LINEAR RELATIONS BETWEEN THE BESSEL POLYNOMIALS

By

A. M. HAMZA*

Department of Mathematics University of Libya, Faculty of Engineering Tripoli, Libya
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Presented by Prof. Dr. M. FARKAS

1. Introduction

The Bessel polynomials were defined by KRALL and FRINCK [1] by the formula

$$y_n(x, a, b) = {}_2F_0\left(-n, a+n-1, \frac{-x}{b}\right) \tag{1}$$

These polynomials arise in the solution of the wave equation in spherical polar coordinates. A large number of papers has been written on these polynomials (see RAGAB, [2] & [3], SALAM [4], AGARWAL [5] and BRAFMAN [6]).

In this paper new recurrence relations as well as some series expansions will be established.

The following formulae will be required in the proofs: DOUGALL [7];

$$ {}_5F_4\begin{bmatrix} a, & 1+1\frac{1}{2}a, & c, & d, & e; & 1 \\ \frac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a-e & & \end{bmatrix} = \tag{2}$$

$$= \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-e)\,\Gamma(1+a-c-d-e)}{\Gamma(1+a)\,\Gamma(1+a-d-e)\,\Gamma(1+a-c-d)\,\Gamma(1+a-c-e)};$$

Vandermond's theorem: if n is a positive integer, then

$${}_2F_1(-n, \beta; \gamma; 1) = \frac{(\gamma-\beta; n)}{(\gamma; n)}; \tag{3}$$

where the symbol

$$(\alpha; r) = \alpha(\alpha+1)\ \ldots(\alpha+r-1) = \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)},$$

$$(\alpha; 0) = 1,$$

And Saalschütz's theorem [8]:

$$ {}_3F_2\left[\begin{matrix}a, b, -n; & 1\\ 1+a+b-c-n, c\end{matrix}\right] = \frac{(c-a;n)(c-b;n)}{(c;n)(c-a-b;n)}; \tag{4} $$

where n is a positive integer.

2. Recurrence formulae for y_n (x, a, b)

The formulae to be proved are:

$$ y_n(x, a, b) + \frac{nx}{b} y_{n-1}(x, a+2, b) = y_n(x, a+1, b); \tag{5} $$

$$ y_n(x, a, b) - (a+n-1)\frac{x}{b} y_{n-1}(x, a+2, b) = y_{n-1}(x, a+1, b); \tag{6} $$

$$ y_n^1(x, a, b) = \frac{n(a+n-1)}{b} y_{n-1}(x, a+2, b). \tag{7} $$

To prove (5), substitute for $y_n(x, a, b)$ and $y_{n-1}(x, a+2, b)$ from (1); then the left hand side of (5) becomes:

$$ \sum_{r=0}^{n}\frac{(-n;r)(a+n-1;r)}{r!}\left(-\frac{x}{b}\right)^r + (-n)\sum_{r=0}^{n-1}\frac{(-n+1;r)(a+n-1;r)}{r!}\left(-\frac{x}{b}\right)^{r+1}. $$

Now the coefficient of $\left(-\frac{x}{b}\right)^r$ in the last expression is

$$ \frac{(-n;r)(a+n-1;r)}{r!} + \frac{(-n)(-n+1;r-1)(a+n;r-1)}{(r-1)!} = $$

$$ = \frac{(-n;r)(a+n-1;r)}{r!} + \frac{(-n;r)(a+n-1;r)}{(a+n-1)(r-1)!} = $$

$$ = \frac{(-n;r)(a+n-1;r)}{r!}\left[1+\frac{r}{a+n-1}\right] = $$

$$ = \frac{(-n;r)(a+n-1;r)}{r!} \times \frac{a+n-1+r}{a+n-1} = $$

$$ = \frac{(-n;r)(a+n;r)}{r!} = $$

$$ = \text{coefficient of } \left(\frac{-x}{b}\right)^r \text{ in } y_n(x, a+1, b). $$

This completes the proof of (5).

In the same way (6) can be proved.

To prove (7), substitute for $y_n(x, a, b)$ from (1), differentiate term by term, so we get:

$$y_n^1(x, a, b) = \frac{d}{dx}\left\{\sum_{r=0}^{n} \frac{(-n; r)(a+n-1; r)}{r!}\left(\frac{-x}{b}\right)^r\right\} =$$

$$= -\sum_{r=1}^{n} \frac{(-n; r)(a+n-1; r)}{b(r-1)!}\left(\frac{-x}{b}\right)^{r-1} =$$

$$= -\sum_{r=0}^{n-1} \frac{(-n; r+1)(a+n-1; r+1)}{r!}\left(\frac{-x}{b}\right)^r =$$

$$= \frac{n(a+n-1)}{b}\sum_{r=0}^{n-1} \frac{(-n+1; r)(a+n; r)}{r!}\left(\frac{-x}{b}\right)^r =$$

$$= \frac{n(a+n-1)}{b} y_{n-1}(x, a+2, b);$$

by applying (1) again. Thus (7) is proved. As KRALL and FRINCK remarked we see that the derivatives of Bessel polynomials are Bessel polynomials with the parameter increased by 2.

3. Representation of y_n (x, a, b) as a series of generalised hypergeometric functions of the form ${}_4F_2$

The expansion to be proved is:

$$y_n(x, a, b) = \sum_{r=0}^{n} {}^nC_r \frac{(a; r)\left(1+\frac{1}{2}a; r\right)(\beta; r)(-n; r)(a+n-1; r)}{\left(\frac{1}{2}a; r\right)(1+a-\beta+n; r)(1+a; 2r)}\left(\frac{-x}{b}\right)^r \times$$

$$\times {}_4F_2\left[\begin{matrix}-n+r, a+n-1+r, 1+a+n+r, 1+a-\beta+r; -x/b\\ 1+a-\beta+n+r; 1+a+2r\end{matrix}\right]. \quad (8)$$

To prove (8), substitute for ${}_4F_2$; then the right-hand side of (8) is equal to:

$$\sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-n; r)(a; r)\left(1+\frac{1}{2}a; r\right)(\beta; r)(-n; r+s)(a+n-1; r+s)}{r!\,s!\left(\frac{1}{2}a; r\right)(1+a-\beta+n; r+s)(1+a; r)(1+a+r; r+s)} \times$$

$$\times (-1)^r \frac{(1+a-\beta; r+s)(1+a+n; r+s)}{(1+a-\beta; r)(1+a+n; r)}\left(\frac{-x}{b}\right)^{r+s}.$$

Here write $s = p - r$, change the order of summation and the right-hand side of (8) becomes

$$\sum_{p=0}^{n} \frac{(-n; p)(a+n-1; p)(1+a-\beta; p)(1+a+n; p)}{p!\,(1+a; p)(1+a-\beta+n; p)} \left(\frac{-x}{b}\right)^p \times$$
$$\times {}_5F_4\left(\begin{matrix} a, 1+\frac{1}{2}a, \beta, -n, -p; 1 \\ \frac{1}{2}a, 1+a-\beta, 1+a+n, 1+a+p \end{matrix}\right).$$

Now sum the ${}_5F_4$ by means of (2) and so obtain the left-hand side of (8). Thus (8) is proved.

4. Series expansions of the Bessel polynomials

The formula to be established is

$$\sum_{k=0}^{n} \binom{\mu}{k} \frac{n!}{(n-k)!} \left(\frac{x}{b}\right)^k y_{n-k}(x, a+2k+2, b) = y_n(x, a+\mu+2, b) \tag{9}$$

where

$$\binom{\mu}{k} = (-1)^k \frac{(-\mu; k)}{k!}.$$

To prove (9) substitute for the Bessel polynomials in the left; then the left-hand side of (9) becomes

$$\sum_{k=0}^{n} \sum_{r=0}^{n-k} (-1)^k \frac{(-\mu; k)(-n; k)(-n+k; r)(n+k+a+1; r)}{k!\,r!} \left(\frac{-x}{b}\right)^{k+r}.$$

Here write $k = p - r$; then the last expression becomes

$$\sum_{p=0}^{n} \frac{(n; p)(-\mu; p)}{p!} \left(\frac{x}{b}\right)^p \sum_{r=0}^{p} \frac{(-a-n-p; r)\,(-p; r)}{r!\,(1+\mu-p; r)} =$$
$$= \sum_{p=0}^{n} \frac{(-1)(-n; p)(-\mu; p)}{(-1)\,p!} \left(\frac{-x}{b}\right)^p {}_2F_1(-p, -a-n-p; 1+\mu-p; 1).$$

Now sum the ${}_2F_1$ by Vandermond's theorem and the last expression becomes

$$\sum_{p=0}^{n} \frac{(-n; p)(-\mu; p)}{p!} \left(\frac{-x}{b}\right)^p \frac{(1+\mu+a+n; p)}{(-\mu; p)} =$$
$$= \sum_{p=0}^{n} \frac{(-n; p)(1+\mu+a+n; p)}{p!} \left(\frac{-x}{b}\right)^p = y_n(x, a+\mu+2, b),$$

which is the right-hand side of (9). Thus (9) is proved.

This was proved for the particular case $b = 2$ in another form by Salam [4] p. 152.

5. The second series expansion to be proved is:

$$\sum_{r=0}^{k} {}^{k}c_r\,(a+2n+1-k;\,r)y_{n+1-k}(x,\,a+r,b)\left(\frac{x}{b}\right)^r = y_{n+1}(x,\,a,\,b) \tag{10}$$

where k, n are any positive integer or zero such that $n+1-k \geqslant 0$.

As before the left-hand side of (10) is

$$\sum_{r=0}^{k} {}^{k}c_r\,(a+2n+1-k;\,r)\left(\frac{x}{b}\right)^r {}_2F_0\left(k-n-1,\,a+n-k+r;\,-\frac{x}{b}\right)=$$

$$=\sum_{r=0}^{k}\sum_{s=0}^{n+1-k}\frac{(-k;r)(a+2n+1-k;\,r)(k-n-1;\,s)(a+n-k+r;\,s)}{r\,!\,s\,!}\times\left(\frac{-x}{b}\right)^{r+s}.$$

Here put $p = r + s$ where p is the new parameter of summation and the last expression becomes

$$\sum_{p=0}^{n+1}\frac{(k-n-1;\,p)(a+n-k;\,p)}{p\,!}\left(\frac{-x}{b}\right)^p {}_3F_2\left(\begin{matrix}a+2n+1-k,\,-p,\,-k;\,1\\ a+n-k,\,2-k+n-p\end{matrix}\right).$$

Now sum the ${}_3F_2$ by Saalschütz's theorem (4) and the left hand side (10) is equal to

$$\sum_{p=0}^{n+1}\frac{(-n-1;\,p)(a+n;\,p)}{p\,!}\left(\frac{-x}{b}\right)^p = y_{n+1}(x,\,a,\,b)\;.$$

Thus (10) is proved.

When $k = 1$; (10) becomes

$$y_n(x,\,a,\,b)+\frac{x}{b}(\mathrm{a}+2n)y_n(x,\,a+1;\,b)=y_{n+1}(x,\,a,\,b). \tag{11}$$

6. Generalisation of (5)

The formula to be proved is

$$\sum_{r=0}^{m}(-1)^r\,{}^{m}c_r\,(-n;r)\left(\frac{x}{b}\right)^r y_{n-r}(x,\,a+2r,\,b)=y_n(x,\,a+m,\,b) \tag{12}$$

where m is any positive integer. This generalises the recurrence relation (5) of §2. because when $m = 1$, (12) becomes (5).

To prove (12), assume it is true for a particular value of m. Thus (12) with $a + 1$ instead of a becomes

$$\sum_{r=0}^{m}(-1)^r\,{}^{m}c_r\,(-n;\,r)\left(\frac{x}{b}\right)^r y_{n-r}(x,\,a+2r+1,\,b)=y_n(x\,,a+m+1,\,b)\,.$$

Now apply (5) to each term on the left and get

$$y_n(x, a+m+1, b) = \sum_{r=0}^{m} (-1)^r \, {}^m c_r \, (-n; r) \left(\frac{x}{b}\right)^r y_{n-r}(x, a+2r, b) +$$
$$+ \sum_{r=0}^{m} (-1)^r \, {}^m c_r \, (-n; r) \left(\frac{x}{b}\right)^{r+1} (n-r) \, y_{n-r-1}(x, a+2r+2, b).$$

In the second of those two series, write $r-1$ for r, add the two series, applying the identity

$${}^m c_r + {}^m c_{r-1} = {}^{m+1} c_r \, ;$$

then the last expression becomes

$$y_n(x, a+m+1, b) = \sum_{r=0}^{m+1} (-1)^r (-n;r) \left(\frac{x}{b}\right)^r {}^{m+1} c_r \, y_{n-r}(x, a+2r, b)$$

which is (12) with $m+1$ in place of m. But (12) is true when $m=1$; therefore it follows by induction that (12) is true for all positive values of m. This completes the proof of (12).

7. Generalisation of Formula (6)

The theorem to be established is:

$$\sum_{r=0}^{m} (-1)^r \, {}^m c_r (a+n-1; r) \left(\frac{x}{b}\right)^r y_{n-r}(x, a+2r, b) = y_{n-m}(x, a+m, b) \qquad (13)$$

where m is any positive integer.

(13) also generalises formula (6) of §2. because when $m = 1$; then (13) becomes (6).

To prove (13), assume it is true for a particular value of m. Thus (13) with $(n-1)$ instead of n and $a+1$ instead of a becomes

$$\sum_{r=0}^{m} (-1)^r \, {}^m c_r (a+n-1; r) \left(\frac{x}{b}\right)^r y_{n-r-1}(x, a+2r+1, b) =$$
$$= y_{n-1-m}(x, a+m+1, b).$$

Now apply (6) to each term on the left and get

$$y_{n-1-m}(x, a+m+1, b) = \sum_{r=0}^{m} (-1)^r \, {}^m c_r (a+n-1; r) \left(\frac{x}{b}\right)^r \times$$
$$\times y_{n-r}(x, a+2r, b) + \sum_{r=0}^{m} (-1)^{r+1} \, {}^m c_r (a+n-1; r) \times$$
$$\times (a+n+r-1) \left(\frac{x}{b}\right)^{r+1} y_{n-r-1}(x, a+2r+2, b).$$

As before, in the second of these two series, write $r-1$ for r, and applying the last identity, namely

$${}^m c_r + {}^m c_{r-1} = {}^{m+1} c_r$$

then the last expression becomes (13) with $m+1$ in place of m.

But (13) is true when $m = 1$; therefore it follows that (13) is true for all positive integer values of m. Thus (13) is proved.

8. The formula now to be proved is

$$\sum_{s=0}^{[n/2]} \frac{1}{s!}\left(\frac{a}{2}+\frac{n}{2}-\frac{1}{2}; s\right)(-n; 2s)\left(\frac{x}{b}\right)^{2s} y_{n-2s}(x, a+4s, b) = \tag{14}$$
$$= y_n\left(x, \frac{a}{2}-\frac{n}{2}+\frac{1}{2}, \frac{b}{2}\right).$$

To prove (14), substitute for $y_{n-2s}(x, a+4s, b)$ from (1); then the left-hand side of (14) is equal to

$$\sum_{s=0}^{[n/2]} \sum_{r=0}^{n-2s} \frac{\left(\frac{a+n-1}{2}; s\right)(-n; 2s)(-n+2s; r)(a+n-1+2s; r)}{s!\,r!}\left(\frac{-x}{b}\right)^{r+2s}.$$

Here write $p = 2s+r$, where p is the new parameter of summation and the last expression becomes

$$\sum_{p=0}^{n} \frac{(-n; p)(a+n-1; p)}{p!}\left(\frac{-x}{b}\right)^{p} {}_2F_1\left(\frac{-p}{2}, \frac{1-p}{2}; \frac{a+n}{2}; 1\right).$$

Again sum the ${}_2F_1$ by Vandermond's theorem, and the left-hand side of (14) becomes

$$\sum_{p=0}^{n} \frac{(-n; p)\,(a+n-1; p)}{p!} \times \frac{\left(\frac{a+n-1}{2}; p\right)}{(a+n-1; p)}\left(\frac{-2x}{b}\right)^{p},$$

by applying the duplication formula for the gamma function. But the last expression is equal to

$$\sum_{p=0}^{n} \frac{(-n; p)\left(\frac{a+n-1}{2}; p\right)}{p!}\left(\frac{-2x}{b}\right)^{p} =$$
$$= {}_2F_0\left(-n, \frac{a+n-1}{2}; \frac{-2x}{b}\right) = y_n\left(x, \frac{a-n+1}{2}, \frac{b}{2}\right) =$$
$$= \text{right-hand side of (14).}$$

Thus (14) is proved.

A particular case of interest is when $n = 2$. In that case (14) gives

$$y_2(x, a, b) + (a+1)\left(\frac{x}{b}\right)^2 y_0(x, a+4, b) = y_2\left(x, \frac{a-1}{2}, \frac{b}{2}\right); \tag{15}$$

which can be obtained directly by comparing coefficients of different powers of x.

In the same way for even n the following formula can be established:

$$\sum_{s=0}^{[n/2]} \frac{\left(\frac{-n}{2}; s\right)(a+n-1; 2s)}{s!} \left(\frac{x}{b}\right)^{2s} y_{n-2s}(x, a+4s, b) = \\ = {}_2F_0\left(\frac{-n}{2}, a+n-1; \frac{-2x}{b}\right). \tag{16}$$

Summary

The Bessel polynomials defined by the formula

$$y_n(x, a, b) = {}_2F_0\left(-n, a+n-1, -\frac{x}{b}\right)$$

arise in the solution of the wave equation in spherical polar coordinates. In this paper some recurrence relationships and series expansions are established for the Bessel polynomials.

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Prof. A. M. HAMZA, Department of Mathematics, University of Libya,
Faculty of Engineering, Tripoli, Libya