LINEAR RELATIONS BETWEEN THE BESSEL POLYNOMIALS

By

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1. Introduction

The Bessel polynomials were defined by KRALL and FRINCK [1] by the formula

\[ y_n(x, a, b) = \binom{2F_0}{-n, a + n - 1, \frac{-x}{b}} \]  

These polynomials arise in the solution of the wave equation in spherical polar coordinates. A large number of papers has been written on these polynomials (see RAGAB, [2] & [3], SALAM [4], AGARWAL [5] and BRAFMAN [6]).

In this paper new recurrence relations as well as some series expansions will be established.

The following formulae will be required in the proofs: DOUGALL [7]:

\[ \binom{5F_4}{a, 1 + \frac{1}{2} a, c, d, e; 1} = \binom{1}{\frac{1}{2} a, 1 + a - c, 1 + a - d, 1 + a - e} \]

\[ \frac{\Gamma(1 + a - c) \Gamma(1 + a - d) \Gamma(1 + a - e) \Gamma(1 + a - c - d - e)}{\Gamma(1 + a) \Gamma(1 + a - d - e) \Gamma(1 + a - c - d) \Gamma(1 + a - c - e)} ; \]

Vandermond’s theorem: if \( n \) is a positive integer, then

\[ \binom{2F_1}{-n, \beta; 1} = \frac{(\gamma - \beta; n)}{(\gamma; n)} ; \]

where the symbol

\[ (x; r) = x(x + 1) \ldots (x + r - 1) = \frac{\Gamma(x + r)}{\Gamma(x)} , \]

\[ (x; 0) = 1 , \]

And Saalschütz’s theorem [8]:

\[ \binom{\gamma - \beta; n}{\gamma; n} \]
\[
\frac{\binom{3F_2}{a, b, -n; 1}{1+a+b-c-n, c}}{(c-a; n)(c-b; n)} = \frac{(c-a; n)(c-b; n)}{(c; n)(c-a-b; n)};
\]
where \( n \) is a positive integer.

2. Recurrence formulae for \( y_n(x, a, b) \)

The formulae to be proved are:

\[
y_n(x, a, b) + \frac{nx}{b} y_{n-1}(x, a+2, b) = y_n(x, a+1, b);
\]
(5)

\[
y_n(x, a, b) - (a+n-1) \frac{x}{b} y_{n-1}(x, a+2, b) = y_{n-1}(x, a+1, b);
\]
(6)

\[
y_n^2(x, a, b) = \frac{n(a+n-1)}{b} y_{n-1}(x, a+2, b).
\]
(7)

To prove (5), substitute for \( y_n(x, a, b) \) and \( y_{n-1}(x, a+2, b) \) from (1); then the left hand side of (5) becomes:

\[
\sum_{r=0}^{n} (-n; r)(a+n-1; r) \left(-\frac{x}{b}\right)^r + (-n) \sum_{r=0}^{n-1} (-n+1; r)(a+n-1; r) \left(-\frac{x}{b}\right)^{r+1}.
\]

Now the coefficient of \(-\frac{x}{b}\) in the last expression is

\[
\frac{(-n; r)(a+n-1; r)}{r!} + \frac{(-n)(-n+1; r-1)(a+n; r-1)}{(r-1)!} = \frac{(-n; r)(a+n-1; r)}{r!} + \frac{(-n; r)(a+n-1; r)}{(a+n-1)(r-1)!} = \frac{(-n; r)(a+n-1; r)}{r!} \left(1 + \frac{r}{a+n-1}\right) = \frac{(-n; r)(a+n-1; r)}{r!} \times \frac{a+n-1+r}{a+n-1} = \frac{(-n; r)(a+n; r)}{r!} = \text{coefficient of } \left(-\frac{x}{b}\right)^r \text{ in } y_n(x, a+1, b).
\]

This completes the proof of (5).

In the same way (6) can be proved.
To prove (7), substitute for \( y_n(x, a, b) \) from (1), differentiate term by term, so we get:

\[
y'_n(x, a, b) = \frac{d}{dx} \left\{ \sum_{r=0}^{n} \frac{(-n; r)(a + n - 1; r)}{r!} \left( \frac{-x}{b} \right)^r \right\} = \]

\[
= - \sum_{r=1}^{n} \frac{(-n; r)(a + n - 1; r)}{b(r - 1)!} \left( \frac{-x}{b} \right)^{r-1} = \]

\[
= - \sum_{r=0}^{n-1} \frac{(-n; r + 1)(a + n - 1; r + 1)}{r!} \left( \frac{-x}{b} \right)^r = \]

\[
= \frac{n(a + n - 1)}{b} \sum_{r=0}^{n-1} \frac{(-n + 1; r)(a + n; r)}{r!} \left( \frac{-x}{b} \right)^r = \]

\[
= \frac{n(a + n - 1)}{b} y_{n-1}(x, a + 2, b); \]

by applying (1) again. Thus (7) is proved. As Krall and Frinck remarked we see that the derivatives of Bessel polynomials are Bessel polynomials with the parameter increased by 2.

### 3. Representation of \( y_n(x, a, b) \) as a series of generalised hypergeometric functions of the form \( \text{F}_2^{4} \)

The expansion to be proved is:

\[
y_n(x, a, b) = \sum_{r=0}^{n} \binom{n}{r} \frac{(a; r) \left( 1 + \frac{1}{2} a; r \right) (\beta; r)(-n; r)(a+n-1; r)}{\left( \frac{1}{2} a; r \right)(1 + a - \beta + n; r)(1 + a; 2r)} \left( \frac{-x}{b} \right)^r \times \]

\[
\times \, _4F_2 \left[ \begin{array}{c}
-n+r, a+n-1+r, 1+a+n+r, 1+a - \beta + r; \\
1+a-\beta+n+r, 1+a+2r
\end{array} \right] (-1)^r \frac{1 + a - \beta; r + s)(1 + a + n; r + s)}{(1 + a - \beta; r)(1 + a + n; r)} \left( \frac{-x}{b} \right)^{r+s}. \]
Here write \( s = p - r \), change the order of summation and the right-hand side of (8) becomes

\[
\sum_{p=0}^{n} \frac{(-n; p)(a + n - 1; p)(1 + a - \beta; p)(1 + a + n; p)}{p! (1 + a; p)(1 + a - \beta + n; p)} \left( \frac{-x}{b} \right)^p \times
\]

\[
\times\binom{a, 1 + \frac{1}{2} a, \beta, -n, -p; 1}{\frac{1}{2} a, 1 + a - \beta, 1 + a + n, 1 + a + p}.
\]

Now sum the \( _5F_4 \) by means of (2) and so obtain the left-hand side of (8). Thus (8) is proved.

4. Series expansions of the Bessel polynomials

The formula to be established is

\[
\sum_{k=0}^{n} \binom{\mu}{k} \frac{n!}{(n - k)!} \left( \frac{x}{b} \right)^k y_{n-k}(x, a + 2k + 2, b) = y_n(x, a + \mu + 2, b) \quad (9)
\]

where

\[
\binom{\mu}{k} = (-1)^k \frac{(-\mu; k)}{k!}.
\]

To prove (9) substitute for the Bessel polynomials in the left; then the left-hand side of (9) becomes

\[
\sum_{k=0}^{n} \sum_{r=0}^{n-k} (-1)^k \frac{(-\mu; k)(-n; k)(-n + k; r)(n + k + a + 1; r)}{k! r!} \left( \frac{-x}{b} \right)^{k+r}.
\]

Here write \( k = p - r \); then the last expression becomes

\[
\sum_{p=0}^{n} \frac{(n; p)(-\mu; p)}{p!} \left( \frac{x}{b} \right)^p \frac{1}{b} \sum_{r=0}^{p} \frac{(-a - n - p; r)(-p; r)}{r! (1 + \mu - p; r)} \binom{a, 1 + a + n; p}{1 + a + n; p} = \binom{(-1)p!}{2F_1(-p, -a - n - p; 1 + \mu - p; 1)}.
\]

Now sum the \( 2F_1 \) by Vandermond's theorem and the last expression becomes

\[
\sum_{p=0}^{n} \frac{(-n; p)(-\mu; p)}{p!} \left( \frac{-x}{b} \right)^p \binom{1 + \mu + a + n; p}{-\mu; p} = \sum_{p=0}^{n} \frac{(-n; p)(1 + \mu + a + n; p)}{p!} \left( \frac{-x}{b} \right)^p = y_n(x, a + \mu + 2, b),
\]

which is the right-hand side of (9). Thus (9) is proved.

This was proved for the particular case \( b = 2 \) in another form by Salam [4] p. 152.
5. The second series expansion to be proved is:
\[ \sum_{r=0}^{k} k \cdot c_r (a + 2n + 1 - k; r) y_{n+1-k}(x, a + r, b) \left( \frac{x}{b} \right)^r = y_{n+1}(x, a, b) \quad (10) \]

where \( k, n \) are any positive integer or zero such that \( n + 1 - k \geq 0 \).

As before the left-hand side of (10) is
\[ \sum_{r=0}^{k} k \cdot c_r (a + 2n + 1 - k; r) \left( \frac{x}{b} \right)^r \frac{\Gamma(k - n - 1, a + n - k + r; - \frac{x}{b})}{r!} = \sum_{r=0}^{k} \sum_{s=0}^{n+1-k} \frac{(k-r)(a+2n+1-k; r)(k-n-1; s)(a+n-k+r; s)}{r! s!} \left( \frac{-x}{b} \right)^{r+s}. \]

Here put \( p = r + s \) where \( p \) is the new parameter of summation and the last expression becomes
\[ \sum_{p=0}^{n+1} \frac{(k-n-1; p)(a+n-k; p)}{p!} \left( \frac{-x}{b} \right)^p \frac{\Gamma(a+2n+1-k, -p, -k; 1)}{\Gamma(a+n-k, 2-k+n-p; 1)}. \]

Now sum the \( _3F_2 \) by Saalschütz's theorem (4) and the left hand side (10) is equal to
\[ \sum_{p=0}^{n+1} \frac{(-n-1; p)(a+n; p)}{p!} \left( \frac{-x}{b} \right)^p = y_{n+1}(x, a, b). \]

Thus (10) is proved.

When \( k = 1; \) (10) becomes
\[ y_n(x, a, b) + \frac{x}{b} (a + 2n) y_n(x, a + 1; b) = y_{n+1}(x, a, b). \quad (11) \]

6. Generalisation of (5)

The formula to be proved is
\[ \sum_{r=0}^{m} (-1)^r \cdot n \cdot c_r (a + 2n + 1 - n; r) \left( \frac{x}{b} \right)^r y_{n-r}(x, a + 2r, b) = y_n(x, a + m, b) \quad (12) \]

where \( m \) is any positive integer. This generalises the recurrence relation (5) of §2. because when \( m = 1, \) (12) becomes (5).

To prove (12), assume it is true for a particular value of \( m \). Thus (12) with \( a + 1 \) instead of \( a \) becomes
\[ \sum_{r=0}^{m} (-1)^r \cdot n \cdot c_r (a + 2n + 1 - n; r) \left( \frac{x}{b} \right)^r y_{n-r}(x, a + 2r + 1, b) = y_n(x, a + m + 1, b). \]

Now apply (5) to each term on the left and get
\[ y_n(x, a + m + 1, b) = \sum_{r=0}^{m} (-1)^r m_c_r (a + n; r) \left( \frac{x}{b} \right)^r y_{n-r}(x, a + 2r, b) + \sum_{r=0}^{m} (-1)^r m_c_r (n; r) \left( \frac{x}{b} \right)^{r-1} (n-r) y_{n-r-1}(x, a + 2r + 2, b). \]

In the second of those two series, write \( r - 1 \) for \( r \), add the two series, applying the identity

\[ m_c_r + m_{c_{r-1}} = m_{c_r} ; \]

then the last expression becomes

\[ y_n(x, a + m + 1, b) = \sum_{r=0}^{m-1} (-1)^r (n; r) \left( \frac{x}{b} \right)^{r-1} Y_n-r(x, a + 2r + 2, b) \]

which is (12) with \( m + 1 \) in place of \( m \). But (12) is true when \( m = 1 \); therefore it follows by induction that (12) is true for all positive values of \( m \). This completes the proof of (12).

### 7. Generalisation of Formula (6)

The theorem to be established is:

\[ \sum_{r=0}^{m} (-1)^r m_c_r (a + n - 1; r) \left( \frac{x}{b} \right)^r y_{n-r}(x, a + 2r, b) = y_{n-m}(x, a + m, b) \quad (13) \]

where \( m \) is any positive integer.

(13) also generalises formula (6) of §2, because when \( m = 1 \); then (13) becomes (6).

To prove (13), assume it is true for a particular value of \( m \). Thus (13) with \( (n - 1) \) instead of \( n \) and \( a + 1 \) instead of \( a \) becomes

\[ \sum_{r=0}^{m} (-1)^r m_c_r (a + n - 1; r) \left( \frac{x}{b} \right)^{r-1} Y_n-r-1(x, a + 2r + 1, b) = y_{n-1-m}(x, a + m + 1, b). \]

Now apply (6) to each term on the left and get

\[ y_{n-1-m}(x, a + m + 1, b) = \sum_{r=0}^{m} (-1)^r m_c_r (a + n - 1; r) \left( \frac{x}{b} \right)^{r-1} Y_n-r-1(x, a + 2r + 1, b) \]
\[ \times y_{n-r}(x, a + 2r, b) + \sum_{r=0}^{m} (-1)^{r+1} m_c_r (a + n - 1; r) \times \]
\[ \times (a + n - r - 1) \left( \frac{x}{b} \right)^{r+1} y_{n-r-1}(x, a + 2r + 2, b). \]

As before, in the second of these two series, write \( r - 1 \) for \( r \), and applying the last identity, namely

\[ m_c_r + m_{c_{r-1}} = m_{c_r} \]
then the last expression becomes (13) with \( m + 1 \) in place of \( m \).

But (13) is true when \( m = 1 \); therefore it follows that (13) is true for all positive integer values of \( m \). Thus (13) is proved.

8. The formula now to be proved is

\[
\sum_{s=0}^{[n/2]} \frac{1}{s!} \left( \frac{a + n - 1}{2} - \frac{1}{2} \right)^s \frac{x}{b} \binom{-n - 2s}{b} y_{n-2s}(x, a + 4s, b) = \quad (14)
\]

To prove (14), substitute for \( y_{n-2s}(x, a + 4s, b) \) from (1); then the left-hand side of (14) is equal to

\[
\sum_{s=0}^{[n/2]} \sum_{r=0}^{n-2s} \frac{(-n; 2s)(-n + 2s; r)(a + n - 1 + 2s; r)}{s! r!} \left( \frac{x}{b} \right)^{r + 2s}.
\]

Here write \( p = 2s + r \), where \( p \) is the new parameter of summation and the last expression becomes

\[
\sum_{p=0}^{n} \binom{-n; p}{p!} (a + n - 1; p) \left( \frac{x}{b} \right)^p \binom{-p + 1 - p}{2} \binom{a + n}{2} \binom{-2x}{b}^p.
\]

Again sum the \( \binom{a + n}{2} \) by Vandermonde's theorem, and the left-hand side of (14) becomes

\[
\sum_{p=0}^{n} \binom{-n; p}{p!} \left( \frac{a + n - 1}{2}; p \right) \times \left( \frac{a + n - 1}{2}; a + n \right) \left( \frac{-2x}{b} \right)^p.
\]

by applying the duplication formula for the gamma function. But the last expression is equal to

\[
\sum_{p=0}^{n} \binom{-n; p}{p!} \left( \frac{a + n - 1}{2}; p \right) \left( \frac{-2x}{b} \right)^p = \binom{-n, a + n - 1}{2} \frac{-2x}{b} y_n \left( \frac{x, a - n + 1}{2}, \frac{b}{2} \right) = \text{right-hand side of (14)}.
\]

Thus (14) is proved.

A particular case of interest is when \( n = 2 \). In that case (14) gives

\[
y_2(x, a, b) + (a + 1) \left( \frac{x}{b} \right)^2 y_0(x, a + 4, b) = y_2 \left( \frac{x, a - 1}{2}, \frac{b}{2} \right); \quad (15)
\]
which can be obtained directly by comparing coefficients of different powers of $x$.

In the same way for even $n$ the following formula can be established:

$$
\sum_{s=0}^{[n/2]} \binom{n}{2s} s! (a + n - 1; 2s) \frac{x^{2s}}{s!} y_{n-2s}(x, a + 4s, b) =
$$

$$
\begin{align*}
2F_0 \left( \frac{-n}{2}, a \pm n - 1; \frac{-2x}{b} \right). 
\end{align*}
$$

(16)

Summary

The Bessel polynomials defined by the formula

$$
y_n(x, a, b) = 2F_0 \left( -n, a + n - 1, \frac{-x}{b} \right)
$$

arise in the solution of the wave equation in spherical polar coordinates. In this paper some recurrence relationships and series expansions are established for the Bessel polynomials.

References


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