THIN ELASTIC LAYERS; CEMENTED JOINTS, COATS

By

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1. Castigliano's variational principle

The basic relationships of classic elasticity theorem can be defined not only as simultaneous differential equations but as a variational problem as well: [1], [2].

The following calculations apply to homogeneous isotropic linearly elastic bodies exposed to small strain. Volumetric force system is omitted. The tested body takes up a spatial volume V, and has a surface S. Displacement vector $\vec{t}$ and surface load $\vec{p}$ are given to surface sections S_t and S_p, respectively: $S = S_t + S_p$; $\vec{n}$ is the normal unit vector to surface S pointing outwards. The inner force system of space V is defined by symmetric stress tensor $\mathbf{F}$.

According to Castigliano's variational principle for

$$\delta K \langle \mathbf{F} \rangle = 0 \tag{1-a}$$

and

$$\mathbf{F} \cdot \vec{n} = p \text{ on surface } S_p, \tag{1-b}$$

$$\mathbf{F} \cdot \nabla = \bar{0} \text{ in space } V, \tag{1-c}$$

and

$$(\delta \mathbf{F}) \cdot \vec{n} = \bar{0} \text{ on surface } S_p, \tag{1-d}$$

$$(\delta \mathbf{F}) \cdot \nabla = \bar{0} \text{ in space } V, \tag{1-e}$$

the solution of the above problem is tensor $\mathbf{F}$.

Functional $K\langle \mathbf{F} \rangle$ is:

$$K\langle \mathbf{F} \rangle = \frac{1}{4G} \int_V \left([\mathbf{F}^2]_I - \frac{1}{m+1} [\mathbf{F}]_I^2 \right) dV - \int_{S_t} \vec{t} \cdot \mathbf{F} \cdot \vec{n} \, dS. \tag{2}$$

Castigliano's theorem includes the compatibility equation and the geometric boundary conditions.

2. Plane strain

Suppose that in the Cartesian system of co-ordinates $[x, y, z]$ fitted to the examined body, in each point of the body the stress tensor $\mathbf{F}$ is

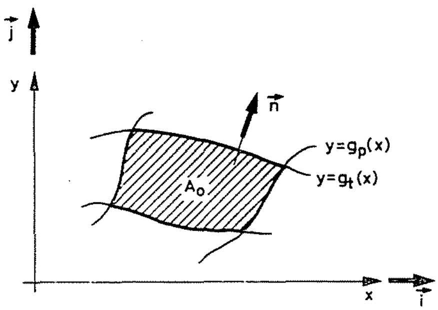

Fig. 1

$$\mathbf{F}=\begin{bmatrix}\sigma_x & \tau_{xy} & 0\\ \tau_{xy} & \sigma_y & 0\\ 0 & 0 & \sigma_z\end{bmatrix}; \quad \frac{\partial \sigma_z}{\partial z}=0. \tag{3}$$

The specific elongation in direction z is $\varepsilon_z = 0$.

The intersection of surface S_t and the co-ordinate plane $[x, y]$ is curve $g_t(x)$ (see Fig. 1). The normal vector of surface S_t in plane $[x, y]$ is:

$$\vec{n}=\frac{1}{\sqrt{1+g_t'^2}}(-g_t'\vec{i}+\vec{j}),$$

$$g_t'=\frac{dg_t}{dx}.$$

Displacement vector:

$$\vec{t}(x, y)=u(x, y)\vec{i}+v(x, y)\vec{j}.$$

Conditions (1-c, e) are satisfied by using Airy's stress function. Writing the stress function in the form:

$$A(x, y)=\sum_{i=1}^{n} f_i(x)\, Y_i(y) \tag{4}$$

the problem will be solved by Kantorovich's method [3]. The function system $Y_i(y)$ is chosen first, taking conditions (1-b, d) into consideration.

Function K can be obtained from

$$\sigma_z=\frac{1}{m}(\sigma_x+\sigma_y),$$

$$E=2G\frac{m}{m-1},$$

as follows:

$$K\langle\sigma_x, \sigma_y, \tau_{xy}\rangle = \frac{1}{2E}\int\limits_V \left(\sigma_x^2 + \sigma_y^2 - \frac{2}{m-1}\,\sigma_x\,\sigma_y + \frac{2m}{m-1}\,\tau_{xy}^2\right) dV -$$

$$-\int\limits_{S_t} [\sigma_x n_x u + \sigma_y n_y v + \tau_{xy}(n_y u + n_x v)]\, dS\,.$$

Stresses are:

$$\sigma_x = \sum_i f_i\, Y_i'' ; \qquad \sigma_y = \sum_i f_i''\, Y_i ; \qquad \tau_{xy} = -\sum_i f_i'\, Y_i'\,.$$

Supposing that the size of the body in direction "z" is constant and stresses are independent of co-ordinate z, the complementary energy is:

$$\widetilde{K}\langle f_i\rangle = \frac{1}{2E}\sum_k \sum_j \int\limits_{A_0} \left(f_k f_j\, Y_k''\, Y_j'' + f_k'' f_j''\, Y_k\, Y_j + \right.$$

$$\left. + 2\,\frac{m}{m-1}\, f_k' f_j'\, Y_k'\, Y_j' - 2\,\frac{1}{m-1}\, f_k f_j''\, Y_k''\, Y_j\right) dA_0 -$$

$$-\sum_k \int\limits_{g_t} [-f_k'\, Y_k'(u - g_t' v) + f_k''\, Y_k v - f_k\, Y_k''\, u g_t']\, dx\,.$$

Introducing notations:

$$D_{kj} = \int Y_j\, Y_k\, dy\,; \qquad B_{kj} = 2\,\frac{m}{m-1}\int Y_k'\, Y_j'\, dy\,;$$

$$C_{kj} = \int Y_k''\, Y_j''\, dy\,; \qquad L_{kj} = -\frac{2}{m}\int Y_k''\, Y_j\, dy\,; \tag{5}$$

$$N_k = (Y_k v)\,|_{y=g_t(x)}\,; \qquad J_k = -(Y_k''\, u g_t')\,|_{y=g_t(x)}\,;$$

$$M_k = [Y_k'(u - g_t' v)]\,|_{y=g_t(x)}\,;$$

(1-a) will be equal to the following simultaneous differential equations:

$$\sum_j \left[\left(D_{kj} f_j'' + \frac{1}{2} L_{kj} f_j\right)'' - (B_{kj} f_j')' + C_{kj} f_j + \frac{1}{2} L_{kj} f_j\right] = E(J_k + M_k' + N_k''),$$

$$(k = 1, 2, \ldots n)\,. \tag{6}$$

Boundary conditions to functions $f_j(x)$ are obtained from (1-b, d).

3. Cemented joint

Let us examine the tensions in adhesive layers of lap joints using the equations above. The adhesive layer of thickness $2h$, and length $2l$ shown in Fig. 2 connects two flat sheets.

Suppose the glue material to be homogeneous, isotropic and elastic, the sheets to be perfectly rigid. The joint is loaded by two forces P of common influence line [4]. They impose the sheets a relative movement u_0.

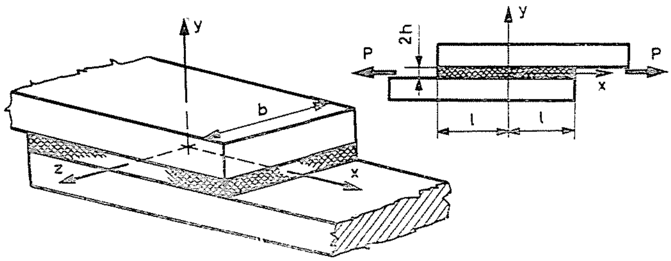

Fig. 2. Cemented lap joint

Except in the environment of the sides of the adhesive layer at co-ordinate $z = b/2$, correlations (3) can be considered as fairly approximate in conformity to [5].

Boundary conditions relating to the glue layer are:

geometric boundary conditions:

$$y = h \quad u = u_0 \ \text{(constant)} \quad v = 0 \qquad (7\text{-a})$$
$$y = -h \quad u = 0 \quad v = 0$$

dynamic boundary conditions:

$$x = \pm l \qquad \tau_{xy} = 0 \qquad \sigma_x = 0 \,. \qquad (7\text{-b})$$

Let us choose the function system such that

$$Y_i = y^i \qquad (i = 1, 3, 5, \ \ldots \ \text{odds}).$$

To find the first approximative solution of the problem, let $i = 1$. The fourth-degree Eq. (6) with determined factors (5) will be as follows:

$$h^2 f_1^{\mathrm{IV}} - c^2 f_1'' = 0; \qquad c^2 = 6 \frac{m}{m-1} \,.$$

On the basis of boundary conditions (7-b)

$$x = \pm l \qquad f_1 = 0 \qquad f_1' = 0.$$

The solution is

$$f_1(x) = -B\left(c\frac{x}{h} - \frac{\mathrm{sh}\left(c\frac{x}{h}\right)}{\mathrm{ch}\left(c\frac{l}{h}\right)}\right).$$

Stresses are

$$\sigma_{1x} = 0; \qquad \sigma_{1y} = By\frac{c^2}{h^2}\frac{\mathrm{sh}\left(c\frac{x}{h}\right)}{\mathrm{ch}\left(c\frac{l}{h}\right)}$$

$$\tau_{1xy} = B\frac{c}{h}\left[1 - \frac{\mathrm{ch}\left(c\frac{x}{h}\right)}{\mathrm{ch}\left(c\frac{l}{h}\right)}\right]. \tag{8}$$

Constant B can be determined from

$$\int_{-l}^{l} \tau_{1xy}\, dx = \frac{P}{b},$$

hence:

$$B = \frac{P}{2b\left[c\frac{l}{h} - \mathrm{th}\left(c\frac{l}{h}\right)\right]}.$$

For $\frac{l}{h} \gg 0$, a good approximation is obtained from

$$\sigma_y = -\bar{\tau}\, c\frac{y}{h} e^{-c\xi},$$

$$\tau_{xy} = \bar{\tau}(1 - e^{-c\xi}), \tag{9}$$

where

$$\bar{\tau} = \frac{P}{2bl}; \qquad \xi = \frac{x+l}{h}.$$

Using general relationships [6] to the previous problem, we get:

$$\sigma_x = \sigma_y = 0 \qquad \tau_{xy} = \bar{\tau}. \tag{10}$$

The calculation shown above gave an important stress concentration. It means that failure in shear of joints is due to tensile stress σ_y:

$$\frac{\sigma_{y\max}}{\bar{\tau}} = \sqrt{6\frac{m}{m-1}}$$

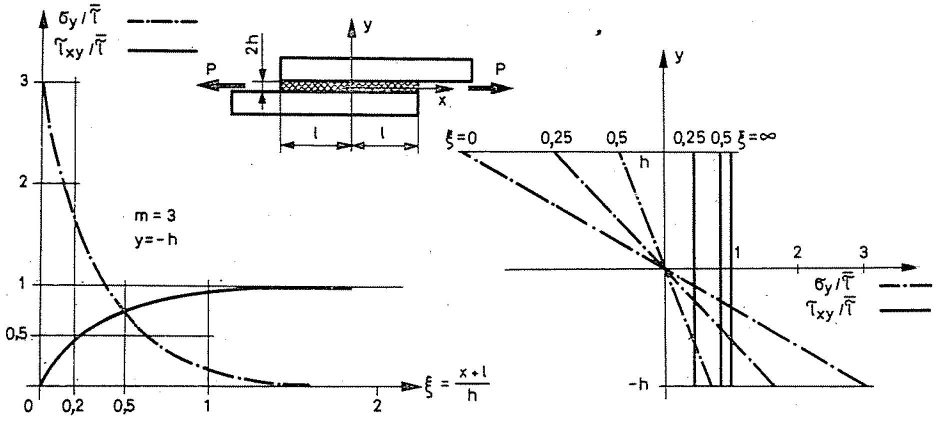

Fig. 3. Stress distribution in the environment of the glue edge

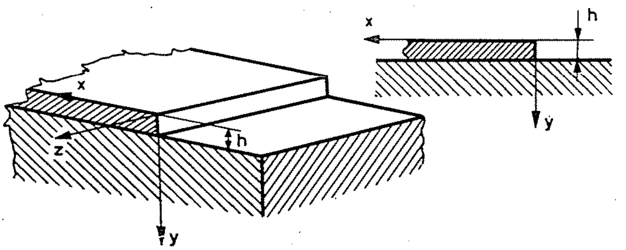

Fig. 4. Flat surface with elastic coating

If the glue is of other than plane, the stress σ_y will be of a different value. From the calculation it is obvious that the shape of adhesive edge (at $x = \pm l$) may be decisive for the strength of joints [6].

4. Surface coatings

In Fig. 4 a plane body is seen, infinite in directions x and z, coated with a layer of thickness h (enamel, synthetic resin, etc.).

Suppose the layer material to be homogeneous, isotropic and elastic. Strains in the body are supposed not to be influenced by the layer stresses, the adherence between layer and body will not fail. These strains meet assumptions (3), $\varepsilon_z = 0$.

Boundary conditions for the layer are geometric boundary conditions:

$$y = h \qquad \frac{\partial u}{\partial x} = \varepsilon_{x0}(x) \tag{11-a}$$

dynamic boundary conditions:

$$y = 0 \qquad \sigma_y = 0 \qquad \tau_{xy} = 0 \tag{11-b}$$

$$x = 0 \qquad \sigma_x = 0 \qquad \tau_{xy} = 0\,. \tag{11-c}$$

Accordingly let us choose the function system $Y_i(y)$ as follows

$$Y_i(y) = y^{1+i} \qquad (i = 1, 2, \ldots).$$

Determine the first approximation of the problem, $i = 1$!

The fourth-degree Eq. (6) will be

$$h^4 f_1^{\mathrm{IV}} - \frac{20}{3} K h^2 f_1'' + 20 f_1 = 10\, E\, \varepsilon_{x0}(x)$$

$$K = (2m + 1)/(m - 1).$$

According to (11-c), the boundary conditions are:

$$x = 0 \qquad f_1 = 0 \qquad f_1' = 0$$

and let limits $(\lim_{x\to\infty} f_1)$ and $(\lim_{x\to\infty} f_2)$ be finite.

If $\varepsilon_{x0}(x)$ is constant then:

$$f_1(x) = \frac{1}{2} \frac{E\, \varepsilon_{x0}}{c_1 - c_2} \left[c_2 e^{-c_1\xi} - c_1 e^{-c_2\xi} + 1\right]$$

where $\xi = \frac{x}{h}$ is a new independent variable,

$$c_1 = \sqrt{\frac{10}{3}\left(K + \sqrt{K^2 + 1.8}\right)},$$

$$c_2 = \sqrt{\frac{10}{3}\left(K - \sqrt{K^2 + 1.8}\right)}.$$

The stresses are

$$\sigma_{1x} = \frac{E\, \varepsilon_{x0}}{c_1 - c_2}\left[c_2 e^{-c_1\xi} - c_1 e^{-c_2\xi} + 1\right],$$

$$\sigma_{1y} = \frac{1}{2} E\, \varepsilon_{x0} y^2 \frac{c_1 c_2}{c_1 - c_2}\left[c_1 e^{-c_1\xi} - c_2 e^{-c_2\xi}\right],$$

$$\tau_{1xy} = E\, \varepsilon_{x0}\, y \frac{c_1 c_2}{c_1 - c_2}\left[e^{-c_1\xi} - e^{-c_2\xi}\right].$$

It is seen from the equations that for a high ξ value

$$\tau_{xy} = 0, \qquad \sigma_y = 0, \qquad \sigma_{x\infty} = E\, \varepsilon_{x0},$$

and at $\xi = 0$

$$\sigma_{y\max} = \sqrt{5}\, E\, \varepsilon_{x0}.$$

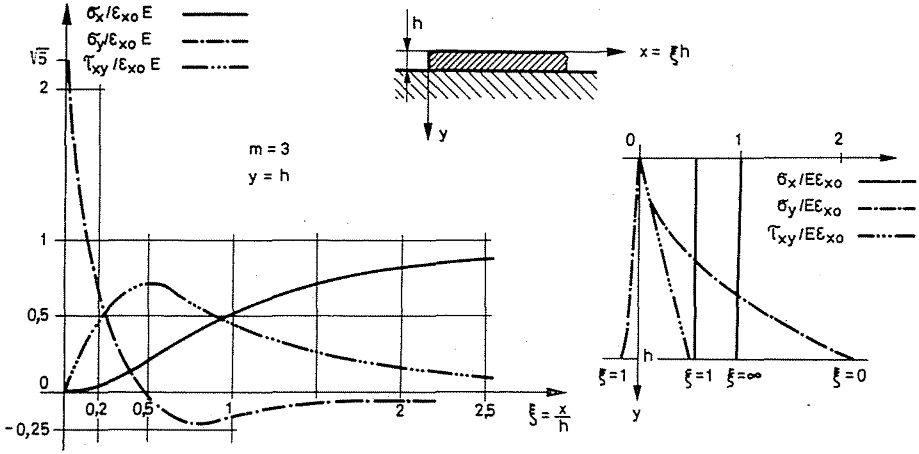

Fig. 5. Stress distribution in the environment of the edge of an elastic coating

Stress concentration in a coated body

$$\frac{\sigma_{y\max}}{\sigma_{x\infty}} = \sqrt{5}\ .$$

Notice that far away from the sides of a layer the stresses tend to reduce. If the coat gets demaged — results in a free edge changing the stress state — be developing $\sigma_{y\max}$ may tear off the coat from the surface.

Summary

Stress states in thin, elastic layers such as cemented joints and surface coatings are determined. The calculations are based on Castigliano's variational theory of classic elasticity theorem, thus the approximative solutions exactly satisfy the equilibrium equations.

The results show significant stress concentrations to develop, in agreement with experimental results.

List of simbols

F	Stress tensor
$\vec{n}$	Normal unit vector to the surface
$\vec{p}$	Surface force system
$\vec{t}$	Displacement vector
$\mathbf{F}^2 = \mathbf{F} \cdot \mathbf{F}$	Product of tensor
$[\ \ ]_I$	First scalar invariant of tensor
$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}$	Stresses
$\varepsilon_z, \varepsilon_x$	Specific strains
$(\ \ )'$	Derivative of a single-variable function
m	Poissons's ratio
G	Shear modulus

E	Young's modulus
P	Shear force applied on a lap joint
b, l, h	Characteristic dimensions of a lap joint
h	Thickness of the coat

$sh(x) = 0{,}5\ (e^x - e^{-x})$; $ch(x) = 0{,}5\ (e^x + e^{-x})$; $th(x) = sh(x)/ch(x)$.

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