SOME CONNECTIONS OF THE BURMESTER DESIGNING PROCESS

By

E. FILEMON

Department of Engineering Mechanics. Faculty of Mechanical Engineering. Technical University, Budapest

(Received March 30, 1970)

Presented by Prof. Dr. Gy. BÉDA

From the kinematical geometry of the plane figure moving in its own plane it is well-known that four-bar linkages applicable to realize four arbitrary positions can be designed by means of the centerpoint and circlepoint curves of BURMESTER [1–6]. The points of these curves are related in a mutually unambiguous way. The distance between two corresponding points of these two third-order algebraic plane curves may be the length of one link of the required four-bar linkage. However, this link is a crank or a rocker, depending on the length of all the four links of the linkage. It may become necessary to construct these curves by finding the correlated points. The method, introduced in this paper, does not only solve this problem but simplifies the usual plotting method, making it quicker, easier to survey and thereby more accurate.

Asymptotes of the centerpoint and circlepoint curves

In Fig. 1 the intersection points of the circles G_1 and G_2 will be the points of the centerpoint curve by geometrical definition: the centerpoint curve is the locus of all points the visual angles of which are the same or the adjacent angles to each two opposite sides of the pole quadrilateral [7]. In the orthogonal normal coordinate system the origin of the coordinates coincides with one of the intersection points of the opposite sides of the pole quadrilateral, and the datum line x coincides with one of these very two sides. With the symbols in Fig. 1 (where always $r < R$):

Equation of the circle g_1:

$$x^2 + y^2 - x(r + 2a) + a^2 + ar = 0 .$$

Equation of the circle G_1:

$$x^2 + y^2 - x(r + 2a) + a^2 + ar + 2n_1 y = 0 .$$

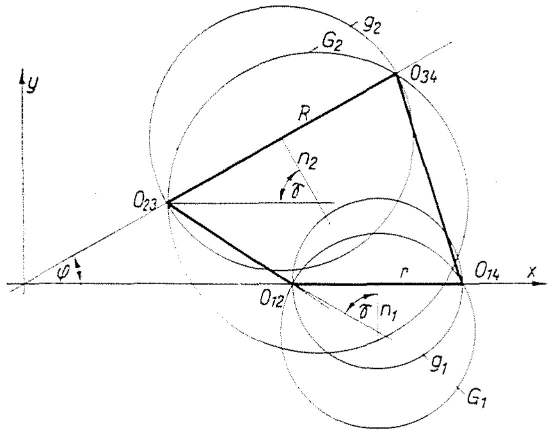

Fig. 1

Equation of the circle g_2:

$$x^2 + y^2 - (R + 2b)(x \cos \varphi + y \sin \varphi) + b^2 + bR = 0 .$$

Equation of the circle G_2:

$$x^2 + y^2 - x(2b + R) \cos \varphi - y(2b + R) \sin\varphi + b^2 + \\ + bR - x\, 2n_2 \sin \varphi + y\, 2n_2 \cos \varphi = 0 .$$

Consequently:

$$G_1(x, y) = g_1(x, y) + \lambda_1 y = 0 . \tag{1}$$

$$G_2(x, y) = g_2(x, y) + \lambda_2 x + \lambda_2' y = 0 . \tag{2}$$

Substituting

$$\lambda_1 = 2n_1 = r \operatorname{ctg} \gamma$$

$$\lambda_2 = -2n_2 \sin \varphi = -R \operatorname{ctg} \gamma \sin \varphi$$

$$\lambda_2' = 2n \cos \varphi = R \operatorname{ctg} \gamma \cos \varphi$$

into Eqs (1) and (2) and reducing Eq. (1) to

$$\operatorname{ctg} \gamma = -\frac{g_1(x, y)}{ry}$$

and putting it into Eq. (2), we get

$$g_2(x, y)\, ry + g_1(x, y)\, R\, [x \sin \varphi - y \cos \varphi] = 0,$$

the equation of the centerpoint curve in the coordinate system $O(x, y)$. Substituting expressions of $g_2(x, y)$ and $g_1(x,y)$:

$$yr\,[x^2 + y^2 - (R + 2b)\,(x \cos \varphi + y \sin \varphi) + b^2 + bR] + \\ + R\,(x \sin \varphi - y \cos \varphi)\,[x^2 + y^2 - x\,(r + 2a) + a^2 + ar] = 0\,. \tag{3}$$

We are interested in the behaviour of the curve in the infinite, so we had better turn to a homogeneous coordinate system.

Substituting them into Eq. (3) and multiplying by x_3^3, then putting $x_3 = 0$ to it, we get:

$$(x_1^2 + x_2^2)\,[x_1\, R \sin \varphi + x_2\,(r - R \cos \varphi)] = 0\,. \tag{4}$$

Besides the imaginary and so-called cyclical points — with coordinates $[x_1, \pm\, ix_1, 0]$ — Eq. (4) has a real solution, too, from the condition:

$$x_1\, R \sin \varphi + x_2\,(r - R \cos \varphi) = 0\,.$$

The homogeneous coordinates of the real point in infinite are

$$\left[x_1, x_1 \frac{R \sin \varphi}{R \cos \varphi - r}, 0\right].$$

The direction at which the real point is in the infinite is determined by the equation:

$$\frac{x_2}{x_1} = \frac{R \sin \varphi}{R \cos \varphi - r}\,.$$

Putting the equation $y = mx + b$ into the Eq. (3) we get a third degree equation:

$$x^3 A\,(m) + x^2 B\,(m) + xC\,(m) + D = 0\,.$$

From the equations $A(m) = 0$ and $B(m) = 0$ the parameters m and b can be determined. Now it is enough for us to solve the equation $A(m) = 0$ as we are looking only for the direction of the real asymptote.

$$A\,(m) = (1 + m^2)\,[R \sin \varphi - m\,(R \cos \varphi - r)] = 0$$

yielding

$$m = \frac{R \sin \varphi}{R \cos \varphi - r} \tag{5}$$

the same as we have got before for $\frac{x_2}{x_1}$.

The general construction of the direction of the real asymptote for marking out the points, to bisect the diagonals of the pole quadrilateral, and to draw a line through these marked points [7] is rather inaccurate, these mid-points being near to each other. Fig. 2 shows the construction of the direction of the real asymptote based on Eq. (5).

$$m = \operatorname{tg} \delta; \qquad \delta < 90^\circ \quad \text{if} \quad r < R \cos \varphi$$

$$\delta = 90^\circ \quad r = R \cos \varphi$$

$$\delta > 90^\circ \quad r > R \cos \varphi .$$

The angle δ can be constructed in several ways.

In Figs 2b and 2c there are two hatched triangles and because of their congruence: $\alpha = \beta$. Besides the intersection point of the opposite sides of the pole quadrilateral can be choosen arbitrarily, so either angle φ_1 with lengths

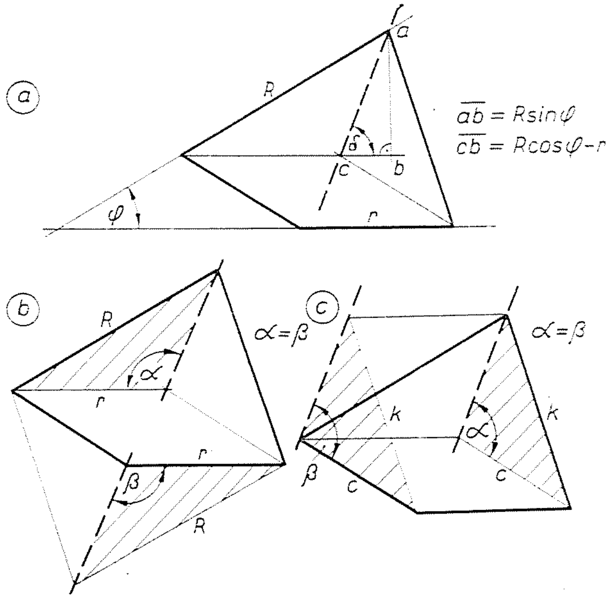

Fig. 2

r_1 and R_1 or angle φ_2 with lengths r_2 and R_2 can be used for the construction. Finally, it can be established that any two neighbouring sides of the pole quadrangle must be completed to a parallelogram and the newly constructed sides of this parallelogram cut out a point which is in the line parallel to the real asymptote. The second point to determine this line is the intersection point of the two sides of the pole quadrangle, close to each other, which is the only point not yet used for any other purpose.

Replacing the pole quadrangle by the mirror pole quadrangle, the above method is applicable for constructing a line parallel to the real symptote of the circlepoint curve.

Construction of the centerpoint and circlepoint curves

The construction shown in Fig. 3 is known from the literature [8]. The mark $\sphericalangle S(BB')$ is interpreted as an angle less than 180° rotated by which the ray SB gets into the position SB'. In accordance with Fig. 3:

$$\sphericalangle S(BB') = \sphericalangle S(CC')$$

$$\overline{SB} = \overline{SB'} \quad \text{and} \quad \overline{SC} = \overline{SC'}$$

therefore
$$\sphericalangle S(BA) = \sphericalangle S(CD)$$

or
$$\sphericalangle [S(BA) + S(CD)] = \pi$$

and
$$\sphericalangle S(CB) = \sphericalangle S(DA)$$

or
$$\sphericalangle [S(CB) + S(DA)] = \pi .$$

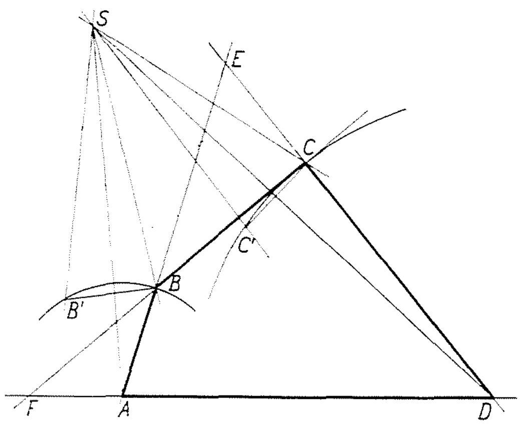

Fig. 3

Consequently, the loci of points S fulfil the geometrical definition of the centerpoint (and circle-point) curve. The points A, B, C, D, E and F are the points of the curve. After the general instructions, an arbitrary pole quadrangle can be chosen (e.g. A, B, C, D correspond to $O_{23}O_{34}O_{14}O_{12}$ and in this case E and F correspond to Q_{24} and Q_{13} and applying the method shown in Fig. 3 we can get the centerpoint curve, which contains the six intersection points of the four sides of the pole quadrangle. After this the mirror poles can be drawn and the whole process must be repeated with an arbitrarily chosen mirror pole quadrangle (e.g. A, B, C, D correspond to $O_{12}O_{13}O_{34}^1O_{24}^1$ and in this case E and F correspond to Q_{23}^1 and Q_{14}^1). So we can get the circlepoint curve, which contains the six intersection points of the four sides of the mirror pole quadrangle.

Instead of this arbitrary designation, it is expedient to arrange the construction as follows:

In Fig. 4 the pole and mirror pole systems, determined by any four

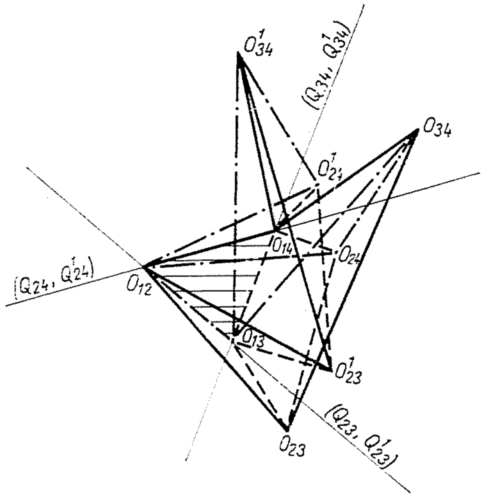

Fig. 4

prescribed positions, are seen. Consider the triangle $O_{12}O_{13}O_{14}$ to be motionless and at the same time frame to the three possible four bar linkages formed from the pole quadrangles. ($O_{12}O_{23}O_{34}O_{14}$; $O_{12}O_{24}O_{34}O_{13}$; $O_{14}O_{24}O_{23}O_{13}$). The mirror pole system can be taken as a new position of the pole mechanisms, determined above, because the next equations are fulfilled:

$$\overline{O_{12}\,O_{23}} = \overline{O_{12}\,O_{23}^1};\quad \overline{O_{14}\,O_{34}} = \overline{O_{14}\,O_{34}^1};\quad \overline{O_{23}\,O_{34}} = \overline{O_{23}^1\,O_{34}^1}$$

$$\overline{O_{12}\,O_{24}} = \overline{O_{12}\,O_{24}^1};\quad \overline{O_{13}\,O_{34}} = \overline{O_{13}\,O_{34}^1};\quad \overline{O_{24}\,O_{34}} = \overline{O_{24}^1\,O_{34}^1}$$

$$\overline{O_{14}\,O_{24}} = \overline{O_{14}\,O_{24}^1};\quad \overline{O_{13}\,O_{23}} = \overline{O_{13}\,O_{23}^1};\quad \overline{O_{24}\,O_{23}} = \overline{O_{24}^1\,O_{23}^1}$$

hence:

$$\sphericalangle\ O_{12}(O_{23}O_{23}^1) = -\varphi_{12};\qquad \sphericalangle\ O_{14}(O_{34}O_{34}^1) = -\varphi_{14} \tag{7a}$$

$$\sphericalangle\ O_{12}(O_{24}O_{24}^1) = -\varphi_{12};\qquad \sphericalangle\ O_{13}(O_{34}O_{34}^1) = -\varphi_{13} \tag{7b}$$

$$\sphericalangle\ O_{14}(O_{24}O_{24}^1) = -\varphi_{14};\qquad \sphericalangle\ O_{13}(O_{23}O_{23}^1) = -\varphi_{13}\,. \tag{7c}$$

On the base of this statement not only the centerpoint curve but also the circlepoint curve can be constructed by means of only one pole quadrilateral and both of them must be the same type (e.g. having a single branch only or being be parted). (The pole triangles can be used for the general determination of the mirror poles so none of the pole quadrilaterals is self-sufficient). As the poles are the results of a previous process, the less of them are involved in the subsequent construction, the more accurate the result will be.

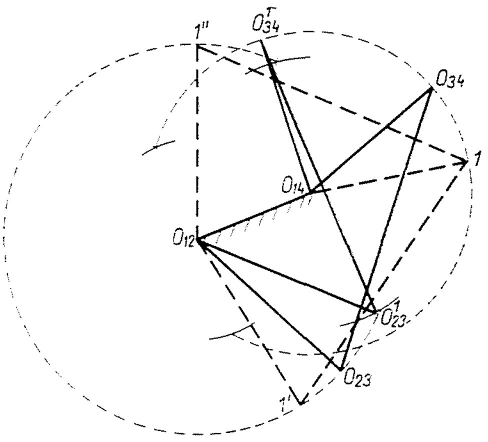

Fig. 5

There is a given pole quadrangle $O_{12}O_{23}O_{34}O_{14}$ in Fig. 5. Let us set down the motion range of the four-bar linkage — $O_{12}O_{23}O_{34}O_{14}$, — and the mirror pole quadrangle — $O_{12}O^1_{23}O^1_{34}O_{14}$. The $\overline{1' O_{12}}$ and $\overline{1'' O_{12}}$ positions of the link $\overline{O_{12}O_{23}}$ belong to $\overline{1\, O_{14}}$ which is one of the arbitrary positions of the link $\overline{O_{34}O_{14}}$. So there are two moving four-bar linkages ($O_{12}-1-1'-O_{14}$; $O_{12}-1-1''-O_{14}$) and two unmovable pole quadrangles. Applying the method shown in Fig. 3, there are two centerpoints (M'; M'') and two circlepoints (K'; K'') belonging to point 1. The midnormals m and k in Figs. 7a, b intersect each other at angles $-\frac{\varphi 12}{2}$ and $-\frac{\varphi 14}{2}$, so K' is the circlepoint to which M' is the unique corresponding centerpoint. This way the points M' K' K'' M'' in Fig. 6 give a theoretically possible solution immediately.

By means of this method of construction, the possible lengths of all cranks and rockers can be determined, and easily plotted against the length of arch of the centerpoint curve.

It is easy to see from this construction that the characteristic points of the centerpoint curve and those of the circle point curve are corresponding

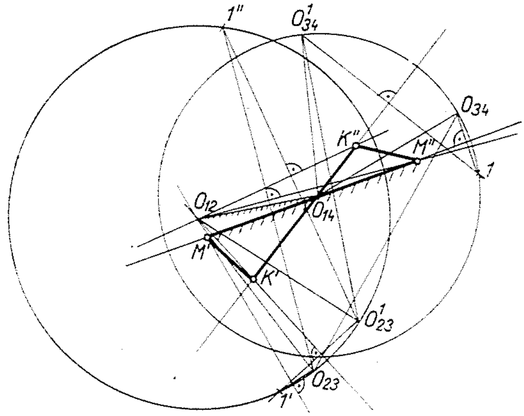

Fig. 6

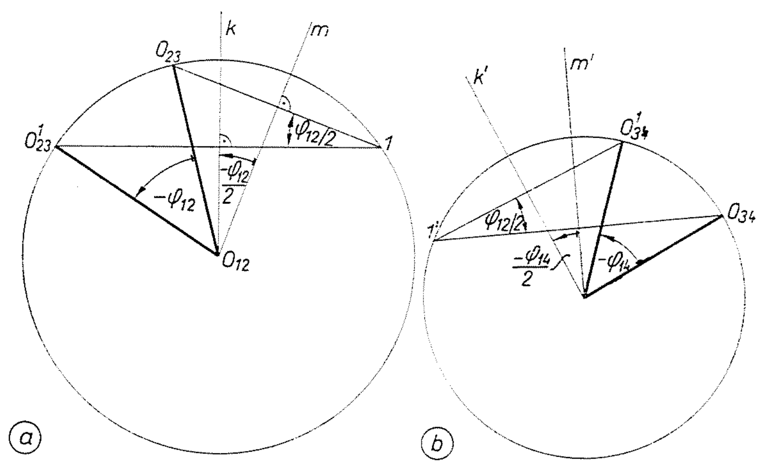

Fig. 7a—b

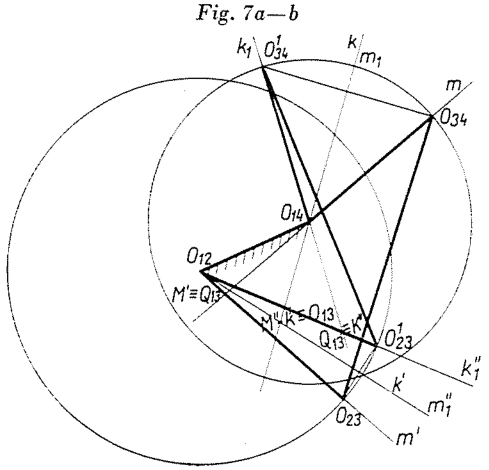

Fig. 8

In Fig. 8 it has been supposed that the $O_{12}-1-1'-O_{14}$ four-bar linkage coincides first, with the unmovable pole quadrilateral, and then with the unmovable mirror pole quadrilateral. The coincidence is possible only between two corresponding links: one of the four-bar linkage and the other of the pole quadrilateral (Fig. 9) or one of the four-bar linkage and the other of the mirror

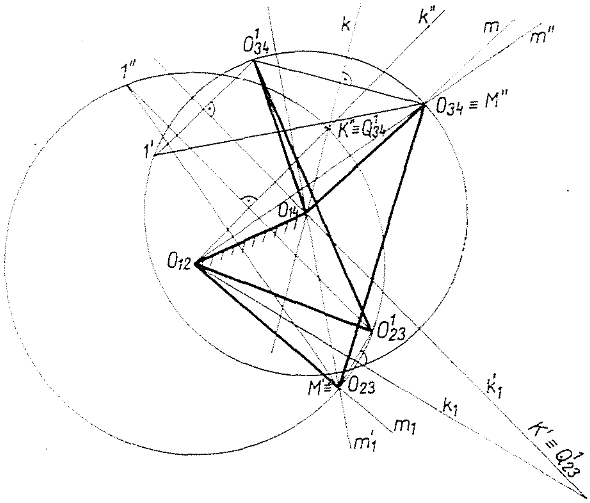

Fig. 9

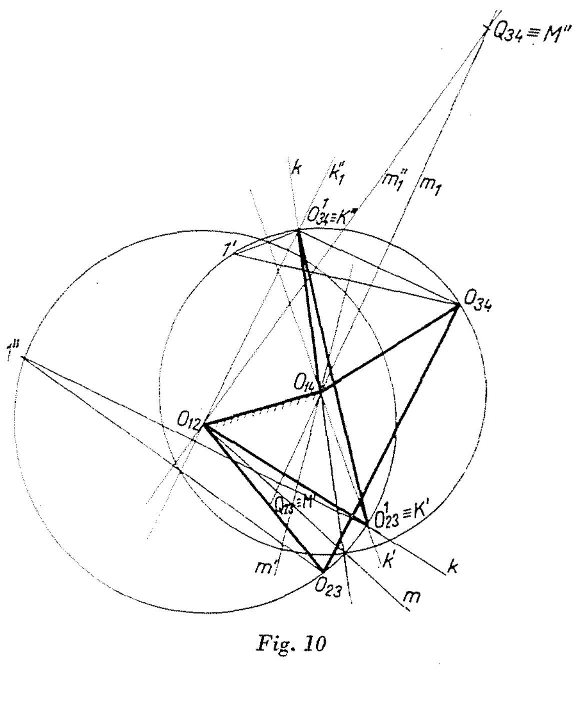

Fig. 10

pole quadrilateral (Fig. 10). In accordance with Figs 8, 9, 10 it is seen that every O^1_{ij} — or Q^1_{ij} — point on the circlepoint curve belongs to the Q_{ij} — or O_{ij} — point on the centerpoint curve — respectively.

In some special cases, without aiming at completeness, assume to have 12 links, given by the pole system, without constructing the centerpoint curve.

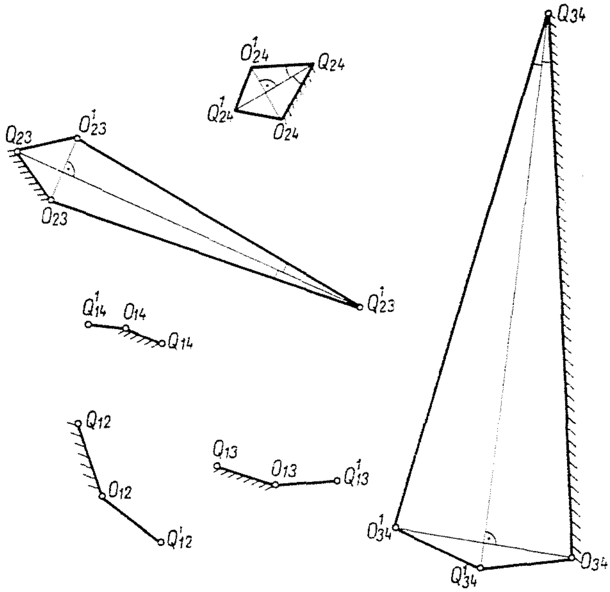

Fig. 11

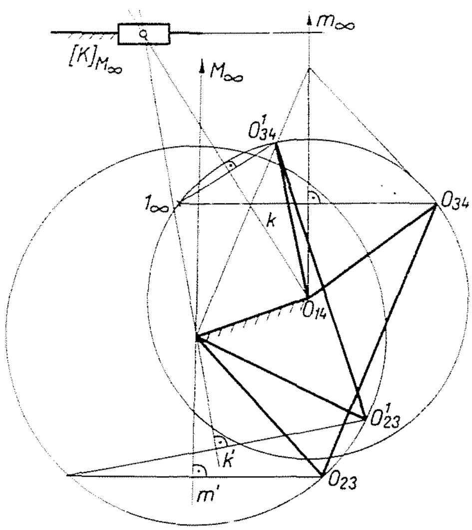

Fig. 12

Among them there are six four-bar linkages for which the Grashof condition becames equality. In the mechanisms — belonging to every frame of type $\overline{O_{ij}\,Q_{ij}}$ — length of the frame is equal to the length of one turning link, and the length of the coupler is equal to the length of the other turning link, thus the sum of the lengths of the longest and shortest links is equal to the sum of the lengths of the other two links. If the vertex of $O_{ij}\,Q_{ij}$ contains the number 1, the links with common length coincide (Fig. 11).

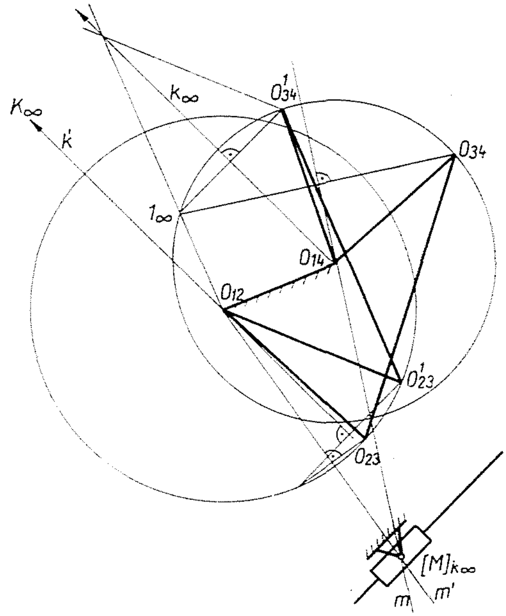

Fig. 13

It is expedient to examine how to construct, with this method, a link of infinite length. There are two cases of infinitely long links: one is when the centerpoint curve point is in the infinite (Fig. 12), the other when the circlepoint curve point is in the infinite (Fig. 13). In both cases it is possible to construct a line parallel to the real asymptote of the centerpoint curve and the circlepoint curve. Using the determination in Fig. 7. the m_∞ and k_∞ correspond to m and k midnormals. According to Fig. 12, $\overline{O_{34}\,1_\infty} \perp m_\infty$, and by means of point 1_∞ points M_∞ and K can be constructed. Point K moves on the straight-line perpendicular to m_∞ and $\overline{M_\infty\,K} = \infty$. In Fig. 13, $\overline{O^1_{34}\,1_\infty} \perp k_\infty$ and by means of the point 1_∞ points M and K_∞ can be constructed and $\overline{K_\infty M} = \infty$. It is necessary to know the M_∞ and K_∞ points, because the solutions in their surroundings are useless for the very great link lengths.

This method is useful even as an auxiliary means for determining the limits of parameters before computation.

Summary

This paper suggests a transformation of the construction known from the literature, making the usual plotting of the centerpoint and circlepoint curves easier to survey, quicker and more accurate. By means of this method the correspondence between the characteristic points of the Burmester curves is very clear, and it is shown that in some cases the Grashof condition must turn to equality. There is an accurate construction method for infinitely long links.

References

1. BURMESTER, L.: Lehrbuch der Kinematik. Verlag von Arthur Felix, Leipzig 1888.
2. HARTENBERG, R. S., DENAVIT, I.: Kinematic Synthesis of Linkages. McGraw-Hill Book Company 1964.
3. TCHERKUDINOV, S. A.: Sintes ploskih sharnirnorichasnic mekhanizmov. Izd. Akad. Nauk SSSR, Moscow 1959.
4. BEYER, R.: Kinematische Getriebesynthese. Springer Verlag, Berlin 1953.
5. SÁLYI, I. JUN.: Some connections of the special case of the Burmester designing process.* Miskolci Nehézipari Műszaki Egyetem közleményei **12**, 433—452 (1963).
6. TERPLÁN, Z.: Mechanismus.* Tankönyvkiadó, Budapest 1959.
7. GERONIMUSZ, I. L.: Geometricheskiy Appart Teorii Sinteza Ploskikh Mekhanismov. Gos. Izd. Fiz.-Mat. Lit. Moscow, 1962.
8. LUCK, K.: Spatial Representation of Plane Kinematics.* Miskolci Nehézipari Műszaki Egyetem magyar nyelvű közleményei **7** (1961).

Dr. FILEMON Józsefné, 1502. Budapest, P. O. B. 91. Hungary

* (in Hungarian)