RAPID CONFORMAL SKETCHING OF THE CENTERPOINT CURVE

By

E. FILEMON

Department of Technical Mechanics, Technical University, Budapest

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Presented by Prof. Dr. GY. BÉDA

Introduction

This paper is concerned with the synthesis of mechanisms by the Burmester's theory which is well known among professionals. The sketching rules given in [1] were developed to aid in the generation of the center and circle point curves on the analog computer. An unexpected result of that work was that the sketching rules were a considerably more powerful tool than the subject computer techniques. A relatively good curve could be sketched in a matter of minutes, while the generation of the somewhat better curves via the computer was a major production requiring increased amounts of both time and equipment.

There is possibly a prescribed territory for the fixed pivots of the four bar linkages. In this case it would be clear without any troublesome trial that there is no solution for the particular problem if the location of the free hand sketch is not in this territory. Expediently changing the location of the centerpoint curve until the sketched curve is convenient as the approximate location of pivots. The amount of precision graphical or analytical work is greatly reduced.

An expedient rapid conformal sketching method must be a very simple one even to the detriment of exactness. This paper suggests the use of a new method [2]. It deals with the application of the sketching rules first while the proof of the rules is given in the appendix to this paper.

Sketching rules of the centerpoint curve

The case of convex pole quadrilaterals

In Fig. 1a, b there is a given pole quadrilateral: $O_{12}-O_{14}-O_{34}-O_{23}$. — Let us introduce the marks U and Z.

U: The sum of the lengths of the shortest and the longest sides of the pole quadrilateral,

Z: The sum of the lengths of the remaining two sides of the pole quadrilateral.

— The middle line v of the centerpoint curve has to be constructed.

— The two intersection points of the opposite sides of the pole quadrilateral have to be constructed (Q_{24}; Q_{13}).

— Be the point Q opposite to the longest side of the pole quadrilateral $Q_{\max}$.

— The visual angle of the longest side of the pole quadrilateral at point $Q_{\max}$ is φ.

— In any case the parts of the curve inside the pole quadrilateral belong to the opposite sides (as chords) of the pole quadrilateral.

1. It has to be established whether the shortest and the longest sides of the pole quadrilateral:

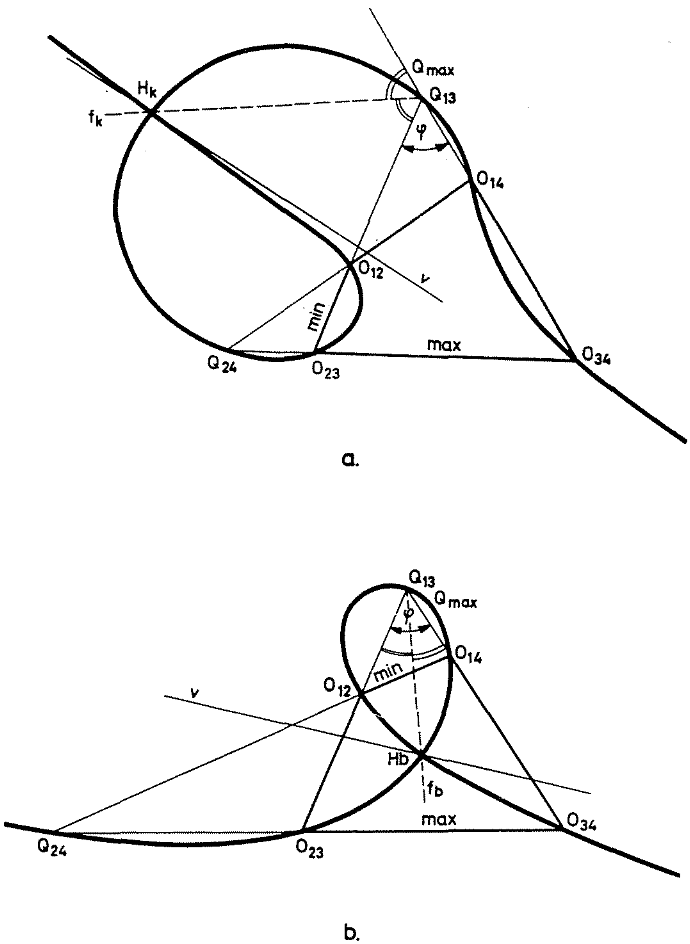


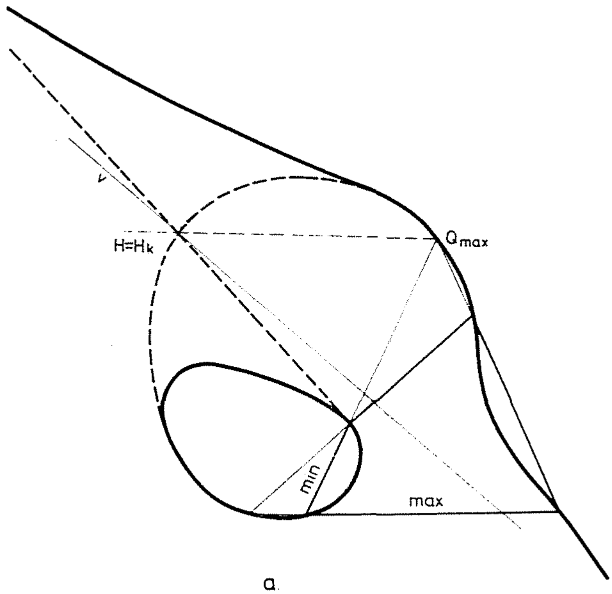
Fig. 1

A) *Are next to each other* (Fig. 1a)

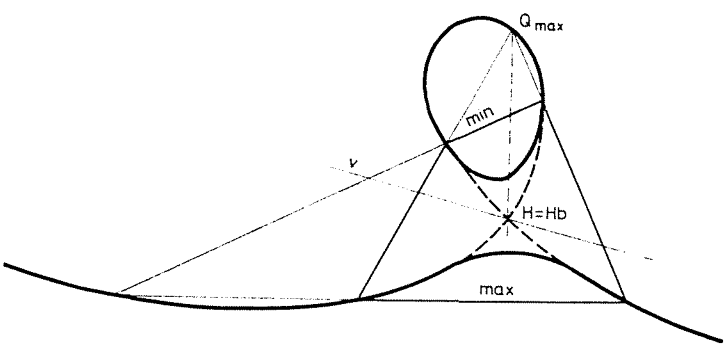
The straight line f_k passes through point Q_{max} and bisects angle $(180° - \varphi)$. The point H_k is the intersection point of lines v and f_k. Considering H_k as a double point of the centerpoint curve, the centerpoint curve can be free-hand sketched. The point Q_{max} is not on the loop of the curve and the shortest side of the pole quadrilateral is a chord of the loop.

B) *Are opposite to each other* (Fig. 1b)

The straight line f_b passes through point Q_{max} and bisects angle φ. The point H_b is the intersection point of lines v and f_b. Considering H_b as a double point of the centerpoint curve, the centerpoint curve can be free-hand sketched. The point Q_{max} is on the loop of the curve and the shortest side of the pole quadrilateral is a chord of the loop.



a.



b.

Fig. 2

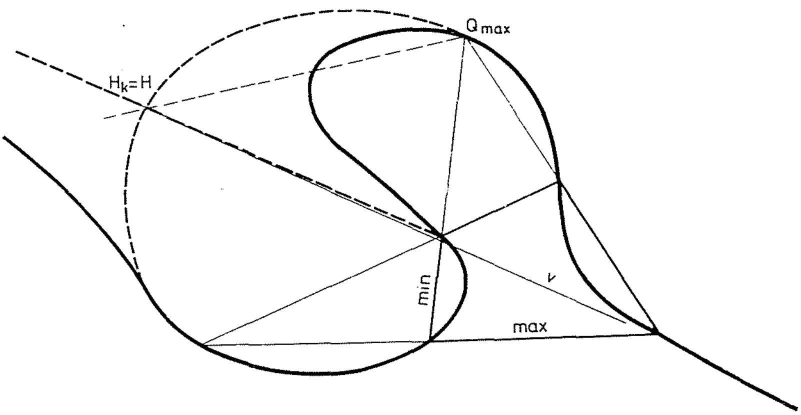
In both cases, at the double point of the curve the existence of two tangents perpendicular to each other, can be assumed.

It can be stated now that the centerpoint curve can be sketched according to the rules 1.A, and 1.B, as follows:

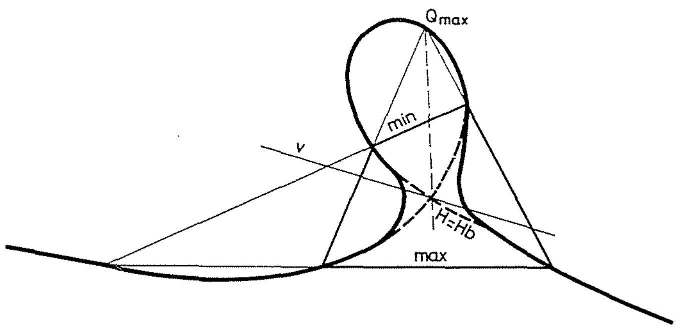
$U = Z$: The conformal sketch is the same as put before (Fig. 1a, b). The centerpoint curve is a nodal one.

$U < Z$: Let $U = Z - \Delta Z$. With an arbitrarily small ΔZ centerpoint curve has two branches. The curve of the section 1.A, B (dashed line in Fig. 2a, b) is only a powerful help to sketch the curve with continuous line in Fig. 2a, b. The helper curve will fall into branches at the disintegrating point H (H_k or H_b) and the shortest side of the pole quadrilateral becomes a chord of the closed part of the curve.

$U > Z$: Let $U = Z + \Delta Z$. With an arbitrarily small ΔZ the centerpoint curve has a single branch without a node. The curve of the section 1.A, B (dashed line in Fig. 3a, b) is only a powerful help to sketch the curve with continuous line in Fig. 3a, b. The disintegrating point



a.



b.

Fig. 3

H (H_k or H_b) of the helper curve will be parted so that the longest side of the pole quadrilateral, a chord, will have no inside arch.

The case of not convex pole quadrilaterals

In Fig. 4a, b there are not convex pole quadrilaterals. Among the six intersection points of the sides of the pole quadrilateral there are always four to determine a convex quadrangle (1—2—3—4). Any point pairs can

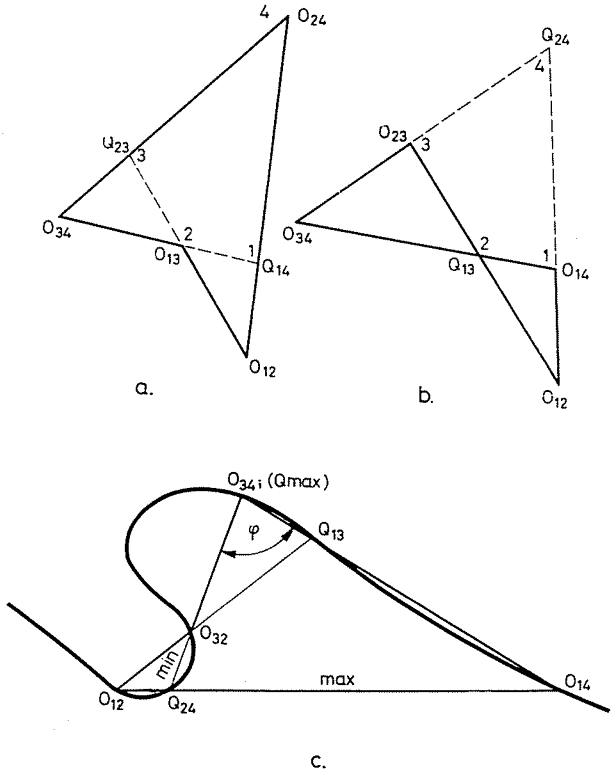


Fig. 4

be changed by each other and all of the rules for the pole quadrilateral are effective for any point pair quadrilateral [3]. In Fig. 4c the point pair Q_{13}—Q_{24} can replace O_{34}—O_{12}.

In this way there is always a convex point pair quadrilateral.

Increasing the accuracy of the sketch

The sketched curve is supposed to be only a conformal one to the centerpoint curve. Some parts of the sketched curve were found to be surprisingly accurate (dashed line in Fig. 5a, b, c) as against that of other parts.

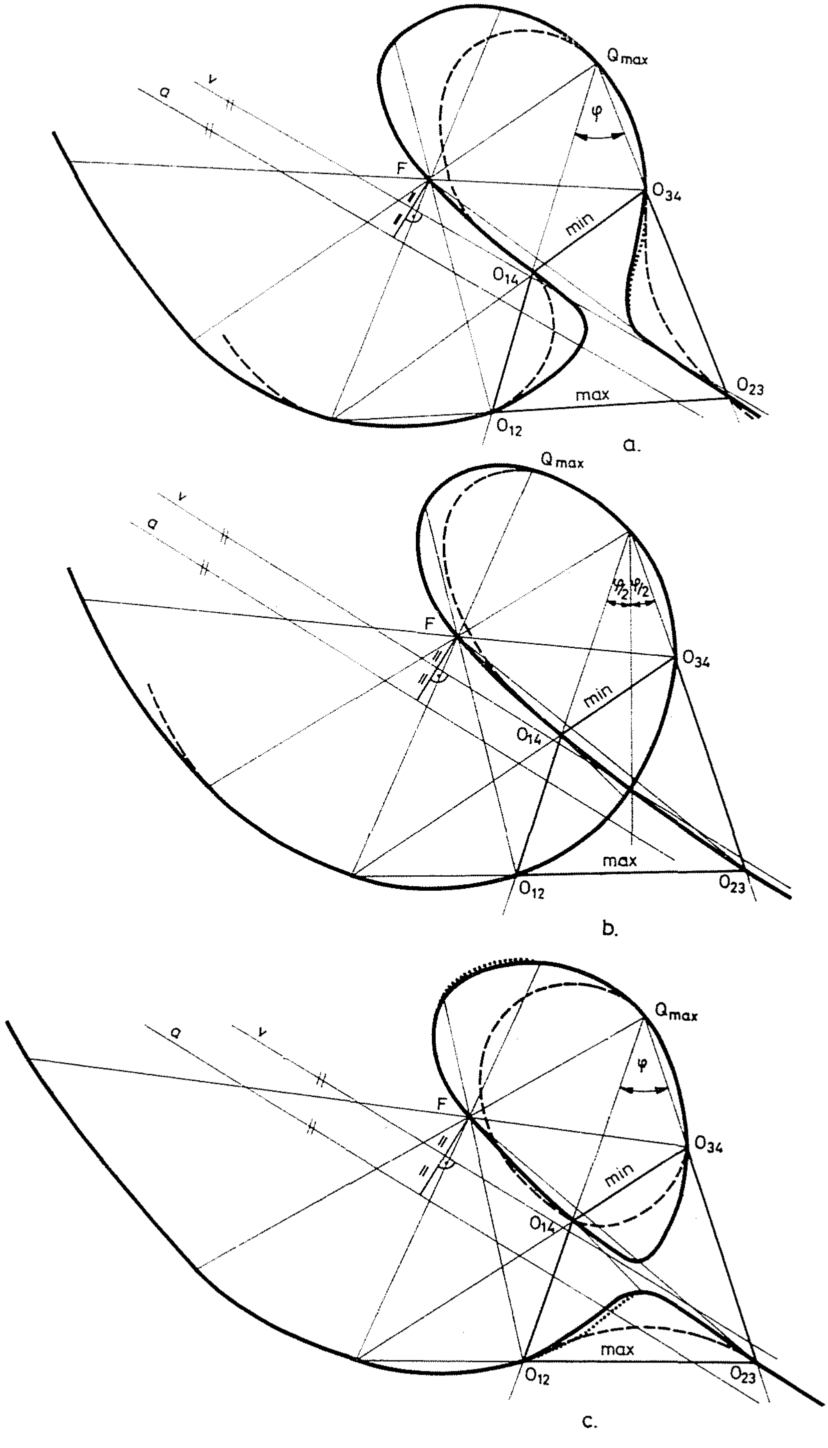

Q max
a
v
φ
F
O34
min
O14
max
O12
O23
a.
Q max
v
a
φ/2 φ/2
F
O34
min
O14
max
O12
O23
b.
Q max
a
v
φ
F
O34
min
O14
max
O12
O23
c.

Fig. 5

To increase the accuracy focus F can be drawn. The asymptote (a) of the curve may also be found by reflecting F about the middle line. There are four poles and two points Q of the curve. Let us reflect these six points about the middle line v on the line passing through the focus F, resulting in six other points of the curve.

In Fig. 5a, b, c a curve has been drawn with continuous line through these 12 points.

The dotted line in Fig. 5a, b, c show that part of the centerpoint curve along which it deviates from the curve sketched by means of the 12 points mentioned above.

To improve accuracy, the six poles and the six points Q can be reflected, so the curve can be sketched through 24 points. It is very likely that a rather good centerpoint curve is given in this way without drawing additional points.

Focus F divides the centerpoint curve into two parts in respect of the ordering of the point pairs of it (Fig. 6a, b, c).

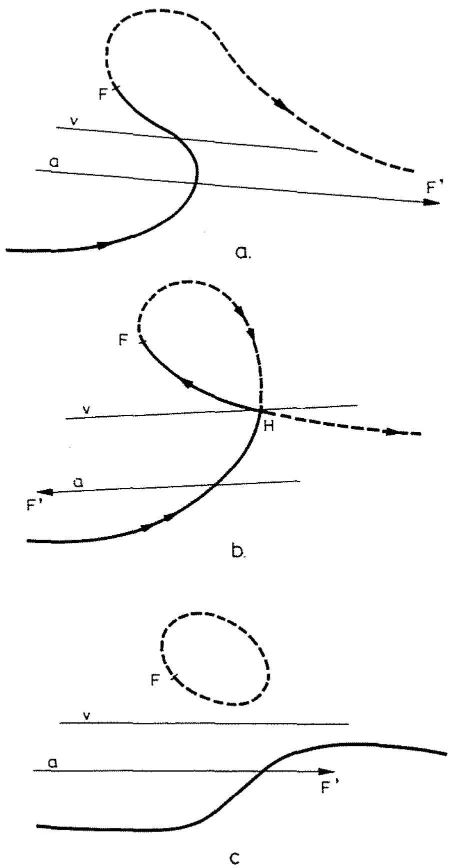


Fig. 6

Appendix: Proof of the rules

Examination of the tangents of the centerpoint curve

Let us infer the type of the centerpoint curve from the existence of the tangents from an arbitrary point of the curve to any other point of the curve. There is a given pole quadrilateral $O_{14}-O_{12}-O_{23}-O_{34}$ in Fig. 7.

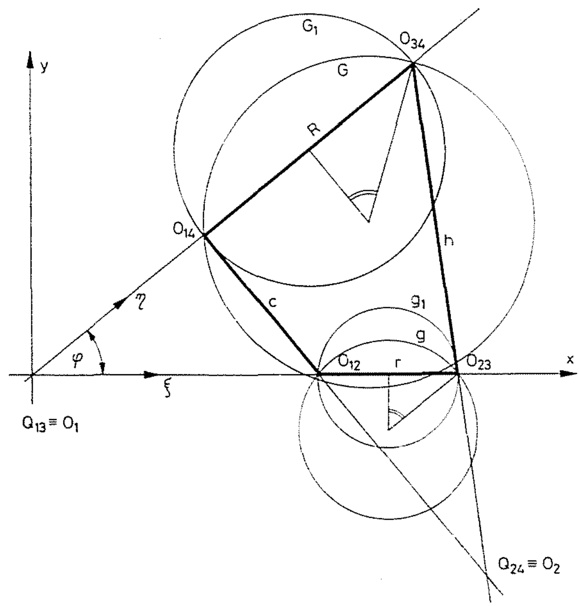


Fig. 7

Be the lengths of the sides of the pole quadrilateral h, R, r and c. We can take without breaking the generality that c is less than h and r is less than R where h and R are sides opposite to points O_1 and O_2, respectively (Fig. 7).

According to Fig. 7 the equation of the centerpoint curve, written in the system $O_1(x, y)$, is:

$$M = \{m : G_1(x, y)ry + g_1(x, y)R\,(\sin\varphi - y\cos\varphi) = 0\} \tag{1}$$

where $G_1(x, y) = \{G_1 : x^2 + y^2 - (x\cos\varphi + y\sin\varphi)(R + 2\eta_0) + \eta_0^2 + \eta_0 R = 0\}$
$g_1(x, y) = \{g_1 : x^2 + y^2 - x\,(r + 2\zeta_0) + \zeta_0^2 + \zeta_0 r = 0\}$

and $\eta_0 = \overline{O_{14}O_1}$; $\zeta_0 = \overline{O_{12}O_1}$

with transformation equations

$$y = \eta\sin\varphi, \quad x = \eta\cos\varphi + \zeta$$

the equation of the centerpoint curve in the system $O_1(\eta, \zeta)$ is:

$$M = \{m : g_1(\eta, \zeta)\, R\zeta + G_1(\eta, \zeta)\, r\eta = 0\} \tag{2}$$

Substituting $\eta = p\zeta$ (where $\zeta \neq 0$) and $L_1 = \eta_0\zeta_0$; $L_2 = (\zeta_0 + r)(\eta_0 + R)$; $L_3 = (\eta_0 + R)\eta_0$; $L_4 = (\zeta_0 + r)\zeta_0$; $f = p + p^{-1}$
into Eq. (2) results in a quadratic equation for ζ. If the line $\eta = p\zeta$ is a tangent to the centerpoint curve, then the discriminant of this equation must be zero. From the zero discriminant we get a fourth-order equation for p and a quadratic one for f. This is:

$$f^2R^2r^2 + f\,4Rr\,[\cos\varphi\,(L_1 + L_2) - L_3 - L_4] + 4\cos^2\varphi\,(L_1 + L_2)^2 - \\ - 8Rr\cos\varphi\,(L_3 + L_4) - 4\,(L_3 - L_4)^2 = 0 \tag{3}$$

From Fig. 7 the discriminant D of Eq. (3) is:

$$D = 16r^2R^2h^2c^2 \tag{4}$$

The roots of Eq. (3) are

$$f_{1,2} = \frac{-4\,rR\,[\cos\varphi\,(L_1 + L_2) - L_3 - L_4] \pm \sqrt{D}}{2\,R^2r^2} \tag{5}$$

From Eq. (5) f_1 and f_2 are:

$$f_1 = \frac{(c + h)^2 - R^2 - r^2}{Rr} \tag{6a}$$

$$f_2 = \frac{(c - h)^2 - R^2 - r^2}{Rr} \tag{6b}$$

with Eqs. (6a) and (6b), $p_{1,2}$ and $p_{3,4}$ are:

$$p_{1,2} = \frac{f_1 \pm \sqrt{f_1^2 - 4}}{2} \tag{7a}$$

$$p_{3,4} = \frac{f_2 \pm \sqrt{f_2^2 - 4}}{2} \tag{7b}$$

If $(f_1^2 - 4)$ and $(f_2^2 - 4)$ in Eqs. (7a) and (7b) are positive or negative the tangents from the origin (O_1) to the centerpoint curve are real or the solutions are imaginary, respectively. There is no possibility for the coincidence of two tangents because the curve is a cubic one. Namely $(f_1^2 - 4) = 0$ and $(f_2^2 - 4) = 0$ give a double point of the curve.

The following can be stated:

The condition of the existence of a double point is:

$$f_1^2 - 4 = 0 \quad \text{or} \quad f_2^2 - 4 = 0 \tag{8}$$

The centerpoint curve is then a nodal cubic and a so-called fourth-class one.

The centerpoint curve is a sixth class one, if

$$f_1^2 - 4 \neq 0 \quad \text{and} \quad f_2^2 - 4 \neq 0 \tag{9}$$

If h and R are exchanged for c and r (in Fig. 1) respectively, an equation similar to Eq. (2) results, referred to point $O_2 = Q_{24}$ as the origin of the co-ordinate system.

Regarding the origin O_1 we get:

$$f_1^2 - 4 \gtreqless 0 \quad \text{if} \quad c + h \gtreqless R + r \tag{10a}$$

$$f_2^2 - 4 \gtreqless 0 \quad \text{if} \quad c + R \gtreqless h + r \tag{10b}$$

Regarding the origin O_2 we get:

$$f_1^2 - 4 \gtreqless 0 \quad \text{if} \quad c + h \lesseqgtr R + r \tag{11a}$$

$$f_2^2 - 4 \gtreqless 0 \quad \text{if} \quad c + R \lesseqgtr h + r \tag{11b}$$

Since O_1 and O_2 are point pairs together ($O_1 = Q_{13}$ and $O_2 = Q_{24}$) the following conclusions are valid for any point pairs.

1. The condition of the existence of a real tangent to the centerpoint curve through one element of the point pair is the same as that of the existence of an imaginary solution to the other element of the point pair.

2. The condition of the existence of the double point is the same for both elements of the point pair.

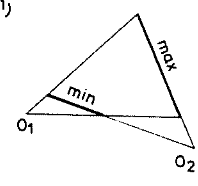
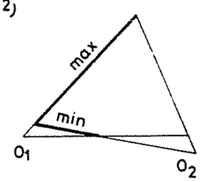
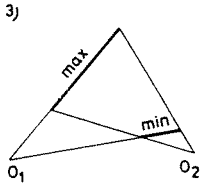
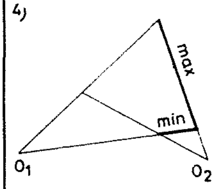
Eqs. (10a, b) and (11a, b) can be examined together with the conditions $U \gtreqless Z$ (Table I), according to the arrangement: r is less than R and c is less than h in any point pair quadrangle.

According to Table I it can be stated that:

$U > Z$ there are two real tangents and two imaginary solutions. There is no double point. All points of the centerpoint curve such as O_1 and O_2 have the same quality as regards tangents. (12a)

$U = Z$ there are two real tangents or two imaginary solutions. There is a double point. (12b)

Table I

 r < R, c < h	relationships (8) and (9)	U > Z		U = Z		U < Z	
		O_1	O_2	O_1	O_2	O_1	O_2
1) 	$f_1^2 - 4 > 0$	2rt					2rt
	$f_1^2 - 4 = 0$			1dp	1dp		
	$f_1^2 - 4 < 0$		2is			2is	
	$f_2^2 - 4 > 0$		2rt		2rt		2rt
	$f_2^2 - 4 = 0$						
	$f_2^2 - 4 < 0$	2is		2is		2is	
2) 	$f_1^2 - 4 > 0$		2rt		2rt		2rt
	$f_1^2 - 4 = 0$						
	$f_1^2 - 4 < 0$	2is		2is		2is	
	$f_2^2 - 4 > 0$	2rt					2rt
	$f_2^2 - 4 = 0$			1dp	1dp		
	$f_2^2 - 4 < 0$		2is			2is	
3) 	$f_1^2 - 4 > 0$					2rt	
	$f_1^2 - 4 = 0$			1dp	1dp		
	$f_1^2 - 4 < 0$	2is					2is
	$f_2^2 - 4 > 0$	2rt		2rt		2rt	
	$f_2^2 - 4 = 0$						
	$f_2^2 - 4 < 0$		2is		2is		2is
4) 	$f_1^2 - 4 > 0$	2rt		2rt		2rt	
	$f_1^2 - 4 = 0$						
	$f_1^2 - 4 < 0$		2is		2is		2is
	$f_2^2 - 4 > 0$		2rt			2rt	
	$f_2^2 - 4 = 0$			1dp	1dp		
	$f_2^2 - 4 < 0$	2is					2is

rt = real tangent
is = imaginary solution
dp = double point

$U < Z$ there are four real tangents or four imaginary solutions. There is no double points. Points of the centerpoint curve are two different assemblies of points. Each assembly of points such as point O_1 and O_2 has a different quality as regards tangents. No one point pair belongs to each of these two assemblies of points. (12c)

Conclusions from the existence of real tangents and imaginary solutions

Taking point O_1 as the origin of the system, and the condition $h > R$, there are only two possibilities:

The shortest and the longest sides of the pole quadrangle are:

A) adjacent: $r < R < c < h$; $r < c < R < h$

B) opposite: $c < r < R < h$

From Table II it is seen that conditions

$$\begin{array}{llll} f_1^2 - 4 < 0 & \text{and} & f_2^2 - 4 > 0; & \text{and} \\ f_1^2 - 4 = 0 & \text{and} & f_2^2 - 4 > 0; & \text{just as} \\ f_1^2 - 4 < 0 & \text{and} & f_2^2 - 4 = 0 & \end{array}$$

never are fulfilled simultaneously.

The possible pair of the discriminants of Eqs. (7a) and (7b) determine the type of the centerpoint curve as well.

The centerpoint curve with a node

In the case $U = Z$ (12b) the curve has a loop and only the locus of its double point has to be constructed.

Determination of the locus of the double point of a nodal centerpoint curve

A) If the shortest and the longest sides of the pole quadrilateral are adjacent:

$$\begin{array}{ll} f_2 < 0 & \text{(from Eq. (6b)), but} \\ f_2^2 - 4 = 0 & \text{(from Table II) thus} \\ f_2 = -2 & \text{therefore} \\ p_{3,4} = -1 & \text{(from Eq. (7b)).} \end{array}$$

The double point is on the line (f_k in Fig. 8a) passing through the origin and its angle in the system $O_1(\eta, \zeta)$ is $-\left(90^\circ - \dfrac{\varphi}{2}\right)$.

Through the origin there are two real tangents (t_1 and t_2) passing, for this reason the origin is not on the loop of the curve.

B) If the shortest and longest sides of the pole quadrilateral are opposite:

$$\begin{array}{ll} f_1 > 0 & \text{(from Eq. (6a)), but} \\ f_1^2 - 4 = 0 & \text{(from Table II) thus} \\ f_1 = 2 & \text{therefore} \\ p_{1,2} = 1 & \text{(from Eq. (7a)).} \end{array}$$

The double point is on the line (f_b in Fig. 8b) passing through the origin and its angle in the system $O_1(\eta, \zeta)$ is $\varphi/2$.

Table II. Sketching of the centerpoint curve

	Relationships (8) and (9)	Real tangents imaginary solutions double points	$U > Z$		$U = Z$		$U < Z$		Type of the curve
			A	B	A	B	A	B	
Fourth-class curves	$f_1^2 - 4 > 0$ $f_2^2 - 4 > 0$	4 real tangents	$+$ $-$	$+$ $-$	$+$ $-$	$-$ $-$	$+$ $+$	$-$ $-$	the curve has two branches
	$f_1^2 - 4 < 0$ $f_2^2 - 4 < 0$	4 imaginary solutions	$-$ $-$	$-$ $-$	$-$ $-$	$-$ $+$	$-$ $-$	$+$ $+$	
	$f_1^2 - 4 > 0$ $f_2^2 - 4 < 0$	2 real tangents and 2 imaginary solutions	$+$ $+$	$+$ $+$	$+$ $-$	$-$ $-$	$+$ $-$	$-$ $+$	the curve has one branch only without loop
	$f_1^2 - 4 < 0$ $f_2^2 - 4 > 0$	never can be fulfilled together	$-$ $-$	$-$ $-$	$-$ $-$	$-$ $-$	$-$ $+$	$+$ $-$	
Sixth-class curves	$f_1^2 - 4 = 0$ $f_2^2 - 4 > 0$	never can be fulfilled together	$-$ $-$	$-$ $-$	$-$ $-$	$+$ $-$	$-$ $+$	$-$ $-$	
	$f_1^2 - 4 > 0$ $f_2^2 - 4 = 0$	2 real tangents and 1 double point	$+$ $-$	$-$ $-$	$+$ $+$	$-$ $-$	$+$ $-$	$-$ $-$	the curve has one branch only, with a loop
	$f_1^2 - 4 = 0$ $f_2^2 - 4 < 0$	1 double point and 2 imaginary solutions	$-$ $+$	$-$ $+$	$-$ $-$	$+$ $+$	$-$ $-$	$-$ $+$	
	$f_1^2 - 4 < 0$ $f_2^2 - 4 = 0$	never can be fulfilled together	$-$ $-$	$-$ $-$	$-$ $+$	$-$ $-$	$-$ $-$	$+$ $-$	

There are two imaginary solutions at the origin, for this reason the origin is on the loop of the curve.

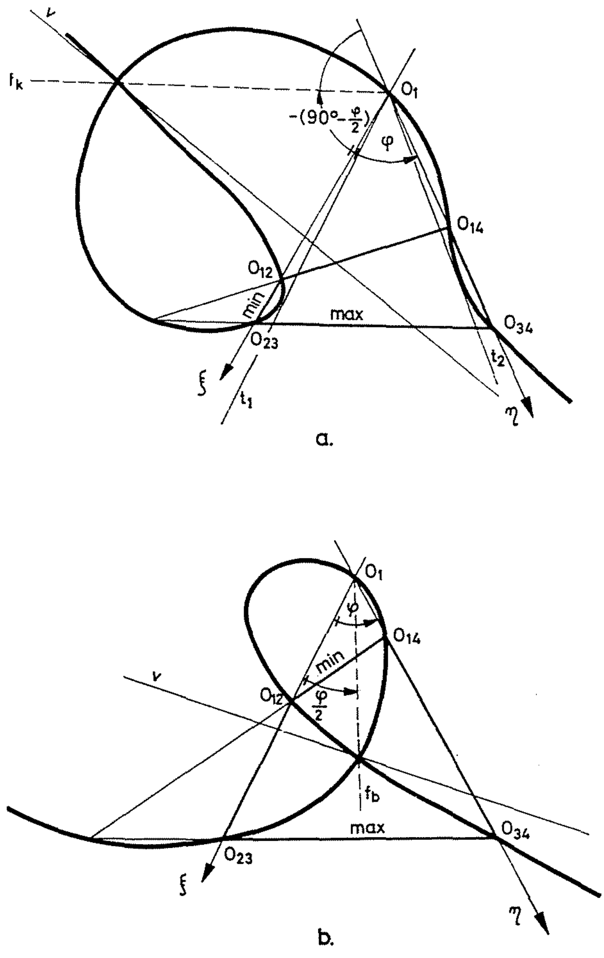
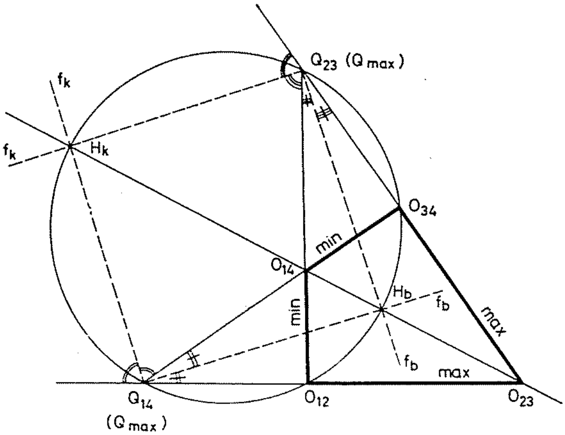


Fig. 8

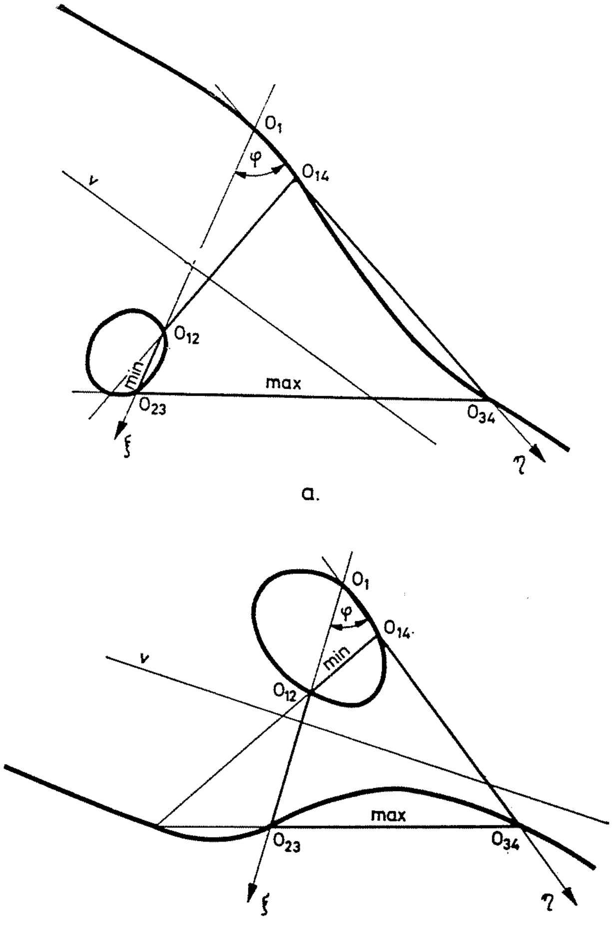
In both cases A) and B), the shortest side of the pole quadrilateral is a chord to the loop.

Cases A) and B) may become true at the same time. For example, two times two nearby sides of the pole quadrilateral have the same length (Fig. 9). In this case the locus of the centerpoint curve will be both a line and circle. It has two double points which can be determined by means of the methods of cases A) and B).



Q_{23} (Q_{max})
f_k
f_k
H_k
O_{34}
min
O_{14}
H_b
f_b
min
max
f_b
max
Q_{14}
(Q_{max})
O_{12}
O_{23}

Fig. 9



O_1
φ
O_{14}
v
O_{12}
min
max
O_{23}
O_{34}
ξ
η
a.
O_1
φ
O_{14}
min
v
O_{12}
max
O_{23}
O_{34}
ξ
η
b.

Fig. 10

The centerpoint curve without a node

If $U < Z$, from (12c) it follows that the centerpoint curve is beparted. From the elements of one point assembly, no real tangents can be drawn to any other point of the curve. It means that one branch is closed in the finity. Any line which intersects this branch can be tangent to no other point for the curve is a cubic one. Since the centerpoint curve has one real point in the infinity and one real asymptote so the other branch must tend to infinity. The asymptote cannot intersect the closed part of the curve so it must intersect the other part. Hence there are three inflexion points on this part. As there cannot be more than three real inflexion points the closed branch has not an inflexion. The closed branch may be called an "even branch" and the other branch an "odd branch".

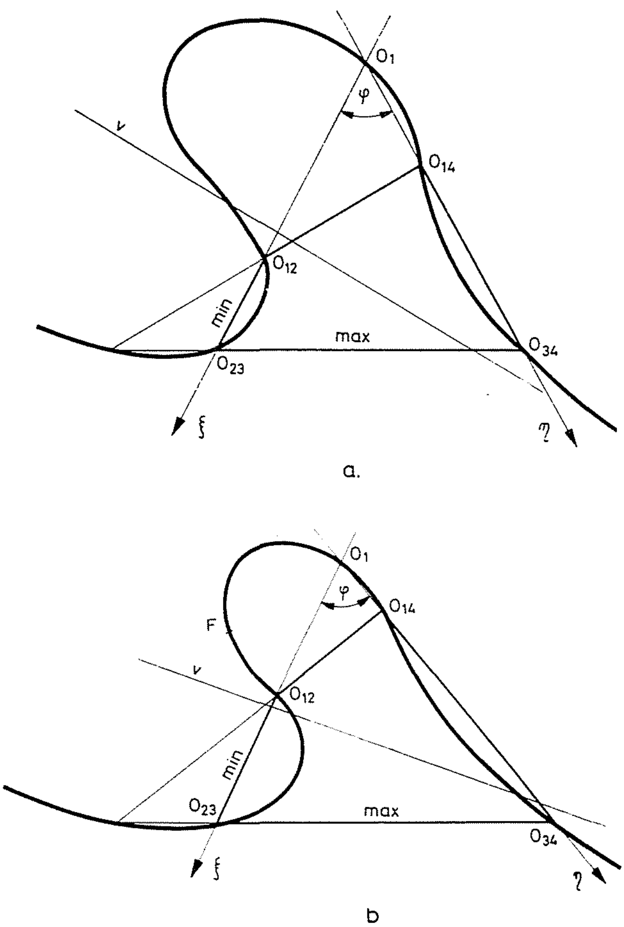


Fig. 11

In case A) the origin is on the odd branch (Fig. 10a) and in case B) it is on the even branch (Fig. 10b).

In both cases the shortest side of the pole quadrilateral is a chord to the even branch.

If $U > Z$ from (12a) it follows that the centerpoint curve has a single branch. The asymptote intersects the curve once so the curve has three inflexion points. At the longest side of the pole quadrilateral the curve can have no inner arch (Fig. 11a, b).

The disintegrating point

It is well known from L. BURMESTER [3] that the centerpoint curve is an assembly of point pairs with the same characteristics as those of the pole pairs. The double point is a point pair, too, with coincident elements.

In Fig. 12 points H_3, H_4 and H_1, H_2 are symmetrical to the point H on the midline v and on the line perpendicular to the midline v respectively.

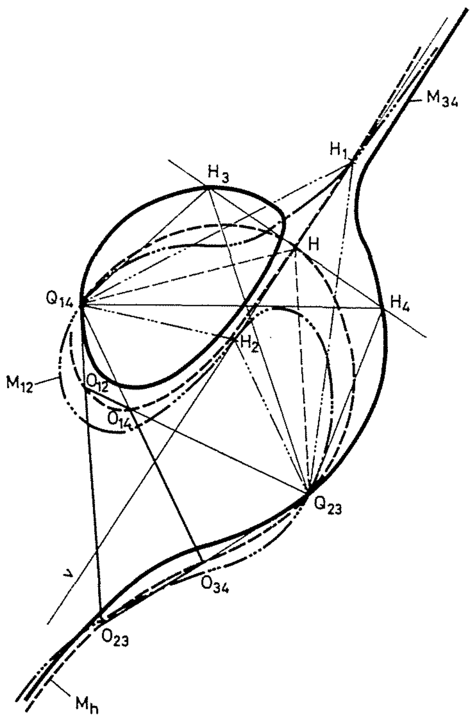


Fig. 12

Let us take now three different point pair quadrangles as $H—Q_{14}—H—Q_{23}$; $H_3—Q_{14}—H_4—Q_{23}$, and $H_1—Q_{14}—H_2—Q_{23}$. Construct the three centerpoint curves M_h, M_{12}, and M_{34} by their means.

Expedience shows that the modifications at point H like this will alter the result curve, mainly in the environment of point H. The more points H_1, H_2 and H_3, H_4 approach point H, the less curves M_{12} and M_{34} deviate from the curve M_h, respectively. Finally if $H_1 = H_2 = H_3 = H_4 = H$ then $M_{12} = M_{34} = M_h$.

The double point of the nodal cubic type centerpoint curve can be taken as a "disintegrating" point, likely to help in sketching the curve if the centerpoint curve is a cubic one without a node.

Summary

This paper has been concerned with the synthesis of mechanisms by means of the Burmester theory. A relatively good centerpoint curve could be sketched in a matter of minutes, while the generation of the somewhat better curves via a computer is a major production requiring increased amount of both time and equipment. There may be a prescribed territory for the fixed pivots of the four bar linkages. In this case it would be clear without any troublesome trial that there is no solution of the particular problem if the location of the free-hand sketch is not in this territory. Expediently changing the initial conditions would result in changing the location of the centerpoint curve until the sketched curve is convenient as the approximate location of pivots. The amount of precision graphical or analytical work is greatly reduced.

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Dr. FILEMON JÓZSEFNÉ, 1143. Budapest, Hungária krt. 39. Hungary.