ON THE MATHEMATICAL FOUNDATION OF THE MOTOR CALCULUS OF R. v. MISES

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(Received May 17, 1974)

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The motor calculus developed by R. v. MISES for handling problems in mechanics is in fact the result of a long evolution. Though the basic ideas of motor calculus are due to R. BALL, a comprehensive theory was constructed first by E. STUDY. This theory, the so-called "Geometrie der Dynamen" is built on projective geometry and especially on line geometry [5]. Adopting the basic geometric concepts of this theory v. MISES has completely remade it by use of vector and tensor calculus [3]. Thus a closer connection of the mechanical concepts and the utilized mathematics has been achieved, moreover the application of the more versatile vector and tensor calculus has brought about some essential new results [4]. Later on the approach of v. MISES has been modified by L. BRAND who introduced an analytic derivation of the basic concepts and by this he managed to obtain a more concise treatment (pp. 63—83 in [2]).

The mathematical concept of motor is applied in mechanics for the description of two essentially different quantities, namely for the velocity of a rigid body and for a system of forces acting on a rigid body. However, the definition of motors is based on a geometric construction which can be motivated only on account of the velocity concept and the fact that a system of forces acting on a rigid body is adequately described by a motor can be verified only at a subsequent stage by a longer argument. In this note a new definition is proposed for the mathematical concept of motor which at the very beginning renders the fact obvious that systems of forces acting on a rigid body are represented by motors. Actually this new definition is but a mathematically precise and up-to-date formulation of the traditional procedure for the reduction of a system of forces acting on a rigid body (see e.g. pp. 41—172 in [1]), and requires only some simple facts from linear algebra.

1. The definition of the mathematical concept of motor

Let the 3-dimensional Euclidean space E be taken as a mathematical model of the physical space and let $\mathfrak{S}$ be the set of *bound vectors*, i.e. the se, the elements of which are the directed segments and points of E. Actually elements of $\mathcal{E}$ are to represent the forces. If X is a point of E, let $\mathcal{E}_X$ be the set whose elements are the oriented segments starting at X and the point X itself. The set $\mathcal{E}_X$ is obviously a 3-dimensional vector space under the natural definition of addition and multiplication by real numbers and thus $\mathcal{E}_X$ will be called the *space of vectors bound as the point* X. Consider now $x_1, \ldots, x_k$ a finite number of elements of $\mathcal{E}$; then the set $\{x_1, \ldots, x_k\}$ is called a *system of bound vectors*. A system of bound vectors is said to be *simple* if neither points nor directed segments having the same origin are contained in it. Any system of bound vectors defines a unique simple system of bound vectors by the following process: Let $\{x_1, \ldots, x_k\}$ be a given system of bound vectors; fix a point $X \in E$, consider those elements $x_{i_1}, \ldots, x_{i_l}$ of the system which have X as origin and form their sum $y = \sum_{j=1}^{l} x_{i_j}$ in the vector space $\mathcal{E}_X$; then discard $x_{i_1}, \ldots, x_{i_l}$ from the system and take instead of them all y alone as a new element of the system; after a finite number of repetitions of this procedure a system is obtained which does not contain vectors with common origin; finally discard the points from the system if there exists any; thus obviously a simple system is obtained. Let $\{y_1, \ldots, y_m\}$ be the simple system constructed above, it will be denoted by $\sigma\{x_1, \ldots, x_k\}$ and will be said to be obtained by *simplifying* the system $\{x_1, \ldots, x_k\}$.

Let now $\mathcal{F}$ be the set of all systems of bound vectors and $\mathcal{G}$ the set of all simple systems of bound vectors. The set $\mathcal{G}$ makes an infinite dimensional vector space over the field of real numbers with the natural definition of addition and multiplication by numbers. In full detail this assertion is meant as follows: Let $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ be simple systems of bound vectors, then their sum is defined by

$$\{x_1, \ldots, x_m\} + \{y_1, \ldots, y_n\} \overset{\text{def.}}{=} \sigma\{x_1, \ldots, x_m, y_1, \ldots, y_n\}$$

and if λ is a real number then the product of λ and $\{x_1, \ldots, x_m\}$ is defined by

$$\lambda\{x_1, \ldots, x_m\} \overset{\text{def.}}{=} \sigma\{\lambda x_1, \ldots, \lambda x_m\}.$$

The fact that the vector space axioms are valid in case of these operations can be verified by obvious elementary arguments.

By the next step an equivalence relation is introduced in the vector space $\mathcal{G}$. This equivalence relation, if the systems of bound vectors represent systems of forces acting on a rigid body, coincides with the well-known equivalence of these systems of forces (see e.g. pp. 48—49 in [1]). In fact this next step is a concise algebraic formulation of the standard constructions for the reduction of a system of forces where the separate forces are translated along their lines of action so as to have a common origin and then summed (see e.g. pp. 143—146 in [1]).

A simple system $\{x_1, x_2\}$ of bound vectors is called *negligible* if the oriented segments x_1, x_2 are congruent, lie on the same line and have opposite directions. Let $\mathfrak{N}$ be the set of negligible systems and $\mathcal{O}$ the subspace of $\mathcal{G}$ generated by the subset $\mathfrak{N}$. In other words $\mathcal{O}$ can be defined as the set of those simple systems of bound vectors which can be obtained as sum of a finite number of negligible systems since this latter set is obviously an infinite dimensional subspace of $\mathcal{G}$. Consider now the quotient vector space

$$\mathbf{M} = \mathcal{G}/\mathcal{O}.$$

Elements of this vector space $\mathbf{M}$ are called *motors* and $\mathbf{M}$ is called the *vector space of motors*. In a more detailed formulation this definition runs as follows: A simple system $\{x'_1, \ldots, x'_l\}$ is said to be *equivalent* to the simple system $\{x_1, \ldots, x_k\}$ if there exists one element $\{z_1, \ldots, z_m\}$ of $\mathcal{O}$ such that

$$\{x'_1, \ldots, x'_l\} = \{x_1, \ldots, x_k\} + \{z_1, \ldots, z_m\}$$

holds. The term equivalence is properly applied here since the relation thus defined is obviously reflexive, symmetric and transitive in consequence of the fact that $\mathcal{O}$ is a subspace of $\mathcal{G}$. Consider now the equivalence classes generated in $\mathcal{G}$ by this equivalence relation. By the above definition such a class is called a motor and an element of the class will be called a *representative* of the motor. Thus, if the simple system $\{x_1, \ldots, x_k\}$ of bound vectors is representative of the mo tor $\mathbf{m}$ then the notation

$$\mathbf{m} = \{x_1, \ldots, x_k\} + \mathcal{O}$$

can be applied. The subspace $\mathcal{O}$ itself forms such a class which will be called the *null-motor* and it will be denoted by $\mathbf{o}$. The set $\mathbf{M} = \mathcal{G}/\mathcal{O}$ of motors is even a vector space according to the above definition. In fact if $\mathbf{m}, \mathbf{n} \in \mathbf{M}$ are motors and $\{x_1, \ldots, x_k\}$, $\{y_1, \ldots, y_l\}$, respectively, their representatives then

$$\{x_1, \ldots, x_k\} + \{y_1, \ldots, y_l\}$$

is representative of one and the same motor, no matter which representatives of $\mathbf{m}, \mathbf{n}$ are considered; therefore it is justified to call this motor the *sum* of the motors $\mathbf{m}, \mathbf{n}$ and to denote it by $\mathbf{m} + \mathbf{n}$. Likewise if λ is a real number then $\lambda \{x_1, \ldots, x_k\}$ is representing one and the same motor for any representative $\{x_1, \ldots, x_k\}$ of $\mathbf{m}$ and accordingly that motor is called the *product* of λ and **m**, in notations: λ**m**. The fact that **M** satisfies the vector space axioms with the above definitions of vector space operations can be verified by rudiments of linear algebra.

2. The classification of motors

Subsequently a classification of motors is given which is essentially the same as the known classification (see e.g. pp. 65—67 in [2]).

The classification given here is based on two facts. The first of these is the following: *A motor has always a representing system consisting of at most two bound vectors.* The proof of this assertion can be obtained by a selfevident algebraic formulation of the well-known argument for the reduction of a system of forces acting on a rigid body (see e. g. pp. 143—146 in [1]). In formulating the second fact the following definition proves convenient: A simple system $\{x_1, x_2\}$ of bound vectors is called a *pair* if x_1, x_2 are congruent segments lying on parallel lines and having opposite directions. The second fact is the following: *If a motor is represented by a pair then it cannot be represented by a system of fewer elements; and conversely if a motor* **m** *is represented by a system* $\{x_1, x_2\}$ *where* x_1, x_2 *lie in one plane and* **m** *cannot be represented by a system of fewer elements then* $\{x_1, x_2\}$ *is a pair.* The proof of this assertion is likewise to be obtained obviously on line of standard arguments in mechanics.

On account of the above observations the motors can be now classified as follows: A motor is called *simple* if it can be represented by a system consisting of a single element. The null-motor **o** which can be represented by the empty set is called simple as well. A motor is called a *couple* if it can be represented by a pair. If a motor can be represented by a system $\{x_1, x_2\}$ where x_1, x_2 do not lie in one plane then it is called a *screw*. It is obvious that any motor falls uniquely into one of the above three classes.

3. The vector space of motors

In what follows a description of the structure of the vector space **M** of motors is given in terms of the preceding classification.

Here again two preliminary observations prove useful. The first of these observations is the following: *If the pair* $\{x_1, x_2\}$ *is representing a couple* **c** *then any parallel translate* $\{x_1', x_2'\}$ *of* $\{x_1, x_2\}$ *is representing the same couple* **c**. In fact by a self-evident geometric construction there exists a bound vector z such that both simple systems $\{z, -x_1, x_2'\}$ and $\{-z, -x_2, x_1'\}$ represent the null-motor (Fig. 1). Consequently

$$\{x_1, x_2\} + \{z, -x_1, x_2'\} + \{-z, -x_2, x_1'\} = \{x'_1, x_2'\}$$

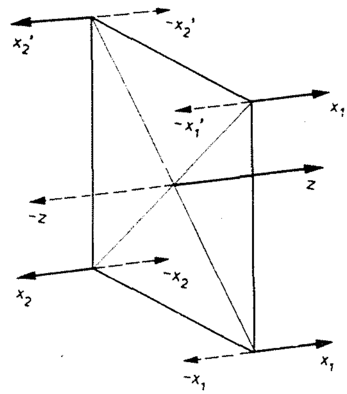

Fig. 1

is representing the same couple **c**. The second observation is the following: *If a pair* $\{x_1, x_2\}$ *represents a couple* **c** *and* x_1' *is an oriented segment in the plane of* $\{x_1, x_2\}$ *with the same origin as* x_1, *then there is an* x_2' *such that* $\{x_1', x_2'\}$ *represents the same couple* **c**. This assertion too can be verified on account of obvious geometric constructions.

Consider now the couples **c**, **e** and pairs $\{x_1, x_2\}$ $\{y_1, y_2\}$ representing them, then the planes of these pairs can be identical, parallel or intersecting. If these planes are identical or parallel then by the above observations there are pairs $\{x_1', x_2'\}$, $\{y_1', y_2'\}$ representing **c**, **e** respectively such that x_1', y_1' and x_2', y_2' are of the same origin. Consequently $\{x_1', x_2'\} + \{y_1', y_2'\}$ is either a pair or the empty set which means that $\mathbf{c} + \mathbf{e}$ is either a couple or the null-motor. If the planes of the pairs intersect then again by the above observations there are pairs $\{x_1', x_2'\}$, $\{y_1', y_2'\}$ representing **c**, **e** respectively, such that x_1', y_1' lie on the line of intersection of the planes, have the same origin and $y_1' = -x_1'$. Therefore $\{x_1', x_2'\} + \{y_1', y_2'\}$ is again a pair and consequently $\mathbf{c} + \mathbf{e}$ is a couple. It is obvious that if λ is a real number and **c** is a couple then the motor $\lambda\mathbf{c}$ is either a couple or the null-motor. Thus the following assertion is obtained: *The set* **C** *whose elements are the couples and the null-motor* **o** *is a* 3-*dimensional subspace of the vector space of motors* **M**. The proof of the fact that the subspace **C** is 3-dimensional being anticipated by the above considerations is not given here. It is to be noted that in mechanics the corresponding fact is generally obtained by means of the moment of the couple, which being a free vector generates a one-to-one correspondence between **C** and the 3-dimensional vector space of free vectors. However, in order to see that this correspondence is a vector space isomorphism it should be proved too that the moment of a sum of couples is the sum of the moments of the couples.

The set **S** whose elements are the simple vectors obviously is not a subspace of the vector space of motors. Fix, however, a point $X \in E$ and consider the set $\mathbf{S}_X$ whose elements are the null-motor and those simple motors which can be represented by a single vector bound at X. This subset $\mathbf{S}_X \subset \mathbf{M}$ is obviously a 3-dimensional subspace of **M**. Now the structure of the vector space **M** can be given by the following proposition: *The vector space of motors* **M** *is the direct sum of subspaces* **C** *and* $\mathbf{S}_X$. Since the validity of $\mathbf{C} \cap \mathbf{S}_X = \{\mathbf{o}\}$ is evident, it suffices for the proof of this proposition to show that any motor $\mathbf{m} \in \mathbf{M}$ can be expressed in the following form:

$$\mathbf{m} = \mathbf{c} + \mathbf{s}, \quad \text{where} \quad \mathbf{c} \in \mathbf{C} \quad \text{and} \quad \mathbf{s} \in \mathbf{S}_X.$$

The existence of such an expression is obvious for couples. If a simple motor **m** is represented by $\{x\}$ then there is a pair $\{x, y\}$ such that the origin of x is X. Thus $\{x\} = \{x, y\} + \{-y\}$ and if **c** is the couple represented by the pair, **s** the simple motor represented by $\{-y\}$ then $\mathbf{m} = \mathbf{c} + \mathbf{s}$ and $\mathbf{s} \in \mathbf{S}_X$ hold. By obvious modification of a standard argument in mechanics (see e.g. pp. 143—145 in [1]) the validity of the following assertion is verified: *If* **m** *is a screw then it has a representing system* $\{x, y\}$ *such that the origin of* x *is an arbitrary point.* Therefore if a screw **m** is given then it has such a representing system $\{x, y\}$ that origin of x is X. But then there is a vector z bound at X such that $\{y, z\}$ is a pair and by putting $u = x - z$ the equality $\{x, y\} = \{x\} + \{y, z\} + \{-z\} = \{y, z\} + \{u\}$ is obtained. Let **c** be the couple represented by $\{y, z\}$ and **s** the simple motor represented by $\{u\}$ then $\mathbf{m} = \mathbf{c} + \mathbf{s}$, where $\mathbf{s} \in \mathbf{S}_X$.

The above established fact that the space of motors admits the direct sum decomposition

$$\mathbf{M} = \mathbf{C} \oplus \mathbf{S}_X$$

has two consequences to be mentioned here. The first of them is immediate: *The vector space of motors* **M** *is 6-dimensional.* The second consequence is actually the equivalence of the motor concept defined in this note to the motor concept of v. MISES and BRAND. Namely on account of the direct sum decomposition $\mathbf{M} = \mathbf{C} \oplus \mathbf{S}_X$ a natural vector space isomorphism is defined between **M** and the space of motors of v. MISES and BRAND such that this isomorphism maps couples to couples, simple motors to simple motors and screws to screws.

Summary

A new definition is given for the mathematical concept of motor which is motivated by the standard treatment of systems of forces acting on a rigid body and thus renders the representability of such systems by motors entirely evident. Beyond this the new definition admits a more concise treatment and uses only some simple concepts from linear algebra.

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