INVESTIGATION OF LAMINAR NON-NEWTONIAN BOUNDARY LAYER USING POLYNOMIALS

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Introduction

The study of the laminar boundary layer is very important from the point of heat and mass transfer. The increasing importance of Non-Newtonian fluids in the chemical and process industry has directed the attention to transfer phenomena in this sort of fluids. To solve transport equations involves generally many difficulties; even no exact solution may be obtained. The same problems arise for Non-Newtonian fluids, with a non-linear relationship between shear stress and deformation rate. Approximation methods have become generalized. Some of them are based upon the transformation of partial differential equations into integral equations applying neglects [1]. The involved error much depends on the type of the chosen function, generally a polynomial describing the change of the intensive quantity according to place.

Now examine the influence of the changing degree of the polynomial on the error of the solution, in case of two-dimensional laminar boundary layer flow of a Non-Newtonian fluid. The rheology of the fluid will be characterized by the power-law.

Kármán—Pohlhausen method

In a rectangular co-ordinate system, according to the boundary layer theory, the following equations arise for two-dimensional laminar boundary layer flow:

for the momentum transfer:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\varrho} \frac{\partial}{\partial y} \tau_{xy} \tag{1}$$

for the heat transfer:

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = a \frac{\partial^2 T}{\partial y^2} \tag{2}$$

for the mass transfer:

$$u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} = D \frac{\partial^2 c}{\partial y^2} . \tag{3}$$

These equations will be solved by the Kármán—Pohlhausen's method. In case of a flow over a solid flat surface the integral forms of these equations are (assuming the velocity caused by mass transfer in y direction can be neglected):

$$\varrho \frac{\partial}{\partial x} \int_0^\delta (U_\infty - u)\, u\, dy = (\tau_{xy})_{y=0} \tag{4}$$

$$\frac{1}{a} \frac{\partial}{\partial x} \int_0^{\delta_T} (T_\infty - T)\, u\, dy = \left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{5}$$

$$\frac{1}{D} \frac{\partial}{\partial x} \int_0^{\delta_c} (c_\infty - c)\, u\, dy = \left(\frac{\partial c}{\partial y}\right)_{y=0} . \tag{6}$$

To solve these equations, variations of velocity u, temperature T and concentration c will be approximated by a polynomial

$$F = \sum_{i=1}^{m} \alpha_i \eta^i \tag{7}$$

with dimensionless variables

$$F = \frac{u}{U_\infty} \qquad \text{and} \qquad \eta = \frac{y}{\delta(x)} , \tag{8}$$

The coefficients of this polynomial of m-th degree can be determined from the boundary conditions, resulting either from physical considerations and experimental work or from mathematical considerations.

Increasing the degree of the polynomial hence the number of boundary conditions is expected to yield more correct results.

Ojha [2] examined six different boundary conditions to test polynomials of various degrees and compared the revelant numerical results with the exact solution for a Newtonian fluid. A higher degree appeared not to give better results in every case.

Below we shall use the same boundary conditions (Table 1) facilitating to determine the coefficients of the polynomials.

As the first step let us study the momentum transfer.

Table 1

Polynomial Symbol	Degree of polynomial	Wall boundary conditions $y=0$		Outer edge boundary conditions $y=\delta$	
$3P_2$	3	I.	$\frac{u}{U_\infty}=0$	I.	$\frac{u}{U_\infty}=1$
		II.	$\frac{\partial^2 u}{\partial y^2}=0$	II.	$\frac{\partial u}{\partial y}=0$
$4P_2$	4	I.	$\frac{u}{U_\infty}=0$	I.	$\frac{u}{U_\infty}=1$
		II.	$\frac{\partial^2 u}{\partial y^2}=0$	II.	$\frac{\partial u}{\partial y}=0$
				III.	$\frac{\partial^2 u}{\partial y^2}=0$
$5P_2$	5	I.	$\frac{u}{U_\infty}=0$	I.	$\frac{u}{U_\infty}=1$
		II.	$\frac{\partial^2 u}{\partial y^2}=0$	II.	$\frac{\partial u}{\partial y}=0$
				III.	$\frac{\partial^2 u}{\partial y^2}=0$
				IV.	$\frac{\partial^3 u}{\partial y^3}=0$
$4P_3$	4	I.	$\frac{u}{U_\infty}=0$	I.	$\frac{u}{U_\infty}=1$
		II.	$\frac{\partial^2 u}{\partial y^2}=0$	II.	$\frac{\partial u}{\partial y}=0$
		III.	$\frac{\partial^3 u}{\partial y^3}=0$		
$5P_3$	5	I.	$\frac{u}{U_\infty}=0$	I.	$\frac{u}{U_\infty}=1$
		II.	$\frac{\partial^2 u}{\partial y^2}=0$	II.	$\frac{\partial u}{\partial y}=0$
		III.	$\frac{\partial^3 u}{\partial y^3}=0$	III.	$\frac{\partial^2 u}{\partial y^2}=0$
$6P_3$	6	I.	$\frac{u}{U_\infty}=0$	I.	$\frac{u}{U_\infty}=1$
		II.	$\frac{\partial^2 u}{\partial y^2}=0$	II.	$\frac{\partial u}{\partial y}=0$
		III.	$\frac{\partial^3 u}{\partial y^3}=0$	III.	$\frac{\partial^2 u}{\partial y^2}=0$
				IV.	$\frac{\partial^3 u}{\partial y^3}=0$

Momentum transfer

Determination of the resistance coefficient will be illustrated on the polynomial $6P_3$. The cipher 6 means the degree of the polynomial the subscript 3 the number of the wall boundary conditions.

The rheological property of the fluid can be described as:

$$(\tau_{xy})_{y=0} = K \left(\frac{\partial u}{\partial y}\right)^n_{y=0} \tag{9}$$

and the velocity profil:

$$F(\eta) = \sum_{i=1}^{6} \alpha_i \eta^i . \tag{10}$$

According to Table 1:

$$\mathrm{F}(\eta) = 2\eta - 5\eta^4 + 6\eta^5 - 2\eta^6 . \tag{11}$$

Considering (8) and (9), the integral equation for the momentum transfer:

$$2^n K \frac{U^n_\infty}{\delta^n} = U^2_\infty \varrho \frac{\partial}{\partial x} \int_0^1 (1 - F) F \delta(x) d\eta . \tag{12}$$

Substituting (11) into (12) and integrating we get an ordinary differential equation:

$$\delta^n \frac{d\delta}{dx} = 2^n \frac{K}{\varrho} \frac{9009}{985} U^{n-2}_\infty \tag{13}$$

solving this for boundary condition

$$\delta(0) = 0 \tag{14}$$

yields relationship of momentum boundary layer thickness and the place:

$$\delta(x) = \left[2^n (n+1) \frac{9009}{985} \frac{K}{\varrho} U^{n-2}_\infty x\right]^{1/(n+1)} . \tag{15}$$

Hence, the resistance coefficient:

$$\xi(x) = \frac{(\tau_{xy})_{y=0}}{U^2_\infty \varrho} = \frac{985}{9009} \delta'(x) = \left[\frac{985}{9009} \frac{2}{n+1}\right]^{n/(n+1)} Re^{-1/(n+1)} \tag{16}$$

where

$$Re = \frac{\varrho x^n U^{2-n}_\infty}{K} . \tag{17}$$

Introducing notation:

$$c(n) = \xi(x) \cdot \mathrm{Re}^{1/(n+1)} \tag{18}$$

we get for the polynomial $6P_3$:

$$c(n) = \left[\frac{985}{9009}\,\frac{2}{n+1}\right]^{n/(n+1)} . \tag{19}$$

Table 2

Polynomial / n	$3P_2$	$4P_2$	$5P_2$	$4P_3$	$5P_3$	$6P_3$	Exact
0.1	0.859842	0.869057	0.874785	0.852836	0.859154	0.863412	0.9853
0.2	0.747256	0.762003	0.771236	0.736131	0.746160	0.753353	0.8712
0.3	0.655816	0.673804	0.685134	0.642336	0.654485	0.668749	0.7577
0.4	0.580715	0.600499	0.613026	0.565973	0.579256	0.588325	0.6592
0.5	0.518365	0.539026	0.552168	0.503045	0.516846	0.526299	0.577
0.6	0.466073	0.487024	0.500402	0.450606	0.464537	0.474107	0.5090
0.7	0.421802	0.442667	0.456036	0.406457	0.420276	0.429792	0.4526
0.8	0.383992	0.404534	0.417737	0.368936	0.382493	0.391849	0.4055
0.9	0.351441	0.371513	0.384450	0.336774	0.349979	0.359110	0.3658
1.0	0.323209	0.324725	0.355355	0.306987	0.321790	0.330659	0.3321
1.1	0.298557	0.317469	0.329716	0.284809	0.297183	0.305769	0.3030
1.2	0.276894	0.295182	0.307050	0.263630	0.275568	0.283863	0.2778
1.3	0.257748	0.275409	0.286891	0.244965	0.256469	0.264474	0.2558
1.4	0.240737	0.257779	0.268878	0.228424	0.239504	0.247222	0.2363
1.5	0.225546	0.241985	0.252708	0.213689	0.224358	0.231798	0.219
1.6	0.211919	0.227775	0.238133	0.200501	0.210774	0.217946	0.2036
1.7	0.199642	0.214938	0.224944	0.188643	0.198538	0.205453	0.1897
1.8	0.188537	0.203298	0.212965	0.177939	0.187473	0.194143	0.1773
1.9	0.178456	0.192706	0.202049	0.168237	0.177430	0.183865	0.1661
2.0	0.169272	0.183035	0.192068	0.159414	0.168281	0.174493	0.1561

In case of the other polynomials the $c(n)$ values and the values of the exact solution [3] are compiled in Table 2. Approximation closeness by each polynomial:

$$\Gamma = \sum_{i=2}^{20} [c(0{,}1 \cdot i) - c_e(0{,}1 \cdot i)]$$

Γ vs. polynomial degree is plotted in Fig. 1. In fact, the increase of the degree appears not to mean unequivocally an improvement, although the least error occurred for $6P_3$.

$c(n)$ values delivered by polynomial $6P_3$ and the exact solution, respectively, are shown in Fig. 2.

A fair approximation appears in the range $0.2 \leq n \leq 2.0$. The curves of the velocity profil will be plotted for the exact and the $6P_3$ solution, for sake of comparison.

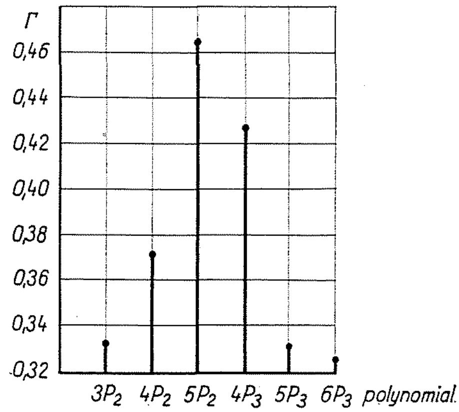

Fig. 1

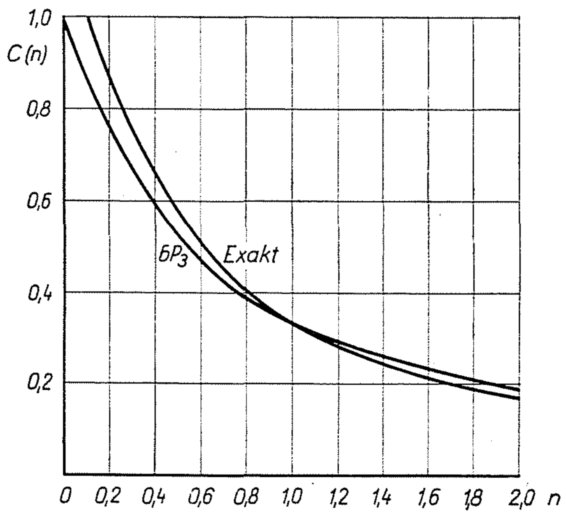

Fig. 2

Heat and mass transfer

By analogy between heat and mass transfer, the following symbols will be introduced:

$$H = \begin{cases} T & \text{for heat transfer} \\ c & \text{for mass transfer} \end{cases} \tag{20.a}$$

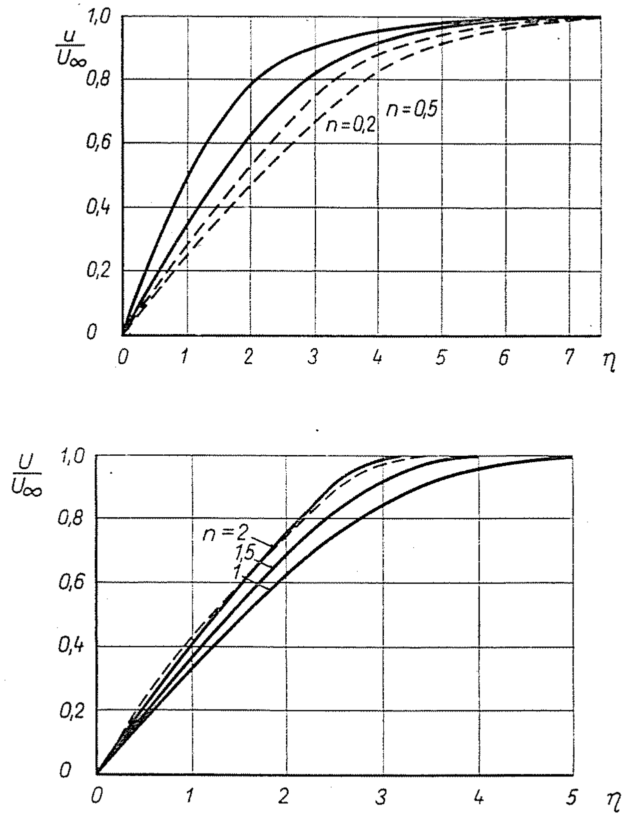

Fig. 3

$$\lambda = \begin{cases} a & \text{for heat transfer} \\ D & \text{for mass transfer} \end{cases} \tag{20.b}$$

$$\delta_\pi = \begin{cases} \delta_T & \text{for heat transfer} \\ \delta_c & \text{for mass transfer} \end{cases} \tag{20.c}$$

and

$$\pi = \frac{H - H_0}{H_\infty - H_0} \quad \text{and} \quad \eta_\pi = \frac{y}{\delta_\pi} \tag{20.d}$$

Using these symbols, Eqs (2) and (3) become

$$u \frac{\partial H}{\partial x} + v \frac{\partial H}{\partial y} = \lambda \frac{\partial^2 H}{\partial y^2}$$

and in integral form:

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$$U_\infty \delta_x(x) \frac{\partial}{\partial x} \int_0^1 (1-\pi) F(\eta)\, \delta_\pi(x)\, d\eta = \lambda \left(\frac{\partial \pi}{\partial \eta_\pi}\right)_0. \tag{21}$$

The procedure of the calculation is illustrated again on $6P_3$. The concentration and the temperature profil are approximated by polynomial:

$$\pi(\eta_\pi) = \sum_{i=0}^{6} \beta_i \eta_\pi^i .$$

The same boundary conditions can be used for π as for u/U_∞

$$\pi(\eta_\pi) = 2\eta_\pi - 5\eta_\pi^4 + 6\eta_\pi^5 - 2\eta_\pi^6 . \tag{22}$$

Introducing symbol:

$$\Delta_\pi = \frac{\eta}{\eta_\pi}$$

becomes:

$$F(\eta) = 2\eta_\pi \Delta_\pi - 5\eta_\pi^4 \Delta_\pi^4 + 6\eta_\pi^5 \Delta_\pi^5 - 2\eta_\pi^6 \Delta_\pi^6 . \tag{23}$$

Substituting (22) and (23) into (21) and integrating we get:

$$\frac{2\lambda}{U_\infty} = \Delta_\pi\, \delta(x) \frac{d}{dx}\left[\delta(x)\left(\frac{5}{42}\Delta_\pi^2 - \frac{2}{99}\Delta_\pi^5 + \frac{1}{77}\Delta_\pi^6 - \frac{20}{8008}\Delta_\pi^7\right)\right].$$

Supposing $\Delta\pi < 1$, and neglecting higher — degree terms

$$\Delta_\pi^3\, \delta\delta' + 2\Delta_\pi^2\, \delta^2\, \Delta_\pi' = \frac{84}{5}\, \frac{\lambda}{U_\infty} . \tag{24}$$

Considering (15):

$$\Delta_\pi^3 + 2(n+1)\, x \Delta_\pi^2 \Delta_\pi' = \frac{84}{5}(n+1)\left[\frac{985}{9009}\, \frac{1}{2^n(n+1)}\right]^{2/(n+1)} \frac{1}{A} \tag{25}$$

where

$$A = \frac{U_\infty x \mathrm{Re}^{-2/(n+1)}}{\lambda} .$$

The homogeneous part of the inhomogeneous differential equation (25), introducing $\Delta_\pi^3 = \varphi(x)$:

$$\varphi + \frac{2}{3}(n+1)\, x\varphi' = 0 . \tag{26}$$

The solution of the homogeneous equation is looked for in form:

$$\varphi_h = c_1 x^r .$$

Substituting this into (26), the homogeneous solution is

$$\varphi_h = c_1\, x^{-3/2(n+1)}.$$

Looking for a particular solution of the inhomogeneous equation of the form

$$\varphi_P = c_2 x^P$$

and substituting into (25) we obtain:

$$c_2\, x^P\left[1+\frac{2}{3}(n+1)\,P\right] = \frac{84}{5}\,(n+1)\left[\frac{985}{9009}\,\frac{1}{2^n\,(n+1)}\right]^{2/(n+1)}\frac{1}{\Lambda}\,.$$

Since

$$\Lambda = [\Lambda]x^{(1-n)/(1+n)}$$

where $[\Lambda]$ is a part of Λ and independent of x,

$$P = \frac{n-1}{n+1}$$

and

$$c_2 = \frac{\dfrac{84}{5}(n+1)\left[\dfrac{985}{9009}\,\dfrac{1}{2^n(n+1)}\right]^{2/(n+1)}\dfrac{1}{[\Lambda]}}{1+\dfrac{2}{3}(n-1)}\,.$$

The general solution:

$$\varphi = \varphi_h + \varphi_P$$

hence

$$\varphi(x) = c_1 x^{-3/2(n+1)} + c_2 x^{(n-1)/(n+1)}\,. \tag{27}$$

The constant c_1 can be calculated from the boundary condition:

$$\delta_\pi(x_0) = 0 \quad \text{i.e.} \quad \varphi(x_0) = 0$$

Thus

$$\varphi(x) = \frac{c'(n)}{\Lambda}\left[1-\frac{\Lambda}{\Lambda_0}\left(\frac{x_0}{x}\right)^{3/2(n+1)}\right]$$

where

$$c'(n) = c_2(n)[\Lambda]$$

and

$$\Lambda_0 = \Lambda x_0^{(1-n)/(1+n)}$$

since

$$\varphi(x) = \Delta_\pi^3.$$

The solution of (25):

$$\Delta_\pi = \frac{c'(n)^{1/3}}{\Delta^{1/3}} \left[1 - \frac{\Lambda}{\Lambda_0} \left(\frac{x_0}{x}\right)^{3/2(n+1)}\right]^{1/3}. \tag{28}$$

Introducing symbols:

$$\varkappa = \begin{cases} \alpha & \text{heat transfer} \\ k & \text{mass transfer} \end{cases}$$

$$\theta = \begin{cases} \lambda_0 & \text{heat transfer} \\ D & \text{mass transfer} \end{cases}$$

The heat or mass flux at the boundary can be expressed as

$$\varkappa(H_\infty - H_0) = \Theta \left(\frac{\partial H}{\partial y}\right)_0$$

hence

$$\varkappa = \frac{\Theta}{\Delta\pi\,\delta(x)} \left(\frac{\partial \pi}{\partial y}\right)_0 .$$

Using $6P_3$:

$$\left(\frac{\partial \pi}{\partial y}\right)_0 = 2 .$$

Considering (15) and (28):

$$\varkappa = \frac{\Theta}{x} c'(n) \frac{\Lambda^{1/3}\,\mathrm{Re}^{1/(n+1)}}{\left[1 - \frac{\Lambda}{\Lambda_0}\left(\frac{x_0}{x}\right)^{3/2(n+1)}\right]^{1/3}} \tag{29}$$

where

$$c^*(n) = \frac{2}{c'(n)^{1/3}} \left[2^n (n+1) \frac{9009}{985}\right]^{-1/(n+1)} .$$

Using the symbol:

$$\sigma = \frac{\varkappa x}{\Theta}$$

(29) can be written in dimensionless form

$$\sigma = c^*(n) \frac{\Lambda^{1/3}\,\mathrm{Re}^{1/(n+1)}}{\left[1 - \frac{\Lambda}{\Lambda_0}\left(\frac{x_0}{x}\right)^{3/2(n+1)}\right]^{1/3}} . \tag{30}$$

If the starting point of the momentum boundary layer coincides with the starting point of the heat and concentration boundary layer, (30) reduces into:

$$\sigma = c^*(n) \Lambda^{1/3}\,\mathrm{Re}^{1/(n+1)} . \tag{31}$$

Values of $c^*(n)$ delivered by the exact solution [4] and by various degree polynomials are compiled in Table 3. Deviations are plotted in Fig. 4.

Table 3

n \ c^*	$3P_2$	$4P_2$	$5P_2$	$4P_3$	$5P_3$	$6P_3$	Exact
0.1	0.262429184	0.287765580	0.305102927	0.244522408	0.259957141	0.271143404	0.418679955
0.2	0.273181794	0.298670637	0.316087657	0.255120473	0.270668626	0.281928431	0.357467377
0.3	0.282815714	0.308429710	0.325912206	0.264625851	0.280266631	0.291586730	0.336427390
0.4	0.291544098	0.317266532	0.334806506	0.273242876	0.288962929	0.300335027	0.328312079
0.5	0.299518527	0.325338448	0.342931094	0.281118029	0.296908241	0.308326794	0.325952633
0.6	0.306852179	0.332761937	0.350404144	0.288361372	0.304215174	0.315676245	0.326505654
0.7	0.313632729	0.339626640	0.357316224	0.295058579	0.310871000	0.322471636	0.328624315
0.8	0.319930043	0.346003647	0.363738936	0.301278172	0.317245290	0.328783164	0.331604828
0.9	0.325801040	0.351950664	0.369730282	0.307076135	0.323094740	0.334667940	0.335040718
1.0	0.331292914	0.357515417	0.375381153	0.312498983	0.328566366	0.340173265	0.338722790
1.1	0.336445360	0.362737969	0.380602712	0.317585899	0.333699721	0.345338896	0.342422797
1.2	0.341292158	0.367652370	0.385558071	0.322370271	0.338528466	0.350198648	0.346118088
1.3	0.345862342	0.372287861	0.390233510	0.326880819	0.343081537	0.354781571	0.349753621
1.4	0.350181077	0.376669767	0.394654386	0.331142452	0.347384015	0.359112840	0.353242693
1.5	0.354270335	0.380820195	0.398842828	0.335176931	0.351457798	0.363214429	0.356612361
1.6	0.358149422	0.384758561	0.402818276	0.339003380	0.355322129	0.367105645	0.359869513
1.7	0.361835397	0.388502024	0.406597909	0.342638703	0.358994012	0.370803547	0.362965420
1.8	0.365343410	0.392065822	0.410196984	0.346097915	0.362488547	0.374323284	0.365984726
1.9	0.368686978	0.395463549	0.413629120	0.349394413	0.365819207	0.377678372	0.368892180
2.0	0.371878208	0.398707387	0.416906517	0.352540196	0.368998060	0.380880919	0.371741551

The values of $c(n)$ given by the closest polynomial and by the exact solution vs. n are shown in Fig. 5. $6P_3$ seems to give a close approximation in the range $0.6 \leq n \leq 2.0$.

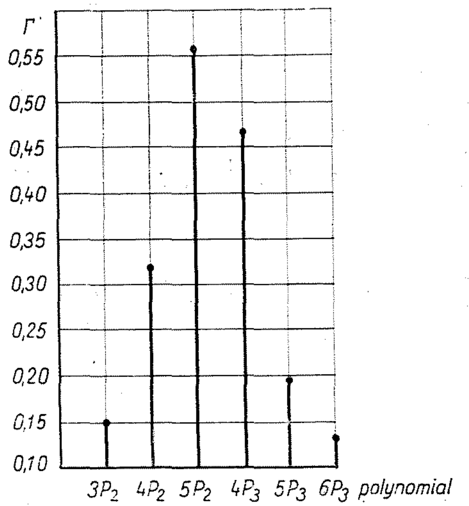

Fig. 4

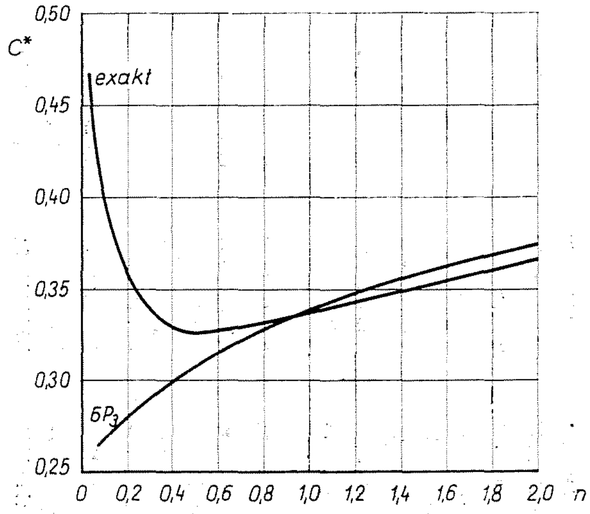

Fig. 5

The results show the practical applicability of $6P_3$ for both the momentum transfer and the heat and mass transfer, and the deviation from the exact solution to be negligible.

Summary

Development of relationships for heat, mass and momentum transfer and the distribution of the intensive quantity for a two-dimensional laminar boundary layer of Non-Newtonian flow has been examined as approximated by various degree polynomials.

Comparison of the coefficients of dimensionless equations showed a basic influence of the changing degree of polynomial on the error of the solution. The increase of the polynomial degree appeared not to unambiguously reduce the error. The approximation by a polynomial appeared practically satisfactory in a certain range of the index of the power law model.

Notations

a	m^2/s	thermal conductivity coefficient
c	kg/m^3	concentration
c_1		constant
c_2		constant
$c(n)$		def. (18)
c^*		def. (29a)
i		summing index
k	m/s	mass transfer coefficient
n		index of the power law model
u	m/s	velocity
v	m/s	velocity
x	m	co-ordinate
y	m	co-ordinate
D	m^2/s	diffusivity
F		symbol of function
K	$kg/m\ s^{2-n}$	rheological parameter
T	K°	temperature
α	$kcal/m^2h°K$ grad	heat transfer coefficient
α, β		coefficient
δ	m	thickness of the momentum boundary layer
δ_T	m	thickness of the thermal boundary layer
δ_c	m	thickness of the concentrate boundary layer
η		dimensionless co-ordinate
λ	kcal/m h°K grad	heat conductivity
ξ		resistance coefficient
ϱ	kg/m^3	density
τ	$kg/m\ s^2$	shear stress
φ		symbol of function
Re		Reynolds-number def. 17.

Index

0	value for $y \geqq 0$
∞	value for $y = \delta_{T,c}$

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