CAPILLARY-GRAVITY WAVES IN A THIN, NON-NEWTONIAN FILM ON A FLAT PLATE*

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Introduction

There are many chemical technological processes involved in the stratified flow of a thin liquid layer on a great, flat plate. This phenomenon had first been investigated by an experimental method.

GRIMLEY [1] considered the critical Reynolds number v.e. where waves appear. For various values of the Reynolds number, three intervals can be distinguished depending on the mean velocity and the mean film thickness.

For $\mathrm{Re} < 30$ there is a thin film flow with constant-thickness layer.

In the interval $30 < \mathrm{Re} < 60$ capillary-gravity waves occur on the surface of the film. The phase velocity to mean velocity relationship and the value of the critical Reynolds number for such an apparent transition were derived by BINNIE [2], KIRKBRIDGE [3], and FRIEDMANN and MILLER [4].

For $\mathrm{Re} > 1500$ the flow of the film becomes turbulent [5].

The first analytical investigation of the film flow was made by JEFFREYS [6].

LANDAU and LEWICH have taken into consideration the effect of the capillary forces too, in investigating the film flow on the surface of a rigid body taken out of a stratified, static fluid [7].

In an analysis by KAPITSA [8] the interval $30 < \mathrm{Re} < 60$ has been investigated in case of a Newtonian fluid.

The present work is aiming at extending KAPITSA's analysis to purely viscous, non-Newtonian fluids.

Formulation of the problem

Let us consider an infinitely long flat plate. A Cartesian co-ordinate system $Oxyz$ is chosen with its origin in the plate, and the y-axis perpendicular to it. A thin liquid layer flows on the plane xz, in the direction of the x-axis.

Our assumptions are as follows:

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1. The flow is laminar.
2. The fluid is a purely viscous, incompressible non-Newtonian liquid.
3. The shear stress — shear rate relationship is of the "power-law"-type.
4. The flow is quasi-one-dimensional:

$$v_x \gg v_y; \qquad v_z = 0.$$

5. Although $v_x \gg v_y$, $v_x\, \partial v_x/\partial x$ and $v_y\, \partial v_x/\partial y$ are of similar order, because the film is very thin.

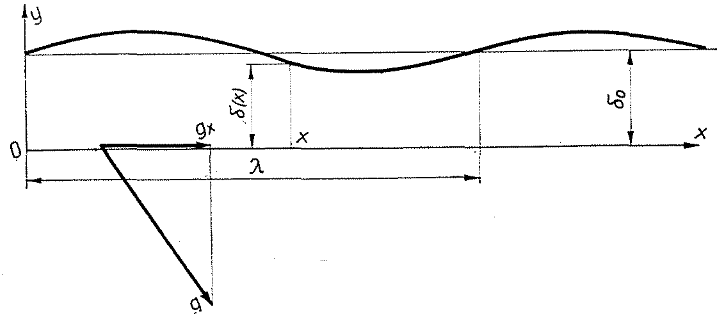

Fig. 1

6. The wave-length is greater than the mean thickness of the film:

$$\lambda \gg \delta_0.$$

7. The waves are not decaying. The decrease of the potential energy of the film is equal to the dissipation.
8. At the surface of the film the pressure equals the capillary pressure:

$$P_{y=\delta} = -\sigma \frac{d^2 \delta}{dx^2}.$$

9. Inside the film the pressure is not varying in y direction:

$$\frac{\partial p}{\partial y} = 0.$$

Consequently, the equations of motion and continuity are the following:

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = gx - \frac{1}{\varrho} \frac{\partial p_\delta}{\partial x} + \frac{\eta_0}{\varrho} \frac{\partial}{\partial y} \left(\frac{\partial v_x}{\partial y} \right)^n \tag{1}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \qquad 2)$$

where p_s is the capillary pressure;

η_0 is the "apparent" viscosity;
n is the flow-behavior index.

The total differential of v_y is:

$$dv_y = \frac{\partial v_y}{\partial x}\,dx + \frac{\partial v_y}{\partial y}\,dy. \qquad (3)$$

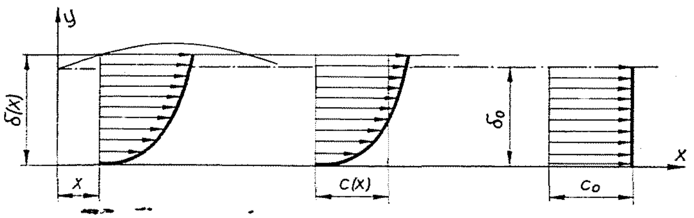

Fig. 2

Owing to the very small film-thickness

$$\frac{\partial v_y}{\partial x} \ll \frac{\partial v_y}{\partial y}$$

thus

$$dv_y = \frac{\partial v_y}{\partial y}\,dy.$$

When continuity is applied:

$$v_y = -\int_0^y \frac{\partial v_x}{\partial x}\,dy. \qquad (4)$$

The equation of motion in such a way is

$$\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} - \left(\int_0^y \frac{\partial v_x}{\partial x}\,dy\right)\frac{\partial v_x}{\partial y} = g_x + \frac{\sigma}{\varrho}\,\frac{d^3\delta}{dx^3} + \frac{\eta_0}{\varrho}\,\frac{\partial}{\partial_y}\left(\frac{\partial v_x}{\partial y}\right)^n \qquad (5)$$

where σ is the surface tension.

Now we assume that the distribution of v_x equals the velocity distribution of a laminar, parallel stratified flow, which is of the same thickness as the wave-surface film at the given place (Fig. 2).

In the latter case:

$$v_x = \frac{2n+1}{n+1}\left[1 - \left(1 - \frac{y}{\delta}\right)^{\frac{n+1}{n}}\right]c \qquad (6)$$

where c is the mean velocity of the constant-thickness layer:

$$c = \frac{1}{\delta} \int_0^\delta v_x \, dy . \tag{7}$$

Solution

Substituting (6) into (5) the equation of motion can be written with the mean velocity:

$$\frac{2n+1}{n+1}\left[1-\left(1-\frac{y}{\delta}\right)^{\frac{n+1}{n}}\right]\frac{\partial c}{\partial t}+\left(\frac{2n+1}{n+1}\right)^2\left[1-\left(1-\frac{y}{\delta}\right)^{\frac{n+1}{n}}\right]c\,\frac{\partial c}{\partial x}-$$

$$-\frac{(2n+1)^2}{n(n+1)}\left(1-\frac{y}{\delta}\right)^{\frac{1}{n}}\left[\frac{y}{\delta}+\frac{n}{2n+1}\left(1-\frac{y}{\delta}\right)^{\frac{2n+1}{n}}-\frac{n}{2n+1}\right]c\,\frac{\partial c}{\partial x}=$$

$$=g_x+\frac{\sigma}{\varrho}\,\frac{d^3\delta}{dx^3}-\frac{\eta_0}{\varrho}\left(\frac{2n+1}{n}\right)^n\frac{c^n}{\delta^{n+1}} . \tag{8}$$

Averaging Eq. (8) in y-direction:

$$\frac{\partial c}{\partial t}+\left(\frac{2n+1}{n+1}\right)^2\frac{2n^2+4n+2}{6n^2+7n+2}\,c\,\frac{\partial c}{\partial x}-\left(\frac{2n+1}{n+1}\right)^2\frac{n(n+1)}{6n^2+7n+2}\,c\,\frac{\partial c}{\partial x}=$$

$$=g_x+\frac{\sigma}{\varrho}\,\frac{d^3\delta}{dx^3}-\left(\frac{2n+1}{n}\right)^n\frac{\eta_0}{\varrho}\,\frac{c^n}{\delta^{n+1}} . \tag{9}$$

After simplifications:

$$\frac{\partial c}{\partial t}+\left(\frac{2n+1}{n+1}\right)^2\frac{n^2+3n+2}{6n^2+7n+2}\,c\,\frac{\partial c}{\partial x}=g_x+\frac{\sigma}{\varrho}\,\frac{d^3\delta}{dx^3}-$$

$$-\left(\frac{2n+1}{n}\right)^n\frac{\eta_0}{\varrho}\,\frac{c^n}{\delta^{n+1}} . \tag{10}$$

The component v_y somewhere on the free surface is the following:

$$v_y=\frac{\partial\delta}{\partial t}+v_x\frac{\partial\delta}{\partial x} .$$

Using (4):

$$\frac{\partial\delta}{\partial t}=-\int_0^\delta\frac{\partial v_x}{\partial x}\,dy-v_x\frac{\partial\delta}{\partial x} .$$

After transformation, reverse the order of derivation and integration:

$$\frac{\partial \delta}{\partial t} = -\delta \frac{\partial}{\partial x}\left(\frac{1}{\delta}\int_0^\delta v_x\, dy\right) - v_x \frac{\partial \delta}{\partial x}.$$

Using the approximation $v_x = c$ in the second term:

$$\frac{\partial \delta}{\partial t} = -\frac{\partial}{\partial x}(c\delta). \tag{11}$$

Small-amplitude waves being concerned, the equation of free surface can be written as:

$$\delta = \delta_0(1+\varphi). \tag{12}$$

The waves are assumed non-decaying as confirmed by the experimental results in region $30 < \mathrm{Re} < 60$. Then φ and c are periodic functions of the variable (x-at). Using the identity of time and space periodicity, the time derivative can be traced back to space derivative:

$$\frac{\partial \delta}{\partial t} = -a\delta_0 \frac{\partial \varphi}{\partial x} \tag{13}$$

and

$$\frac{\partial c}{\partial t} = -a\frac{\partial c}{\partial x}. \tag{14}$$

In respectively (13) and (14), a is the phase velocity of the waves. Then the equation of motion is:

$$-a\frac{\partial c}{\partial x} + \alpha c\frac{\partial c}{\partial x} = g_x + \frac{\delta_0 \sigma}{\varrho}\frac{\partial^3 \varphi}{\partial x^3} - \frac{\eta_0}{\varrho}\beta\frac{c^n}{\delta_0^{n+1}(1+\varphi)^{n+1}} \tag{15}$$

where α and β are constants, depending on the behaviour index and viscosity.

$$\alpha = \left(\frac{2n+1}{n+1}\right)^2 \frac{n^2+3n+2}{6n^2+7n+2} \tag{16}$$

$$\beta = \left(\frac{2n+1}{n}\right)^n. \tag{17}$$

Using (11), (12) and (13)

$$-\delta_0 a \frac{\partial \varphi}{\partial x} = -\frac{\partial}{\partial x}[\delta_0(1+\varphi)c] \tag{18}$$

and

$$\frac{\partial}{\partial x}[a\,\delta_0(1+\varphi) - c\delta_0(1+\varphi)] = 0 \tag{19}$$

hence:

$$\frac{\partial}{\partial x}\left[(a-c)\delta_0(1+\varphi)\right]=0\,. \tag{20}$$

After integration:

$$\delta_0\,(a-c_0)(1+\varphi)=\delta_0\,(a-c_0) \tag{21}$$

thus

$$c=a-\frac{a-c_0}{1+\varphi} \tag{22}$$

where c_0 is the mean velocity in the cross section determined by thickness δ_0.

Since $\varphi \ll 1$ in third-order magnitude

$$c=c_0+(a-c_0)\,\varphi-(a-c_0)\,\varphi^2\,. \tag{23}$$

Derivating:

$$\frac{\partial c}{\partial x}=(a-c_0)\,(1-2\varphi)\,\frac{\partial\varphi}{\partial x}\,. \tag{24}$$

Substituting (24) for (15) we the expression for φ is obtained:

$$a(a-c_0)(1-2\varphi)\frac{d\varphi}{dx}+\alpha\left[c_0+(a-c_0)\varphi-(a-c_0)\varphi^2\right]\cdot$$

$$\cdot(a-c_0)(1-2\varphi)\frac{d\varphi}{dx}=g_x+\frac{\delta_0\,\sigma}{\varrho}\,\frac{d^3\varphi}{dx^3}-\frac{\eta_0}{\varrho}\,\frac{\left[c_0\,(a-c_0)\,\varphi-(a-c_0)\,\varphi^2\right]^n}{\delta_0^{n+1}(1+\varphi)^{n+1}}\,. \tag{25}$$

This expression in x is a linear, third-order differential equation. It is necessary for non-decay, periodic solution that the coefficient of φ and the 0-order terms is zero.

In the first approximation:

$$\begin{aligned}-a(a-c_0)\frac{d\varphi}{dx}+\alpha(c_0-a)c_0\frac{d\varphi}{dx}=g_x+\frac{\delta_0\,\sigma}{\varrho}\,\frac{d^3\psi}{dx^3}-\\-\frac{\eta_0}{\varrho}\,\beta\left[\frac{nc_0^{n-1}(a-c_0)-c_0^n(n+1)}{\delta_0^{n+1}}\,\varphi+\frac{c_0^n}{\delta_0^{n+1}}\right].\end{aligned} \tag{26}$$

Thus

$$g_x-\frac{\eta_0}{\varrho}\,\beta\,\frac{c_0^n}{\delta_0^{n+1}}=0 \tag{27}$$

and

$$\frac{\eta_0}{\varrho}\left(\frac{2n+1}{n}\right)^n\,\frac{nc_0^{n-1}\,(a-c_0)-c_0(n+1)}{\delta_0^{n+1}}=0\,. \tag{28}$$

The latter delivers the phase velocity of the waves:

$$a = \frac{2n+1}{n} c_0 \, . \tag{29}$$

The mean thickness from (27):

$$\delta_0 = \left(\frac{\beta \eta_0}{\varrho g_x} Q^n \right)^{\frac{1}{2n+1}} \tag{30}$$

where $Q = c_0 \delta_0$, the flow rate per time.

The non-decay, periodic solving for φ, can be derived from

$$\frac{d^3 \varphi}{dx^3} + \frac{\varrho}{\delta_0 \sigma} (a - c_0)(a = \alpha c_0) \frac{d\varphi}{dx} - 0 \, . \tag{31}$$

Assuming sinusoidal free surface:

$$\varphi = A \sin (kx - \omega t) \, . \tag{32}$$

Substituting (31) we obtain for the wave-number:

$$k = \sqrt{\frac{\varrho}{\delta_0 \sigma} (a - c_0)(a - \alpha c_0)} \, . \tag{33}$$

These are relationships for the wave number, phase velocity, the mean film thickness to constants ratio characterizing the fluid behaviour (σ, η_0, n). It makes possible a fast checking of rheological measurements.

The energy equation is:

$$\frac{dt}{d} \int\limits_V \left(\frac{v^2}{2} + U \right) \varrho \, dV = \int\limits_V p \operatorname{div} \vec{v} \, dV + \int\limits_{(A)} \vec{v} \, \mathbf{F} \, \overrightarrow{dA} - \int\limits_V \Phi \, dV \tag{34}$$

where Φ is the function of dissipation; $\mathbf{F}$ is the stress tensor.

The fluid is incompressible:

$$\operatorname{div} \vec{v} = 0$$

and the field is gravitational:

$$U = gh \, .$$

After arranging:

$$\int\limits_V \frac{\partial t}{\partial} \left(\frac{v^2}{2} + gh \right) \varrho \, dV + \int\limits_{(A)} \vec{v} \left(\frac{v^2}{2} + gh \right) \varrho \, \overrightarrow{dA} = \int\limits_{(A)} \vec{\imath} \, \mathbf{F} \, \overrightarrow{dA} - \int\limits_V \Phi \, dV. \text{ (35}$$

Since the gravitational field is steady, and the free surface is sinusoidal:

$$\frac{\partial}{\partial t} \left(\frac{v^2}{2} + gh \right) = 0 \, .$$

The surface integrals are zero in sections AB and CD because there $\vec{v}\vec{dA} = 0$. The velocity, and shear stress distributions are the same in sections AD and BC, but the directions of $\vec{dA}$ are opposite.

Thus

$$\varrho Q g(h_2 - h_1) = -\eta_0 \int_{x=0}^{\lambda} \int_{y=0}^{\delta} \left(\frac{\partial v_x}{\partial y}\right)^{n+1} dy\, dx \tag{36}$$

therefore

$$\varrho g_x Q \lambda = \eta_0 \beta \int_0^{\lambda} \frac{c^{n+1}}{\delta^n}\, dx\,. \tag{37}$$

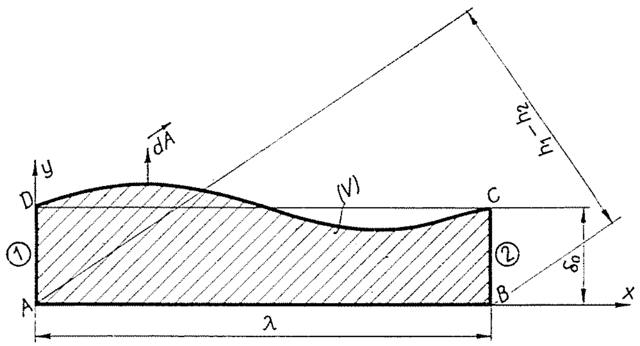

Fig. 3

Substituting (22) and (12), the energy equation of the unit length film is:

$$\varrho g_x Q = \eta_0 \beta \frac{c_0^{n+1}}{\delta_0^n} \frac{1}{\lambda} \int_0^{\lambda} \frac{\left(1 + \frac{g}{c_0}\varphi\right)^{n+1}}{(1+\varphi)^{2n+1}}\, dx\,. \tag{38}$$

The decrease of potential energy is equal to the energy dissipation.

The right side of (38) may be integrated by expanding in series the powers of φ, because $\varphi \ll 1$.

$$\frac{\left(1 + \frac{a}{c_0}\varphi\right)^{n+1}}{(1+\varphi)^{2n+1}} = 1 + (2n+1)\,\varphi - \frac{2n^2 + 3n + 1}{n}\,\varphi^2\,. \tag{39}$$

Taking (32) into consideration, after integration:

$$\varrho g_x Q = \eta_0 \beta \frac{c_0^{n+1}}{\delta_0^n} \left(1 - \frac{2n^2 + 3n + 1}{2n} A^2\right). \tag{40}$$

From (40) the amplitude, is the following:

$$A = \sqrt{\frac{2n}{2n^2 + 3n + 1}\left(1 - \frac{\varrho\, g_x\, \delta_0^{n+1}}{\beta\, \eta_0\, c_0^n}\right)}. \tag{41}$$

Consequently, from (41), if

$$1 < \frac{\varrho_x\, g\, \delta_0^{n+1}}{\beta\, \eta_0\, c_0^n} \tag{42}$$

is no real solution, no wawes can occur at the free surface of the film.

The boundary value of stability of the constant-thickness layer.

$$1 = \frac{\varrho\, g_x\, \delta_0^{n+1}}{\beta\, \eta_0\, c_0^n}. \tag{43}$$

If the flow rate is increasing for a constant mean thickness, then:

$$1 > \frac{\varrho\, g_x\, \delta_0^{n+1}}{\beta\, \eta_0\, c_0^n} \tag{44}$$

here the capillary-gravity wawes will occur on the surface of the film.

Similar Results obtained results may be derived for the case $n = 1$. Our expressions are in good agreement with experimental results [1], [2], [3].

Summary

An approximate analysis is presented for determining the motion of capillary-gravity waves in thin, non-Newtonian liquid layers, flowing down on an inclined, flat plate. The equations of motion are averaged across the film thickness, after some simplifications the mean velocity, the phase velocity of the waves, the mean thickness of the film, the wave-number are obtained. At last the boundary value of the stability of the constant-thickness layer is obtained. Thereby a fast control of rheological measurements becomes possible.

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