by a simple linear coordinate transformation.) Let us denote by

$$p_2(t) = \text{col}\,(p_{12}(t), p_{22}(t), p_{32}(t)),$$
$$p_3(t) = \text{col}(p_{13}(t), p_{23}(t), p_{33}(t))$$

the two solutions of (9), for which

$$p_2(0) = \text{col}\,(0, 1, 0),$$
$$p_3(0) = \text{col}\,(0, 0, 1).$$

Using Liouville's formula the Wronskian $W(t)$ is given by

$$W(t) = \det \begin{bmatrix} p_{11}(t) & p_{12}(t) & p_{13}(t) \\ p_{21}(t) & p_{22}(t) & p_{23}(t) \\ p_{31}(t) & p_{32}(t) & p_{33}(t) \end{bmatrix} = \exp \int_0^t - g'_{\ddot{u}}(u_0(s), \dot{u}_0(s), \ddot{u}_0(s))ds.$$

Since $p_1(t)$ is a non-trivial periodic solution of (9), $\varrho_1 = 1$ is a characteristic multiplier of (9). The two other characteristic multipliers are denoted by ϱ_2, ϱ_3.

As in section 1, for $|t_0|$, $\|x^0 - p^0\|$, $|\mu|$ and $|\tau - \tau_0|$ sufficiently small the solution $x(t; t_0, x^0, \mu, \tau)$ of (7) can be written in the form

$$x = x(t; \vartheta, p^0 + h, \mu, \tau).$$

Here

$$x(\vartheta; \vartheta, p^0 + h, \mu, \tau) = p^0 + h,$$

$$p^0 = p(0) = \text{col}\,(u_0^0, \dot{u}_0^0, \ddot{u}_0^0) = \text{col}\,\big(u_0(0), \dot{u}_0(0), \ddot{u}_0(0)\big),$$
$$h = \text{col}\,(0, h_2, h_3)\,.$$

Theorem 5. If the conditions

$$\varrho_2 \neq 1, \qquad \varrho_3 \neq 1$$

are satisfied, then to all μ, ϑ which are in modulus sufficiently small there belongs a unique period $\tau = \tau(\mu, \vartheta)$ and a unique vector $h(\mu, \vartheta) = \text{col}\,(0, h_2(\mu, \vartheta), h_3(\mu, \vartheta))$ such that

$$\omega(t;\mu,\vartheta) = x_1(t;\vartheta, p^0 + h(\mu,\vartheta), \mu, \tau(\mu,\vartheta)) \tag{10}$$

is a D-periodic solution with period $\tau(\mu, \vartheta)$ and coefficient $a(\mu,\vartheta) = \dfrac{a_0\,\tau_0}{\tau(\mu,\vartheta)}$ of the equation (6). The functions $\tau(\mu,\vartheta)$, $h_2(\mu,\vartheta)$ and $h_3(\mu,\vartheta)$ are analytic in a neighbourhood of $\{\mu = \vartheta = 0\}$, the solution (10) is analytic in a neighbourhood of $\{\mu = \vartheta = 0\}$ for all t; $\tau(0,0) = \tau_0$, $h_2(0,0) = h_3(0,0) = 0$, $\omega(t;0,0) = u_0(t)$.

Theorem 6. Under the conditions of theorem 5, the period $\tau(\mu,\vartheta)$, the coefficient a (μ,ϑ) and the D-periodic solution $\omega(t;\mu,\vartheta)$ of the perturbed equation (6) are given by

$$\tau(\mu,\tau) = \tau_0 + \mu\tau_1(\vartheta) + o(\mu)\ ,$$

$$a(\mu,\vartheta) = a_0 - \mu\frac{a_0\,\tau_1(\vartheta)}{\tau_0} + o(\mu)$$

and

$$\omega(t;\mu,\vartheta) = u_0(t-\vartheta) + \mu\left\{\frac{W(\tau_0)}{D(\tau_0)}\sum_{i,j=2}^{3} p_{1i}(t-\vartheta)[p_{ij}(\tau_0) - \delta_{ij}]\int_0^{\tau_0}\frac{\gamma^*(q)}{W(q)}P_{3j}(q)dq + \sum_{i=1}^{3} p_{1i}(t-\vartheta)\int_0^{t-\vartheta}\frac{\gamma^*(q)}{W(q)}P_{3i}(q)\,dq\right\} + o(\mu)\ ,$$

where

$$\tau_1(\vartheta) = -\frac{1}{D(\tau_0)}\sum_{i=2}^{3}[W(\tau_0)p_{1i}(\tau_0) + p_{i1}(\tau_0)]\int_0^{\tau_0}\frac{\gamma^*(q)}{W(q)}P_{3i}(q)\,dq - \int_0^{\tau_0}\frac{\gamma^*(q)}{W(q)}P_{31}(q)dq,\ \gamma^*(q) = \gamma\left(\frac{q+\vartheta}{\tau_0}, p(q), 0, \tau_0\right),$$

$P_{ij}(q)$ is the cofactor of $p_{ij}(q)$ in $W(q)$, $D(\tau_0) = W(\tau_0) + 1 - (p_{22}(\tau_0) + p_{33}(\tau_0))$, δ_{ij} is Kronecker-symbol.

Theorem 7. Under the conditions of theorem 5 if, in addition,

$$|\varrho_2| < 1, \qquad |\varrho_3| < 1,$$

then there exist $\alpha_1 > 0$, $\alpha_2 > 0$ for which the D-periodic solution $\omega(t;\mu,\vartheta)$ of (6) is asymptotically stable for all (μ,ϑ) satisfying the conditions

$$|\mu| < \alpha_1, \qquad |\vartheta| < \alpha_2$$

and

$$\mu\varrho'_\mu(0,0) < 0.$$

Here

$$\varrho_\mu'(0, 0) = -\int_0^{\tau_0} \frac{\gamma_t'\left(\frac{t}{\tau_0}, p(t), 0, \tau_0\right)}{W(t)} \Big\{P_{31}(t) +$$

$$+ \frac{P_{21}(\tau_0) + p_{12}(\tau_0)}{D(\tau_0)} P_{32}(t) + \frac{P_{31}(\tau_0) + p_{13}(\tau_0)}{D(\tau_0)} P_{33}(t)\Big\} dt,$$

where γ_t' is the partial derivative of γ with respect to t.

The proofs of theorems 5, 6, 7 are based on elementary considerations as in section 1 and no additional difficulties arise but longer manipulations are needed in carrying through the calculus. For a more detailed consideration of the above theorems we may refer to [6].

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Summary

M. FARKAS has discussed the existence and stability of a D-periodic solution of perturbed autonomous systems under periodic and non-autonomous perturbation. In this paper Farkas' method has been applied to perturbed second order and third order equations. Poincaré's method has been used for the determination of the first approximation of the D-periodic solution and an effective form has been given to the stability criterion.

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