THE EQUATION OF STATE-CHANGE OF STRUCTURES*

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1. Introduction

The *structures* dealt with in this paper consist of a general kind of *bars* (often called finite elements), connected by *constraints*. The actual position of the structure is described by the co-ordinates of the nodal points in a global orthogonal system x, y, z, furthermore by the directions of the local co-ordinate systems ξ ζ, η, fixed to the connection points of the bars. (The analysis may obviously be performed by the use of other consistent co-ordinate systems, too.) The *response* of the structure will be described by the generalized *displacement* ($\mathbf{u}$) of the nodal points related to the initial position and by the generalized constraint forces acting at the connection points of the bars, viz. the generalized *stresses* ($\mathbf{s}$). The structure may be *loaded* by generalized *forces* ($\mathbf{q}$) acting either at the nodal points or directly on the bars, and also by the initial *strains* of the bars ($\mathbf{t}$). The latter are independent of the stress resultants of the bars.

The equation of state-change of the structure expresses the relationship between the infinitesimal increment of the generalized load and the response

$$\begin{bmatrix} \mathbf{D}(\mathbf{u}, \mathbf{s}) & \mathbf{G}^*(\mathbf{u}) \\ \mathbf{G}(\mathbf{u}) & \mathbf{F}(\mathbf{u}, \mathbf{s}) \end{bmatrix} \cdot \begin{bmatrix} \mathrm{d}\ \mathbf{u} \\ \mathrm{d}\ \mathbf{s} \end{bmatrix} + \begin{bmatrix} \mathrm{d}\ \mathbf{q} \\ \mathrm{d}\ \mathbf{t} \end{bmatrix} = \mathbf{0}$$

$\mathbf{F}(\mathbf{u}, \mathbf{s})$ being the flexibility of the system belonging to the actual position and response, and $\mathbf{D}(\mathbf{u}, \mathbf{s})$ the matrix defined by

$$\mathbf{D}(\mathbf{u}, \mathbf{s}) = [D_{j,k}] = \frac{\partial G_{i,j}}{\partial u_k} s_i = \frac{\partial \mathbf{G}^*(\mathbf{u})}{\partial \mathbf{u}} = \mathbf{s}\,.$$

2. Bars

The structure is composed of a general kind of bars i.e. finite solid bodies with unambiguous geometry and strength characteristics, for instance a common bar (a), a plate (b), a tetrahedron (c) or any other finite element (Fig. 1).

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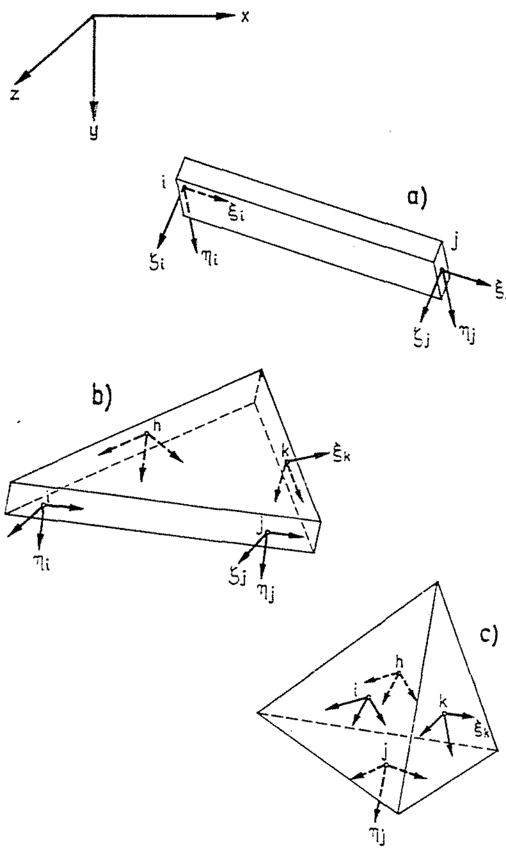

Fig. 1

Some points of the bar or at least one of them, are to be designated as joints, their number being finite. The bar may be connected to other bar(s) merely at the joints. To each joint there is fixed a local frame ξ, η, ζ which may be oriented arbitrarily yet it is advisably directed into the actual principal directions of strain. One of the joints of the bar, e.g. that of the lowest serial number must be distinguished and considered as a point of origin. The local frame (ξ, η, ζ) and the global one (x, y, z) are interconnected by orthogonal transformation. One and the same vector described in the global frame and denoted by $\mathbf{a}(X)$ may be transformed into the local system ξ_i, η_i, ζ_i belonging to the point i, by means of

$$\mathbf{a}(\Xi_i) = \mathbf{T}_{i;0} \cdot \mathbf{a}(X),$$

where

$$\mathbf{T}_{i;0} \cdot \mathbf{T}^*_{i;0} = \mathbf{E}$$

holds. ($\mathbf{E}$ = unit matrix; transposing is indicated by an asterisk).

2.1. *Equilibrium condition of a bar*

The bar shown in Fig. 2 is acted upon by generalized stresses s, consisting of three forces and three couples. The stress $\mathbf{s}_i$, for instance, acting at point i (Fig. 2) is as follows:

$$\mathbf{s}_i^* = [P_{i\xi}\, P_{i\eta}\, P_{i\zeta} \quad M_{i\xi}\, M_{i\eta}\, M_{i\zeta}].$$

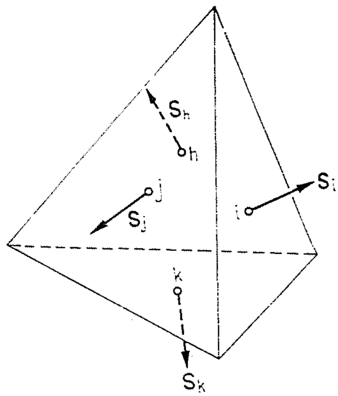

Fig. 2

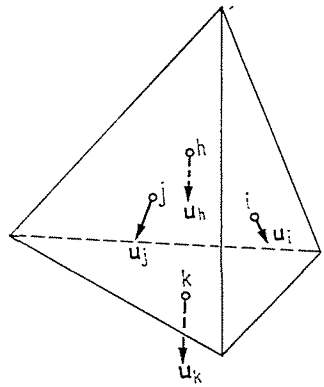

Fig. 3

The bar may be acted upon by forces (e.g. mass-forces) $\mathbf{q}(\Xi_h)$, at arbitrary points besides its joints, described by the local frame belonging to its point of origin, say the point h in Fig. 2:

$$\mathbf{q}(\Xi_h) = \begin{bmatrix} R_x(\Xi_h) \\ R_y(\Xi_h) \\ R_z(\Xi_h) \\ N_x(\Xi_h) \\ N_y(\Xi_h) \\ N_z(\Xi_h) \end{bmatrix}.$$

The bar acted upon by forces at the joints and its other points must be in equilibrium, hence

$$(\mathbf{s}_h, \mathbf{s}_i, \mathbf{s}_j, \mathbf{s}_k, \Sigma\mathbf{q}(\Xi_h)) = 0 \tag{1}$$

(in the case of the bar sketched in Fig. 2).

The symbolic equation (1), expressing the equilibrium condition of the bar, is equivalent to the matrix equation

$$\mathbf{T}_h^*\mathbf{s}_h + \mathbf{A}_{hi}^*\mathbf{s}_i + \mathbf{A}_{hj}^*\mathbf{s}_j + \mathbf{A}_{hk}^*\mathbf{s}_k + \Sigma\mathbf{T}_h^*\mathbf{B}_h^*\mathbf{T}_h\mathbf{q}(\Xi_h) = 0 \tag{2}$$

described in the frame X.

Here

$$\mathbf{T}_h^* = \begin{bmatrix} \mathbf{T}_{h;0}^* & \\ & \mathbf{T}_{h;0}^* \end{bmatrix},$$

$$\mathbf{B}_h^* = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & -\zeta_h & \eta_h & 1 & & \\ \zeta_h & & -\xi_h & & 1 & \\ -\eta_h & \xi_h & & & & 1 \end{bmatrix}.$$

$$\mathbf{A}_{hi}^* = \mathbf{T}_g^* \mathbf{B}_{hi}^* \mathbf{T}_h \mathbf{T}_i^*, \quad \mathbf{B}_{hi}^* = \mathbf{B}_h^* \Bigg|_{\substack{\xi_h = \xi_{hi} \\ \eta_h = \eta_{hi} \\ \zeta_h \equiv \zeta_{hi}}}$$

ξ_{hi}, η_{hi}, ζ_{hi} being the co-ordinates of point i in the co-ordinate syste.. .

2.2. *Relative bar displacements*

In case of rigid body movement, and assuming small displacements, the displacement vectors of the joints of the bar may be related to each other by the simple transformation

$$\mathbf{T}_i \mathbf{u}_i = \mathbf{A}_{hi} \mathbf{u}_h ,$$

where

$$\mathbf{u}_h^* = [V_{ix} V_{iy} V_{iz} \varphi_{ix} \varphi_{iy} \varphi_{iz}] .$$

If the bar undergoes elastic deformations due to forces acting at the joints i, j, k and to other direct forces $\mathbf{q}$ (Ξ_h), furthermore there exist some initial strains $\mathbf{t}_0$ (Ξ_h) along it, the compatibility equations valid for the bar may be assembled into a single matrix equation:

$$\mathbf{G}\,\mathbf{u} + \mathbf{F}\,\mathbf{s} + \mathbf{t} = 0 , \tag{3}$$

where

$$\mathbf{G} = \begin{bmatrix} \mathbf{A}_{hi} & -\mathbf{T}_i & & \\ \mathbf{A}_{hj} & & -\mathbf{T}_j & \\ \mathbf{A}_{hk} & & & -\mathbf{T}_k \end{bmatrix},$$

$$\mathbf{u}^* = [\mathbf{u}_h^* \mathbf{u}_i^* \mathbf{u}_j^* \mathbf{u}_k^*],$$

$$\mathbf{s}^* = [\mathbf{s}_i^* \mathbf{s}_j^* \mathbf{s}_k^*],$$

$$\mathbf{F} = \begin{bmatrix} \mathbf{T}_i\mathbf{T}_h^* & & \\ & \mathbf{T}_j\mathbf{T}_h^* & \\ & & \mathbf{T}_k\mathbf{T}_h^* \end{bmatrix} \cdot \begin{bmatrix} \mathbf{F}_{hii} & \mathbf{F}_{hij} & \mathbf{F}_{hik} \\ \mathbf{F}_{hji} & \mathbf{F}_{hjj} & \mathbf{F}_{hjk} \\ \mathbf{F}_{hki} & \mathbf{F}_{hkj} & \mathbf{F}_{hkk} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{T}_h\mathbf{T}_i^* & & \\ & \mathbf{T}_h\mathbf{T}_j^* & \\ & & \mathbf{T}_h\mathbf{T}_k^* \end{bmatrix},$$

$$\mathbf{t} = \mathbf{t}_0 + \Sigma \mathbf{F}_h \mathbf{T}_h \mathbf{q}(\Xi_h);$$

$$\mathbf{F}_h = \begin{bmatrix} \mathbf{F}_{hi} \\ \mathbf{F}_{hj} \\ \mathbf{F}_{hk} \end{bmatrix}; \qquad \mathbf{t}_0 = \begin{bmatrix} \mathbf{t}_{i;0} \\ \mathbf{t}_{j;0} \\ \mathbf{t}_{k;0} \end{bmatrix},$$

$$\mathbf{t}_{i;0} = [t_{i\xi;0} t_{i\eta;0} t_{i\zeta;0} \vartheta_{i\xi;0} \vartheta_{i\eta;0} \vartheta_{i\zeta;0}],$$

$$\mathbf{F}_{hii} = \mathbf{F}_{hi} \Bigg|_{\substack{\xi_h = \xi_{hi} \\ \eta_h = \eta_{hi} \\ \zeta_h = \zeta_{hi}}} ; \quad \ldots \mathbf{F}_{hjk} = \mathbf{F}_{hj} \Bigg|_{\substack{\xi_h = \xi_{nk} \\ \eta_h = \eta_{nk} \\ \zeta_h = \zeta_{nk}}} ; \quad \ldots,$$

$$\mathbf{F}_{hi} = \begin{bmatrix} F_{hi11} & F_{hi12} & F_{hi16} \\ F_{hi21} & F_{hi22} & F_{hi26} \\ \cdots & \cdots & \cdots \\ F_{hi61} & F_{hi62} & F_{hi66} \end{bmatrix}.$$

$\mathbf{t}_0$ being the vector of the initial strains of the bar, which are independent of the forces. Each column of $\mathbf{F}_{hi}$ contains the displacements of the point i, due to a unit load vector acting at a point ξ_h, η_h, ζ_h of the system Ξ_h.

The elements of $\mathbf{F}_{hi}$ may be computed at the prescribed accuracy by means of energy theorems.

It is especially easy to determine the flexibility matrix **F** of the prismatic bar (Fig. 4) if the shear deformations are neglected.

$$\mathbf{F} = \begin{bmatrix} \frac{l}{EA} & & & & & \\ & \frac{l^3}{3EI_\zeta} & & & & \frac{l^2}{2EI_\xi} \\ & & \frac{l^3}{3EI_\eta} & & -\frac{l^2}{2EI_\eta} & \\ & & & \frac{}{GI_\xi} & & \\ & & -\frac{l^2}{2EI_\eta} & & \frac{l}{EI_\eta} & \\ & \frac{l^2}{2EI_\zeta} & & & & \frac{l}{EI_\zeta} \end{bmatrix}.$$

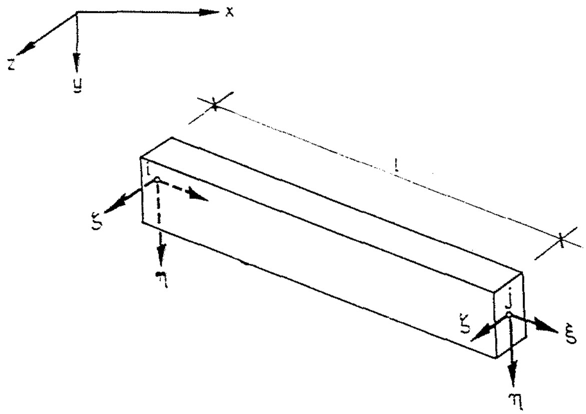

Fig. 4

3. The structure

The structure is built up by the connection of bars. The connection of two bars involves the identity of at least one of the corresponding displacement co-ordinates of at least one of their joints. If all generalized displacements of the interconnected points agree, the joint is a rigid one. The connecting points of the structure are called *nodes*. The dimension of the nodal displacement vector corresponds to the sum of the numbers of the independent displacement co-ordinates belonging to the connecting points of the bars coupled at the node.

For example, the displacement vectors of the bars shown in Fig. 5 are

$$\mathbf{u}_a = \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{bmatrix}; \qquad \mathbf{u}_b = \begin{bmatrix} \mathbf{u}_4 \\ \mathbf{u}_5 \\ \mathbf{u}_6 \\ \mathbf{u}_7 \end{bmatrix}; \qquad \mathbf{u}_c = \begin{bmatrix} \mathbf{u}_8 \\ \mathbf{u}_9 \\ \mathbf{u}_{10} \\ \mathbf{u}_{11} \end{bmatrix}.$$

The conditions

$$\mathbf{u}_1 = \mathbf{u}_4, \quad \mathbf{u}_2 = \mathbf{u}_5 = \mathbf{u}_8, \quad \mathbf{u}_3 = \mathbf{u}_6 = \mathbf{u}_9, \quad \mathbf{u}_7 = \mathbf{u}_{10}$$

mean that three joints of each bar are rigidly connected to one of the nodes so the entity of the nodal displacement vectors of the structure (a, b, c) may be expressed by the vector

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{u}_7 \\ \mathbf{u}_{11} \end{bmatrix} .$$

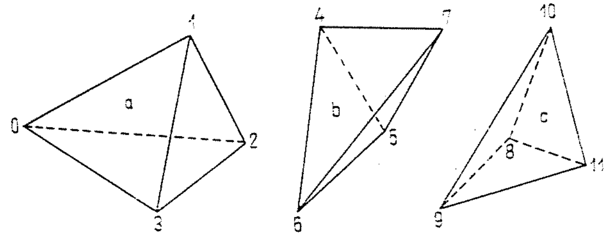

Fig. 5

3.1 Compatibility equation of the structure

The compatibility equation of the structure agrees formally to (3) and what regards to its content, it may be composed from the matrices belonging to the separate bars according to the following rules: The coefficient-hypermatrix **G** of the structure contains the factor-matrices **G** of the separate bars as blocks beneath each other, the columns of the matrices **G** of the bars belonging to common displacement co-ordinates of the joints situated below each other. The flexibility matrix **F** of the structure must be assembled from the flexibility matrices **F** of the bars, while the vectors **s** and **t** of the structure are composed as hypervectors containing the corresponding vectors **s** and **t** of the bars, respectively.

For the case shown in Fig. 5. the compatibility equations of the separate bars are

$$\mathbf{G}_a\mathbf{u}_a + \mathbf{F}_a\mathbf{s}_a + \mathbf{t}_a = 0 ,$$
$$\mathbf{G}_b\mathbf{u}_b + \mathbf{F}_b\mathbf{s}_b + \mathbf{t}_b = 0 ,$$
$$\mathbf{G}_c\mathbf{u}_c + \mathbf{F}_c\mathbf{s}_c + \mathbf{t}_c = 0 .$$

while the equation valid for the structure as a whole:

$$\mathbf{G}\,\mathbf{u} + \mathbf{F}\,\mathbf{s} + \mathbf{t} = 0 ,$$

where

$$\mathbf{G} = \begin{bmatrix} \boxed{\mathbf{G}_a} & & \\ & \boxed{\mathbf{G}_b} & \\ & & \boxed{\mathbf{G}_c} \end{bmatrix} ; \quad \mathbf{F} = \begin{bmatrix} \boxed{\mathbf{F}_a} & & \\ & \boxed{\mathbf{F}_b} & \\ & & \boxed{\mathbf{F}_c} \end{bmatrix} .$$

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{u}_7 \\ \mathbf{u}_{11} \end{bmatrix}; \qquad \mathbf{s} = \begin{bmatrix} \mathbf{s}_a \\ \mathbf{s}_b \\ \mathbf{s}_c \end{bmatrix}; \qquad \mathbf{t} = \begin{bmatrix} \mathbf{t}_a \\ \mathbf{t}_b \\ \mathbf{t}_c \end{bmatrix};$$

$$\mathbf{s}_a = \begin{bmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \\ \mathbf{s}_3 \end{bmatrix}; \qquad \mathbf{s}_b = \begin{bmatrix} \mathbf{s}_5 \\ \mathbf{s}_6 \\ \mathbf{s}_7 \end{bmatrix}$$

provided the conditions of joining quoted above hold.

3.2 Equilibrium equation of the structure

The forces acting at the nodal points of the structure must be balanced in every node by the inverses of the stress resultants acting at the joints of the bars. The entity of the equilibrium equation of the nodes constitutes the equilibrium condition of the structure. It is easy to understand without further instructions that the equilibrium equations of a structure containing a single bar (e.g. that shown in Fig. 2) are:

$$-\mathbf{T}_h^*\mathbf{s}_h + \mathbf{q}_{h;0} = 0$$
$$-\mathbf{T}_i^*\mathbf{s}_i + \mathbf{q}_{i;0} = 0$$
$$-\mathbf{T}_j^*\mathbf{s}_j + \mathbf{q}_{j;0} = 0$$
$$-\mathbf{T}_k^*\mathbf{s}_k + \mathbf{q}_{k;0} = 0.$$

(See also Fig. 6)

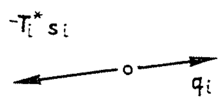

Fig. 6

Thus, by considering (2), the equilibrium equation of the structure may be written in the shorthand form

$$\mathbf{G}^*\mathbf{s} + \mathbf{q} = 0 \tag{4}$$

$\mathbf{q}$ denoting the vector of the reduced nodal loads.

$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_{h;0} + \Sigma\mathbf{T}_h^*\mathbf{B}_h^*\mathbf{T}_h\mathbf{q}(\Xi_h) \\ \mathbf{q}_{i;0} \\ \mathbf{q}_{j;0} \\ \mathbf{q}_{k;0} \end{bmatrix}.$$

The character of the equilibrium equation of a structure composed by joining several bars together, agrees formally with (4), the hypermatrix-factor ($\mathbf{G}^*$) of the equation equals the transpose of the hypermatrix $\mathbf{G}$ of the overall compatibility equation. The order of the vector of the reduced nodal loads is equal to the number of the independent force and couple elements acting at the connected joints.

3.3 *Small displacement state equation of the structure*

Hyperequation

$$\begin{bmatrix} & \mathbf{G}^* \\ \mathbf{G} & \mathbf{F} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{u} \\ \mathbf{s} \end{bmatrix} + \begin{bmatrix} \mathbf{q} \\ \mathbf{t} \end{bmatrix} = 0 \tag{5}$$

constructed by assembling the equilibrium equations of the nodes and the compatibility equations of the bars, is related to a real structure only if it meets prescribed boundary conditions, that is, if some of the nodal displacement co-ordinates are specified.

Let us distinguish a vector containing the prescribed displacement co-ordinates and another one containing the remainder. Denoting the first of these vectors by $\mathbf{u}_p$ and the second one by $\mathbf{u}$ (not identical to the former vector of the displacements), and rearranging the columns of the factor $\mathbf{G}$ simultaneously, the hyperequation of the structure could be written as:

$$\begin{bmatrix} & & \mathbf{G}_p^* \\ & & \mathbf{G}^* \\ \mathbf{G}_p & \mathbf{G} & \mathbf{F} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{u}_p \\ \mathbf{u} \\ \mathbf{s} \end{bmatrix} + \begin{bmatrix} \mathbf{q}_p \\ \mathbf{q} \\ \mathbf{t} \end{bmatrix} = 0.$$

Taking in mind, however, that the vector $\mathbf{G}_p\mathbf{u}_p$ is a straightforward computable one, i.e. practically of the same kind as $\mathbf{t}$, and in what follows, denoting $\mathbf{t} + \mathbf{G}_p\mathbf{u}_p$ by $\mathbf{t}$, the equation of the structure agrees once again with (5), apart from the fact that a separate equation

$$\mathbf{q}_p = -\mathbf{G}_p^*\mathbf{s} \tag{6}$$

arises. The latter serves to compute the nodal forces corresponding to the character of the prescribed displacement co-ordinates, i.e. to the determination of the reactions.

3.4 *Static and kinematic description of the structure*

Eq. (5) of the structure involves the joining specifications, and the boundary conditions, the equilibrium equations of both the bars and the nodes, as well as the compatibility equations of the bars in a concise form. It is valid for the range of small displacements i.e. for any case where the relationship between the *loads*

$$\mathbf{b} = \begin{bmatrix} \mathbf{q} \\ \mathbf{t} \end{bmatrix}$$

and the *response*

$$\mathbf{x} = \begin{bmatrix} \mathbf{u} \\ \mathbf{s} \end{bmatrix}$$

of the structure yields a correct result within the specified limit of error. The hypermatrix-coefficient of the response vector Eq. (5) is symmetric provided $\mathbf{F} = \mathbf{F}^*$. In consequence of the heuristic choice of the reference frames, the

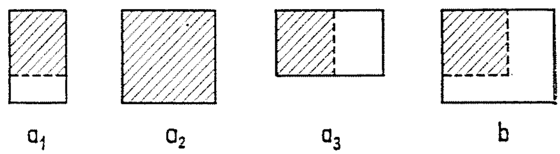

Fig. 7

symmetry may be recognized easily. It is also easy to understand that the static and kinematic features of the structure depend merely on the rank of matrix **G**, that is, on the number

$$\varrho = \varrho(\mathbf{G}).$$

Considering a matrix **G** consisting of m rows and n columns, there are two possibilities.

a) If $\underset{m,n}{\varrho}(\mathbf{G}) = \text{Min} \cdot (m, n)$

then

I.	for $m > n$	the structure is	hyperstatic
II.	$m = n$		statically and kinematically determinate
III.	$m < n$		hyperkinematic

b) $\underset{m,n}{\varrho}(\mathbf{G}) < \text{Min}\,(m, n)$

holds and the structure is both hyperstatic and hyperkinematic. Non-singular minors separable from matrices **G** for each of the cases are shown in Fig. 7.

3.5 *Solution methods*

In the range of small displacements (primary theory), procedures of general validity can only be given for cases a/I and a/II in the previous item. A general presentation of the procedures may be read in [1]. Solution methods of case a/II (hyperstatic structure) can be interpreted by partitioning the hypermatrix coefficient in Eq. (5). Two main groups of the more familiar procedures involve partitioning of Eq. (5) as follows (representing each block on the scale)

$$\left[\begin{array}{c|c} & \mathbf{G}^* \\ \hline \mathbf{G} & \mathbf{F} \end{array}\right] \cdot \left[\begin{array}{c} \mathbf{u} \\ \hline \mathbf{s} \end{array}\right] + \left[\begin{array}{c} \mathbf{q} \\ \hline \mathbf{t} \end{array}\right] = 0\,; \quad \text{original equation}$$

$$\left[\begin{array}{c|c|c} & \mathbf{G}_k^* & \mathbf{G}_s^* \\ \hline \mathbf{G}_k & \mathbf{F}_k & \\ \hline \mathbf{G}_s & & \mathbf{F}_s \end{array}\right] \cdot \left[\begin{array}{c} \mathbf{u} \\ \hline \mathbf{s}_k \\ \hline \mathbf{s}_s \end{array}\right] + \left[\begin{array}{c} \mathbf{q} \\ \hline \mathbf{t}_k \\ \hline \mathbf{t}_s \end{array}\right] = 0\,; \quad \text{force method}$$

$$\left[\begin{array}{cc|c} & & \mathbf{G}_k^* \\ \hline & & \mathbf{G}_s^* \\ \hline \mathbf{G}_k & \mathbf{G}_s & \mathbf{F} \end{array}\right] \cdot \left[\begin{array}{c} \mathbf{u}_k \\ \hline \mathbf{u}_s \\ \hline \mathbf{s} \end{array}\right] + \left[\begin{array}{c} \mathbf{q}_k \\ \hline \mathbf{q}_s \\ \hline \mathbf{t} \end{array}\right] = 0\,; \quad \text{displacement method}$$

subscript s indicating stresses viz. nodal displacements to be treated as free parameters in the procedure that have to meet compatibility conditions defined by the appropriate part of the hyperequation.

4. Equations for large displacements

Eq. (5) is valid for small displacements where it is tacitly taken for granted that during the state change of the structure the matrix **G** involving position parameters may be considered invariable. In case of large displacements, Eq. (5) is only valid to an infinitesimal change of state and even then with the following supplement.

If loads **q** acting at the nodal points cause stresses **s** and nodal displacements **u** (related to a certain initial position), the equilibrium equation may be written as

$$\mathbf{G}^*(\mathbf{u})\mathbf{s} + \mathbf{q} = 0$$

emphasising $\mathbf{G}^*$ as a function of the displacements.

Postulating that the infinitesimal increase of the load causes just an infinitesimal increment of the response, and referring to (5), a simultaneous system of equilibrium and compatibility equations

$$\mathbf{G}^*(\mathbf{u} + d\mathbf{u}) \cdot (\mathbf{s} + d\mathbf{s}) + \mathbf{q} + d\mathbf{q} = 0 ,$$
$$\mathbf{G}(\mathbf{u}) \cdot d\mathbf{u} + \mathbf{F}(\mathbf{u}, \mathbf{s}) \cdot d\mathbf{s} + d\mathbf{t} = 0$$

holds. Assuming matrix $\mathbf{G}$ to be a continuous function possessing derivatives with respect to $\mathbf{u}$, and neglecting terms small of the second order, on the ground of this latter system the *differential equation of the state change* of the structure was developed. It belongs to the gross-deflection theory (tertiary theory), formulated as

$$\begin{bmatrix} \mathbf{D}(\mathbf{u}, \mathbf{s}) & \mathbf{G}^*(\mathbf{u}) \\ \mathbf{G}(\mathbf{u}) & \mathbf{F}(\mathbf{u}, \mathbf{s}) \end{bmatrix} \cdot \begin{bmatrix} d\mathbf{u} \\ d\mathbf{s} \end{bmatrix} + \begin{bmatrix} d\mathbf{q} \\ d\mathbf{t} \end{bmatrix} = 0 , \tag{6}$$

where the elements of the matrix $\mathbf{D}(\mathbf{u}, \mathbf{s})$ are defined by the tensor

$$D_{j,k}(\mathbf{u}, \mathbf{s}) = \frac{\partial G_{i,j}(\mathbf{u})}{\partial u_k} s_i .$$

Assuming

$$\mathbf{G} = \text{const.}, \quad \text{i.e.} \quad \mathbf{D} = (\mathbf{u}, \mathbf{s}) = 0 \quad \text{and} \quad \mathbf{F}(\mathbf{u}, \mathbf{s}) = \text{const.},$$

Eq. (6) involves Eq. (5) as a special case.

Eq. (6) is strictly valid only to infinitesimal increments nevertheless it is over and over applied to a fair approximate determination of the change in the finite surrounding of a prescribed initial condition.

Equation

$$\begin{bmatrix} \mathbf{D}(\mathbf{u}, \mathbf{s}) & \mathbf{G}^*(\mathbf{u}) \\ \mathbf{G}(\mathbf{u}) & \mathbf{F}(\mathbf{u}, \mathbf{s}) \end{bmatrix} \cdot \begin{bmatrix} \Delta\mathbf{u} \\ \Delta\mathbf{s} \end{bmatrix} + \begin{bmatrix} \Delta\mathbf{q} \\ \Delta\mathbf{t} \end{bmatrix} = 0 \tag{7}$$

describing this approximation is a state change equation interpreted according to the so-called secondary theory.

The importance of Eq. (7) is increased by its applicability to the iterative determination of state change due to large displacements at the desired accuracy. Namely, if validity conditions of Eq. (6) are met, then it is always possible to apply the specified load $\mathbf{q}, \mathbf{t}$ in steps $\Delta\mathbf{q}, \Delta\mathbf{t}$ to the structure so that the load compatible with the condition defined by (7) differs by less than $\Delta\mathbf{q}, \Delta\mathbf{t}$ from load $\mathbf{q} + \Delta\mathbf{q}, \mathbf{t} + \Delta\mathbf{t}$.

5. Applications

Application possibilities of the state change equation are illustrated in [2]. Let us present altogether the following two examples:

5.1. *Large displacements of a rigid chain mechanism*

The plane chain mechanism traced with a continuous thin line in Fig. 8 is acted upon by a single concentrated load 2 H. The bars are supposed to be infinitely rigid, i.e. **F** = 0. Nodal displacements due to the load are sought for.

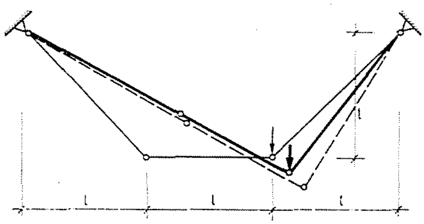

Fig. 8

In conformity with data of the initial position:

$$\mathbf{G} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & & \\ 1 & & -1 & \\ & & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$\mathbf{u} = 0$, and $\mathbf{s}$ can be chosen arbitrarily.
Assume e.g.

$$\mathbf{s} = H \begin{bmatrix} \sqrt{2} \\ 1 \\ \sqrt{2} \end{bmatrix}.$$

The load compatible with $\mathbf{s}$ may be computed as

$$\mathbf{q} = -\mathbf{G}^*\mathbf{s} = H \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}; \qquad \mathbf{t} = 0 .$$

Matrix $\mathbf{D}(\mathbf{u}, \mathbf{s})$ belonging to the initial position and condition is:

$$\mathbf{D} = \frac{H}{2l} \begin{bmatrix} -1 & 1 & & \\ 1 & -3 & & 2 \\ & & -1 & -1 \\ & 2 & -1 & -3 \end{bmatrix}.$$

Since the specified load

$$\mathbf{q} = \mathrm{H} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}; \qquad \mathbf{t} = 0$$

and the load compatible with the initial condition differ by

$$\Delta \mathbf{q} = H \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}; \qquad \mathbf{t} = 0$$

this increment will be used to solve Eq. (7):

$$\Delta \mathbf{u} = \frac{l}{4} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}; \qquad \Delta \mathbf{s} = \frac{H\sqrt{2}}{4} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

The displacement of the chain mechanism obtained by this approximation is indicated by dashed line.

Let us consider the position obtained by this first approximation as initial position, being again $\mathbf{u} = 0$ and choose $\mathbf{s}$ arbitrarily once more. Be e.g. the horizontal component (in direction x) of the stress equal to H, hence:

$$\mathbf{s} = H \begin{bmatrix} 1.1662 \\ 1.1180 \\ 1.9436 \end{bmatrix}.$$

In the new position

$$\mathbf{G} = \begin{bmatrix} -0.8575 & -0.5145 & & \\ 0.8944 & 0.4472 & -0.8944 & -0.4472 \\ & & -0.5145 & -0.8575 \end{bmatrix}$$

and

$$\mathbf{D} = \frac{H}{l} \begin{bmatrix} -0.4118 & 0.7529 & 0.2 & -0.4 \\ 0.7529 & -1.3882 & -0.4 & 0.8 \\ 0.2 & -0.4 & -1.1804 & -0.1882 \\ -0.4 & 0.8 & -0.1882 & -1.1529 \end{bmatrix}.$$

The load compatible with $\mathbf{s}$ amounts to:

$$\mathbf{q} = H \begin{bmatrix} 0 \\ 0.1 \\ 0 \\ 2.1667 \end{bmatrix}; \qquad \mathbf{t} = \begin{bmatrix} 0.0430 \\ 0.1180 \\ 0.0435 \end{bmatrix}$$

($\mathbf{t}$ is obtained as the difference between the bar lengths calculated from the position and the specified bar lengths.)

The prescribed load and the load compatible with the initial condition differ by:

$$\Delta \mathbf{q} = H \begin{bmatrix} 0 \\ -0.1 \\ 0 \\ -0.1667 \end{bmatrix}; \qquad \Delta \mathbf{t} = \begin{bmatrix} -0.0430 \\ -0.1180 \\ -0.0435 \end{bmatrix}.$$

Making use of this increment in solving Eq. (7):

$$\Delta \mathbf{u} = l \begin{bmatrix} 0.0068 \\ -0.0959 \\ -0.1136 \\ -0.1190 \end{bmatrix}; \qquad \Delta \mathbf{s} = H \begin{bmatrix} 0.1601 \\ 0.2076 \\ -0.0171 \end{bmatrix}.$$

Displacements in the state from the second approximation are indicated by continuous thick line. Bar lengths corresponding to this position agree with the specified bar lengths to three decimals. Stress vector for this state is:

$$\mathbf{s} + \Delta\mathbf{s} = H \begin{bmatrix} 1.3263 \\ 1.3256 \\ 1.9265 \end{bmatrix}.$$

As shown by the simple example outlined above, this procedure is rapidly convergent even for large displacements.

5.2 Stability of the change of state

Restricting ourselves to one-parameter loads, it is assumed that in the load vector

$$\mathbf{b} = \begin{bmatrix} \mathbf{q} \\ \mathbf{t} \end{bmatrix}$$

$\mathbf{t} = 0$ and $\mathbf{q} = \mathbf{R} \cdot \mathbf{f}(\mathbf{u})$

$$\| \mathbf{f}(\mathbf{u}) \| = \text{constant},$$

that is, vector of nodal forces may be produced by multiplication of vector $\mathbf{f}$ of constant norm by scalar R. Initial strain is omitted. For such loads, the response vector

$$\mathbf{x} = \begin{bmatrix} \mathbf{u} \\ \mathbf{s} \end{bmatrix}$$

defines a spatial curve (Fig. 9) depending on scalar R. Change of state is considered to be stable as long as to each value of monotonously increasing

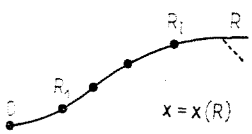

Fig. 9

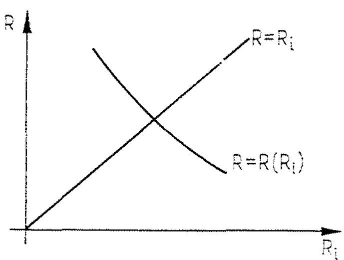

Fig. 10

scalar parameter R an unambiguously determined point of the spatial curve $\mathbf{x} = \mathbf{x}(R)$ belongs. Points of the stable interval of the spatial curve can be determined at the desired accuracy in an arbitrary density according to the secondary theory — i.e. Eq. (7). Spatial curve point $\mathbf{x} = \mathbf{x}(R)$ belonging to parameter R, directly adjacent to a point of response $\mathbf{x} + d\mathbf{x}$ belonging to the same parameter R, is termed *branching point* of the spatial curve — i.e. of the change of state. If the point belonging to parameter value R_i within the small but finite surrounding of the branching point -- hence, the corresponding response characteristics $\mathbf{u}_i$, $\mathbf{s}_i$ — are known and accordingly, the characteristics belonging to parameter value $R = R_i + \Delta R_i$ — hence, belonging to the branching point — are indicated by

$$\mathbf{u} = \mathbf{u}_i + \Delta \mathbf{u}_i$$

$$\mathbf{s} = \mathbf{s}_i + \Delta \mathbf{s}_i$$

then equilibrium and compatibility equations for the branching point and its surrounding can be written as:

$$\mathbf{G}^* (\mathbf{u}_i) \cdot \mathbf{s}_i + R_i \cdot \mathbf{f} (\mathbf{u}_i) = 0$$

$$\mathbf{G}^* (\mathbf{u}) \cdot \mathbf{s} + R\mathbf{f} (\mathbf{u}) = 0$$

$$\mathbf{G}^* (\mathbf{u} + d\mathbf{u}) \cdot (\mathbf{s} + d\mathbf{s}) + R \cdot \mathbf{f} (\mathbf{u} + d\mathbf{u}) = 0$$

$$\mathbf{G} \ (\mathbf{u}_i) \cdot \Delta \mathbf{u}_i + \mathbf{F} \cdot \Delta \mathbf{s}_i = 0$$

$$\mathbf{G} \ (\mathbf{u}_i) \cdot (\Delta \mathbf{u}_i + d\mathbf{u}) + \mathbf{F} \cdot (\Delta \mathbf{s}_i + d\mathbf{s}) = 0 \, .$$

These five equations — allowing the approximation according to the secondary theory for interval ΔR_i — lead to the homogeneous equation

$$(\mathbf{D}_1 + R_i \mathbf{D}_2 + R\mathbf{D}_3) \cdot d\mathbf{u} = 0 \, , \tag{8}$$

an eigenvalue problem related to the scalar parameter R determining the branching point. From the equation it is apparent that the root of the eigenvalue problem is the function of the parameter value R_i of a point chosen in the stable section of the spatial curve (Fig. 10). Thus, exact computation of parameter value R belonging to the branching point requires the determination of the intersection point of the planar curve $R = R(R_i)$ and the straight line $R = R_i$.

Examples on the analysis of the state change stability have been developed in [2].

Summary

Relationship between responses and loads of a structure composed of generally interpreted elastic or rigid bars is described by a linear differential equation of variable hypermatrix coefficient. In case of suitably chosen reference co-ordinate systems, the hypermatrix coefficient is symmetrical and composed of blocks corresponding to the bars, by making use of the bar connection data, involving also boundary conditions specified for the structure. The rank of the geometry block of the coefficient indicates unambiguously the static and kinematic prop-

erties of the structure. Solution methods for hyperstatic structures can be interpreted by partitioning the coefficient hypermatrix. The state-change equation offers the large deflection theory of the structural analysis, involving the primary and secondary theories as special cases, and enable to deduce directly the eigenvalue problem of the branching phenomenon of the structure belonging to a one-parameter load.

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