STRENGTH ANALYSIS OF THICK-WALLED SPHERES IN SPHEROID SYMMETRICAL THERMAL STATE

By

A. SZEKERES

Department of Engineering Mechanics, Faculty of Mechanical Engineering, Technical University, Budapest

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Presented by Prof. Dr. Gy. BÉDA

The strength conditions of a thick-walled sphere are investigated in case of stationary thermal conduction. According to the spheroid symmetry (neglecting the inertia force), the equilibrium equation is expressed in spherical coordinates [1]:

$$\frac{d\,(r^2\sigma_r)}{dr} - 2r\sigma_\Theta = 0\,. \tag{1}$$

From HOOKE's general law:

$$\begin{aligned} \varepsilon_r &= \frac{1}{E}(\sigma_r - 2\,\mu\sigma_\Theta) + \alpha\,t\,(r), \\ \varepsilon_\Theta &= \frac{1}{E}\left[(1-\mu)\,\sigma_\Theta - \mu\sigma_r\right] + \alpha t\,(r); \end{aligned} \tag{2}$$

where α is the coefficient of thermal expansion, $t(r)$ is the relative thermal gradient along the radius. Due to the spheroid symmetry, each point will be radially displaced. The relationship between strains and the radial displacement is:

$$\begin{aligned} \varepsilon_r &= \frac{du}{dr}, \\ \varepsilon_\Theta &= \frac{u}{r}. \end{aligned} \tag{3}$$

Accordingly, the strains fulfill the equation of compatibility:

$$\frac{d}{dr}(r\varepsilon_\Theta) - \varepsilon_r = 0\,. \tag{3'}$$

From the equation systems (1), (2) and (3), the functions $\sigma_r(r)$, $\sigma_\Theta(r)$, $\varepsilon_r(r)$, $\varepsilon_\Theta(r)$ and $u(r)$ can be determined. From (2) and (3') we obtain by substitution:

$$(1-\mu)\frac{d(r\sigma_\Theta)}{dr}-\mu\frac{d(r\sigma_r)}{dr}+\alpha_1\frac{d(rt)}{dr}-\sigma_r+2\mu\sigma_\Theta-\alpha_1 t=0, \qquad (4)$$

$$\alpha_1=E\alpha\,.$$

Eliminating σ_Θ from (1) and (4) and introducing the symbol $y=r^2\sigma_r$ and arranging yields:

$$\frac{d^2y}{dr^2}-2\frac{y}{r^2}+\frac{2\alpha_1}{1-\mu}\frac{d(rt)}{dr}-\frac{2\alpha_1}{1-\mu}t=0. \qquad (5)$$

As for a stationary thermal conduction state [2]:

$$t=\frac{\Delta tr_k}{r_k-r_b}\frac{r-r_b}{r},$$

where r_b and r_k are internal and external radii of the sphere, resp., and $\Delta t = T(r_k)-T(r_b)$; (5) may be replaced by:

$$\frac{d^2y}{dr^2}-2\frac{y}{r^2}+\frac{2\alpha_1}{1-\mu}\Delta t\varrho r_b\frac{1}{r}=0, \qquad \varrho=\frac{r_k}{r_k-r_b}. \qquad (6)$$

The solution of this differential equation:

$$y=C_1r^2+\frac{C_2}{r}+\frac{C}{2}r \qquad (7)$$

thus

$$\sigma_r(r)=C_1+\frac{C_2}{r^3}+\frac{C}{2}\frac{1}{r},$$
$$\sigma_\Theta(r)=C_1-\frac{C_2}{2r^3}+\frac{C}{4r}. \qquad (8)$$

The integration constants follow from the boundary conditions $\sigma_r(r_b) = \sigma_r(r_k)=0$

$$C_2=-\frac{C}{2}r_k^2r_b^2\varrho_1,$$

$$C_1=\frac{C}{2r_k}[r_b^2\varrho_1-1],$$

where

$$\varrho_1=\frac{r_k-r_b}{r_k^3-r_b^3}, \qquad C=\frac{2\alpha_1}{1-\mu}\Delta t\varrho r_b.$$

The stresses:

$$\sigma_r(r) = \frac{C}{2\,r_k}\left[r_b^2\varrho_1 - 1\right] - \frac{C}{2} r_k^2 r_b^2 \varrho_1 \frac{1}{r^3} + \frac{C}{2}\frac{1}{r},$$
$$\sigma_\Theta(r) = \frac{C}{2\,r_k}\left[r_b^2\varrho_1 - 1\right] + \frac{C}{4} r_k^2 r_b^2 \varrho_1 \frac{1}{r^3} + \frac{C}{4}\frac{1}{r}; \qquad (9)$$

the strains:

$$\varepsilon_r(r) = \frac{C}{2\,E}\left[(1-2\mu)\frac{r_b^2\varrho_1 - 1}{r_k} - (1+\mu) r_k^2 r_b^2 \varrho_1 \frac{1}{r^3} + (1-\mu)\frac{1}{r}\right] + \alpha t(r),$$
$$\varepsilon_\Theta(r) = \frac{C}{2\,E}\left[(1-2\mu)\frac{r_b^2\varrho_1 - 1}{r_k} + r_k^2 r_b^2 \varrho_1 \frac{1+\mu}{2}\frac{1}{r^3} + \frac{1-3\mu}{2}\frac{1}{r}\right] + \alpha t(r); \qquad (10)$$

and the displacement:

$$u(r) = \frac{C}{2\,E}\left[(1-2\mu)\frac{r_b^2\varrho_1 - 1}{r_k} r + r_k^2 r_b^2 \varrho_1 \frac{1+\mu}{2}\frac{1}{r^2} + \frac{1-3\mu}{2}\right] + \alpha r t(r). \qquad (11)$$

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The application of previous relationships can be illustrated by the following example. A hydroglobe (water container) can be modelled as a sphere. At low temperature there is a risk of freezing so that the container deteriorates.

Let us take the model in Fig. 1. Suppose the initial condition that the container is filled up with water at 0 °C. Then, an ice layer at 0 °C, of thickness v_j is formed, so that no stresses develop, as the filled-up condition permits a cubic expansion. Subsequently the ice layer is cooled and — according to the supposed stationary state — there is a temperature distribution such that stresses develop. These stresses can be calculated in two steps. Let us disregard for a moment of the steel container and suppose that the ice may distort unconfined. According to the negative coefficient of thermal expansion concomitant to cooling, an increase in diameter will come about, causing stresses. Imposing now the steel container on this ice layer of increased diameter, a

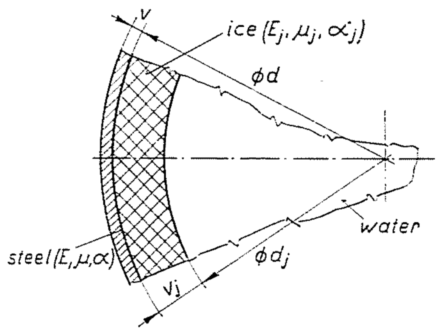

Fig. 1

compressive stress will develop between the two surfaces. The state of stress is a resultant of the two states. In the calculations, the contraction of the container wall is omitted, resulting in a 1 or 2 per cent error.

From (11) the change in the ice-thickness v_j:

$$\Delta v_j = u\left(\frac{d}{2}\right) = \frac{C_j}{2E_j}\left[(1-2\mu_j)\left(\frac{d_j^2}{4}\varrho_{1j}-1\right)+\frac{d_j^2}{4}\varrho_{1j}\frac{1+\mu_j}{2}+\frac{1-3\mu_j}{2}\right]+\alpha_j\frac{d}{2}\Delta t. \tag{12}$$

From this overlapping the thin-walled steel container takes up [3]:

$$w_1 = p\frac{d^2}{8vE}(1-\mu) = pw_1'$$

and the ice sphere:

$$w_2 = -p\frac{d}{2E_j}\left\{\frac{\left(\frac{d_j}{d}\right)^3+2}{2\left[1-\left(\frac{d_j}{d}\right)^3\right]}(1-\mu_j)-\mu_j\right\} = -pw_2'. \tag{14}$$

The pressure evolving between the two spheres is:

$$p = \frac{\Delta v_j}{w_1'-w_2'}. \tag{15}$$

From the above the resultant stresses can be derived.

According to calculations carried out with the above relationships, the ice sphere is in an ultimate stress state at a v_j value where no ultimate stresses develop as yet in the steel container. This method has been checked by ultrasonic and strain gauge tests.

Summary

The problem concerned the strength of thick-walled spheres in stationary thermal conduction state. The resulting displacements and stress functions have been applied to determine the potential behaviour of the hydroglobe (water container) during freezing. The calculation results were supported by tests.

References

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Dr. András Szekeres, 1502. Budapest, P. O. B. 91. Hungary