EXAMINATION OF SHOCK WAVES IN A BAR PRESTRESSED TO PLASTICITY

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A) Exerting additional tension on a prismatic bar which is prestressed into the plastic range in such a way as to cause a discontinuity along one or several lines in the velocity field resulting from loading, velocity waves or in other words shock waves will arise in the bar. Similarly, if the acceleration range suffers a discontinuity along one or several lines, an acceleration wave develops.

When studying the velocity and acceleration waves produced in a prismatic bar, generally an axial stress condition is assumed. In what follows, we shall set out from this assumption.

For the examination of the acceleration and velocity waves the constitutive equation describing the mechanical behaviour of the bar material under dynamic plastic load is wanted.

A constitutive equation may assume the form:

$$\Phi(\sigma_t, \varepsilon, \varepsilon_t, \varepsilon_x) \equiv -\sigma_t + EG_1(\varepsilon_t) + D_0(\varepsilon)\{g(\varepsilon_x)[G_2(\varepsilon_t) - \alpha_0^*] + \alpha_0^*\} \tag{1}$$

where

σ stress arising in the bar cross-section;

σ_t derivative by time of the stress;

ε specific strain along the bar axis;

ε_t and ε_x derivatives by time of the specific strain and of that along the bar respectively;

E Young modulus (modulus of elasticity);

D_0 derivative of the equation of the diagram of static tension, $\frac{d\sigma}{d\varepsilon} = D_0(\varepsilon)$;

$G_1(\varepsilon_t)$ and $G_2(\varepsilon_t)$ are odd functions,

$$G_1(0) = G_2(0) = 0, \qquad \lim_{\varepsilon_t \to \infty} G_2(\varepsilon_t) = \alpha_0$$

and

$$\left.\frac{dG_1}{d\varepsilon_t}\right|_0 = 0, \qquad \left.\frac{dG_2}{d\varepsilon_t}\right|_0 = 1 \qquad \lim_{\varepsilon_t \to \infty} \frac{dG_1}{d\varepsilon_t} = 1$$

$$\lim_{\varepsilon_t \to \infty} \frac{dG_2}{d\varepsilon_t} = 0$$

$g(\varepsilon_x)$ is an even function and

$$g(0) = 1, \quad \left.\frac{dg}{d\varepsilon_x}\right|_0 = 0$$

Finally,

$$\alpha_0^* = \alpha_0 \text{ sign } \varepsilon_t, \quad \varepsilon_t \neq 0$$

and

$$\alpha_0^* = \alpha_0; \qquad \varepsilon_t = 0$$

This constitutive equation can be established by examining the acceleration wave under the following conditions [1]:

1. the bar material is isotropic and homogeneous;
2. the constitutive equation is a function of the type $\sigma_t = \varphi(\sigma, \varepsilon, \varepsilon_t, \varepsilon_x)$;
3. the propagation velocity of the acceleration wave is finite and nonzero;
4. there are forward and return going acceleration waves;
5. if $\varepsilon_t \to 0$ then the propagation velocity of the wave is $\to \sqrt{\frac{D_0}{\varrho}}$;
6. if $\varepsilon_t \to \infty$ then the propagation velocity of the wave is $\to \sqrt{\frac{E}{\varrho}}$;

where

ϱ density of the bar material.

B) Let us connect the coordinate axis x to the bar axis in such a way that the $x = 0$ coordinate should be ordered to the one and the coordinate $x = l$ to the other end of the bar (Fig. 1).

Let the constitutive equation of the shock wave front be

$$\psi(x, t) = 0 \quad \text{or} \quad x = x(t),$$

designated also as

$$\psi \equiv x(t) - x = 0$$

Let us plot the function

$$\psi(x, t) = 0$$

in the coordinate system x, t (Fig. 2). ψ divides the quadrant $x > 0$, $t > 0$ into two parts with the mechanical quantities ahead of the wave front pertaining to part 1, those behind the wave front to part 2. In case of a shock wave, the values of the velocity v, the specific elongation ε and the stress σ will abruptly change beyond the curve $\psi(x, t) = 0$.

For instance, be the velocity v_1 and v_2 at the side of ψ facing the range 1 and 2 resp., then the velocity jump along ψ will be $v_2 - v_1$. This jump is designated by $[v]$, viz.:

$$[v] = v_2 - v_1$$

In case of such discontinuity in a function f along ψ, in its derivatives by x and t the following kinematic conditions will hold for the discontinuities $\left[\frac{\partial f}{\partial x}\right] \equiv [f_x]$; $\left[\frac{\partial f}{\partial t}\right] \equiv [f_t]$; along ψ [2]:

$$[f_x] = \lambda_f \psi_x + \frac{\partial [f]}{\partial x}$$

and

$$[f_t] = \lambda_f \psi_t + \frac{\partial [f]}{\partial t} \tag{2}$$

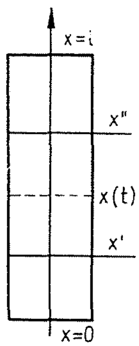

Fig. 1

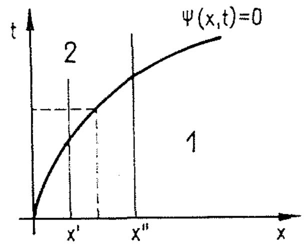

Fig. 2

Accordingly, the kinematic conditions can be written down also for v, ε and σ in the following manner:

$$[v_x] = \lambda_v \psi_x + \frac{\partial [v]}{\partial x}$$

$$[v_t] = \lambda_v \psi_t + \frac{\partial [v]}{\partial t}$$

$$[\varepsilon_x] = \lambda_\varepsilon \psi_x + \frac{\partial [\varepsilon]}{\partial x} \tag{2a}$$

$$[\varepsilon_t] = \lambda_\varepsilon \psi_t + \frac{\partial [\varepsilon]}{\partial t}$$

$$[\sigma_x] = \lambda_\sigma \psi_x + \frac{\partial [\sigma]}{\partial x}$$

$$[\sigma_t] = \lambda_\sigma \psi_t + \frac{\partial [\sigma]}{\partial t}$$

Making use of (2) and (2a) the derivate forms of the equations of motion and of compatibility along $\psi = 0$ will take the following form:

$$\varrho\lambda_v \psi_t + \varrho \frac{\partial[v]}{\partial t} = \lambda_\sigma \psi_x + \frac{\partial[\sigma]}{\partial x} \tag{3}$$

and

$$\lambda_v \psi_x + \frac{\partial[v]}{\partial x} = \lambda_\varepsilon \psi_t + \frac{\partial[\varepsilon]}{\partial t} \tag{4}$$

The quantities $[v]$, $[\varepsilon]$ and $[\sigma]$ are associated by kinematic and dynamic conditions also along the curve $\psi(x, t) = 0$.

The kinematic condition stems from the displacement u being a continuous function beyond ψ, i.e. $[u] = 0$. From u we get v and ε by derivation:

$$v = \frac{\partial u}{\partial t}; \qquad \varepsilon = \frac{\partial u}{\partial x}$$

Making use of (2):

$$\left[\frac{\partial u}{\partial t}\right] \equiv [v] = \lambda_u \psi_t$$

$$\left[\frac{\partial u}{\partial x}\right] \equiv [\varepsilon] = \lambda_u \psi_x$$

From the two equations

$$[v] = \frac{\psi_t}{\psi_x} \cdot [\varepsilon]$$

or, introducing the symbol $c = -\dfrac{\psi_t}{\psi_x}$

$$[v] = -c[\varepsilon] \tag{5}$$

As it is obvious from the form $\psi \equiv x(t) - x$, $c = \dfrac{dx}{dt}$, i.e. c represents the wave rate.

To write down the dynamic condition, take a bar stretch $x'x''$ which contains the wave front $x(t)$ (Fig. 1) and write down the relevant equation of motion [3, 4]:

$$\frac{d}{dt} \int_{x'}^{x''} \varrho v dx = \sigma'' - \sigma'$$

Or else, taking into consideration that $x' \leq x(t) \leq x''$ and performing the limit transitions $x' \to x(t)$ and $x'' \to x(t)$ we obtain:

$$\varrho c v_1 - \varrho c v_2 = \sigma_2 - \sigma_1$$

viz.

$$\varrho c[v] = -[\sigma] \tag{6}$$

i.e., the dynamic condition thought for.

Let us complement Eqs (3) to (6) with the constitutive equation in form (1) which holds along ψ. Again, Eqs (2a) will be applied, however, with the proviso that the σ_{t1}, ε_1, ε_{t1}, ε_{x1} values ahead of the wave front are known and they fulfil the constitutive equation. So do the values $\sigma_{t2} = \sigma_{t1} + \lambda_\sigma \psi_t + \dfrac{\partial[\sigma]}{\partial t}$, ε_2, ε_{t2} and ε_{x2} behind the wave front:

$$\Phi\left(\sigma_{t1} + \lambda_\sigma \psi_t + \frac{\partial[\sigma]}{\partial t},\ \ \varepsilon_1 + [\varepsilon],\right.$$

$$\left.\varepsilon_{t1} + \lambda_\varepsilon \psi_t + \frac{\partial[\varepsilon]}{\partial t},\ \ \varepsilon_{x1} + \lambda_\varepsilon \psi_x + \frac{\partial[\varepsilon]}{\partial x}\right) = 0 \tag{7}$$

If λ_v, λ_ε, λ_σ, $[v]$, $[\varepsilon]$ and $[\sigma]$ are known and fulfil Eqs (3) to (6), then Eq. (7) will be a non-linear partial differential equation of the first order with respect to function $\psi(x, t)$. From the equation of the characteristic curves we have:

$$\frac{dx}{dt} = \frac{\Phi_{\varepsilon x} \lambda_\varepsilon}{\Phi_{\sigma t} \lambda_\sigma + \Phi_{\varepsilon t} \lambda_\varepsilon} \tag{8}$$

the rate of the shock wave.

C) Taking into consideration (3), (4), (5), (6) and

$$c = -\frac{\psi_t}{\psi_x}$$

we may write down that

$$\frac{\lambda_\sigma}{\lambda_\varepsilon} = c\, \frac{\varrho \lambda_v c + \varrho \dfrac{\delta[v]}{\delta t}}{\lambda_v + \dfrac{\delta[\varepsilon]}{\delta t}}$$

$\frac{\delta[v]}{\delta t}$ and $\frac{\delta[\varepsilon]}{\delta t}$ being derivatives of $[v]$ and $[\varepsilon]$, resp., along the curve $\psi(x, t) = 0$. Thereby (8) becomes:

$$c = \frac{\Phi_{\varepsilon x}}{\Phi_{\sigma_t} c \dfrac{\varrho\lambda_v c + \varrho \dfrac{\delta[v]}{\delta t}}{\lambda_v + \dfrac{\delta[\varepsilon]}{\delta t}} + \Phi_{\varepsilon t}} \tag{8a}$$

Introducing symbols $k_0 = \frac{[\varepsilon]}{\lambda_v}$ and $k_1 = \frac{\frac{\delta[\varepsilon]}{\delta t}}{\lambda_v}$ and using Eq. (5) to eliminate $[v]$, after appropriate reduction on (8) the derivative by time of the velocity of wave will be:

$$\dot{c} = \frac{\varrho(1-k_1)\,\Phi_{\sigma_t} c^3 + (1+k_1)\,\Phi_{\varepsilon_t} c - (1+k_1)\,\Phi_{\varepsilon_x}}{\varrho k_0\, \Phi_{\sigma_t} c^2} \tag{9}$$

yielding for c a common non-linear differential equation has been obtained.

D) The differential equation (9) will only have a resolution if the right hand side is continuous and limited (the Peano premise) in the range $0 \le x \le l$; $t \ge 0$. This means that the acceleration of the wave front $\dot{c}$ must be a finite value. If $k_0 \neq 0$ and $\Phi_{\sigma_t} \neq 0$, this will at the same time mean the fulfilment of condition (3), item A.

Accordingly, functions $\Phi_{\sigma_t} \equiv -1$, $\Phi_{\varepsilon_t} \equiv EG_1'(\varepsilon_t) + D_0(\varepsilon)G_2'(\varepsilon_t)g(\varepsilon_x)$ and $\Phi_{\varepsilon_x} = D_0(\varepsilon)g'(\varepsilon_x)\,\{G_2(\varepsilon_t) - \alpha_0^*\}$ are continuous and limited, thereby also functions D_0, G_1', G_2, G_2', g, g' must be continuous and limited.

The fulfilment of condition (4), item A, depends, however, also on functions k_0 and k_1. Without putting constraints on them or without their experimental determination, no further limitation can be made for the constitutive equation Φ.

Summary

Analysing the acceleration wave in a prismatic bar prestressed into the plastic range, the constitutive equation may assume the form:

$$\sigma_t = EG_1(\varepsilon_t) + D_0(\varepsilon)\{g(\varepsilon_x)[G_2(\varepsilon_t) - \alpha_0^*] + \alpha_0^*\}\,.$$

The velocity wave in the bar may be analysed by this expression as well. To keep the velocity of the first front of the velocity wave finite, the functions and their first derivatives in the equation have to be continuous and limited.

References

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