DYNAMICS OF PARAMETRICALLY EXCITED VIBRATIONS OF VEHICLES*

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1. Introduction. The problem

Parametrically excited vibrations are those where one or more characteristics of the vibrating system are specified, usually as a periodical function of time. The differential equation (or system of differential equations) describing the problem has usually periodical, variable coefficients.

It is a well-known fact that the examination of the torsional vibration of an internal combustion piston vehicle engine's crankshaft would require to take such an exact model. From practical purposes, however, it is sufficient to substitute somehow averaged constant coefficients for the variable ones.

The motion of selective gears, other important part of a vehicle, is also such a parametrically excited vibration. Examination of this is especially important from the view-point of noise reduction. Recent methods, however, refrain from the substitution of constant coefficients for the variable ones.

Treating the motion of the whole vehicle, the chassis and the whole engine are usually treated as one rigid body, and the observed transient and steady state motion properties closely agree with the results of the simplified calculation. This model is hard to apply for noises arising in or around the passenger cabin. This problem is, however, of importance from the aspect of preserving or even extending marketableness of our buses by improving travel comfort.

With a view to rationalize noise reduction, equations may be derived by means of the more complicated model, and a solution method developed for them.

A solution of the problem is possible by observing the motion of the main parts of an engine's driving system relative to the engine case and elastic properties of the suspension. This is a problem of parametrically excited vibrations in connection with a motion-problem of the whole body of a vehicle.

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2. Scope. Choice of model

For a bus in urban traffic a most frequent speed is no-load speed. The noise in the passenger cabin of such a bus is mostly due to the operation of the engine rather than to the joltiness of the road. Below, the vibrations caused by the engine of a bus in no-load run will be examined. The results are, however, valid for all and not only for no-load speeds. Model of the engine and its suspension is quite different from those made so far. Therefore, no equations referring to the usual modelled part will be quoted.

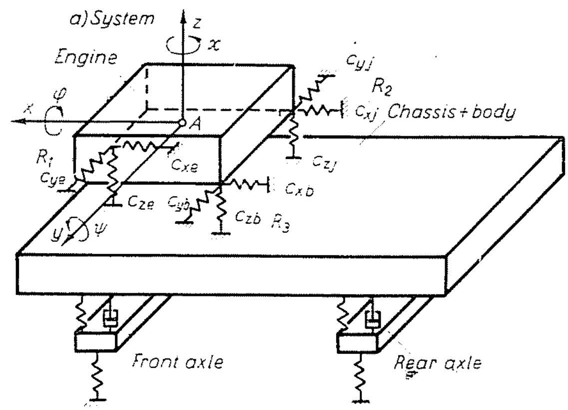

Fig. 1

The model of engine and its suspension consists of the engine case fixed to the chassis at three points by means of rubber springs R_1, R_2, R_3. Rubber springs, assumed to be linearly elastic, are linking both the chassis and the engine case; the force systems at these points, due to rubber springs, are equivalent to a force vector each. Three spring constants are assigned to every rubber spring in three mutually perpendicular directions.

For the sake of illustration three springs are substituted for each rubber spring in our model (Fig. 1). This does not influence our previous assumption, that one rubber spring links the chassis at one point.

The engine consists of the following rigid parts:

a) engine case,
b) crankshaft with balance,
c) connecting rods,
d) pistons.

The angular velocity $\Omega = \Omega(t)$ of the crankshaft relative to the engine case is considered as an empirically known function of time. Theoretically it would be possible to derive and to solve such equations to obtain Ω. But this would require the knowledge of the explosion forces and the frictional forces generated by the moving parts as a function of the crank angle. At present, these forces are not satisfactorily known, therefore equations exempt of these forces should be derived. Thereby Ω cannot be calculated and has to be determined by way of measurements.

This procedure involves a further problem. Namely, the influence of variations of some certain, practically variable, parameters of the engine on the developing vibrations is sought for. In the calculations below, the measured Ω relative to the engine of some given parameters is allowed to be used only if variations of the examined parameters can be proved not to considerably influence Ω.

Fig. 1 shows the model of the whole engine not discussed here: the engine is only pictured as a parallelepiped.

Fig. 2 shows a more detailed model of a six-cylinder in-line engine consisting of the previously specified parts.

3. Frame of references and notations

The frame of references and co-ordinates of some specific points as well as the notations are seen in Fig. 2.

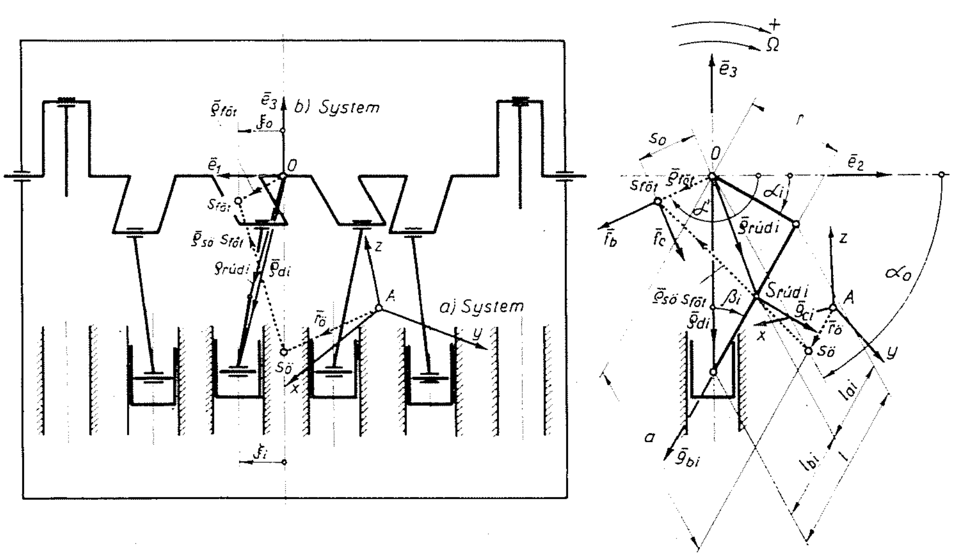

Fig. 2.

1. *Co-ordinate systems*

a) *"Absolute" system.* Its origin is the rest position A of the mass-centre $S_ö$ of the engine case. Directions of axes: x-axis is longitudinal, y-axis is transversal, z-axis is vertical. They make a right-handed co-ordinate system in the order above. Unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are constant in time.

b) *System linked to the engine case.* Its origin is the intersection point O of the geometric axis of the crankshaft on the plane parallel to the x, y-plane in resting state and containing the engine case mass-centre $S_ö$. Axes in the rest state are parallel to x-; y-; and z-axes, respectively. Unit vectors are $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

c) *System linked with the crankshaft.* Its origin is the mass-centre $S_{föt}$ of the crankshaft supplied with balance weights. Unit vectors are $\mathbf{f}_a$, $\mathbf{f}_b$ and $\mathbf{f}_c$. Directions of $\mathbf{f}_a$ and $\mathbf{e}_1$ are the same. Directions of $\mathbf{f}_b$ and $\overrightarrow{OS}_{föt}$ are also the same; $\mathbf{f}_c = \mathbf{f}_a \times \mathbf{f}_b$.

d) *System linked with the i-th connecting rod.* Its origin is the mass-centre $S_{rud\,i}$ of the i-th connecting rod. Unit vectors are $\mathbf{g}_{ai}, \mathbf{g}_{bi}, \mathbf{g}_{ci}$. Directions of $\mathbf{g}_{ai}$ and $\mathbf{e}_1$ are the same. $\mathbf{g}_{bi}$ is parallel to the centre line of the i-th connecting rod and points toward the i-th piston pin. $\mathbf{g}_{ci} = \mathbf{g}_{ai} \times \mathbf{g}_{bi}$.

2. *Position vectors or co-ordinates of special points*

That of mass-centre $S_ö$ in the co-ordinate system a):

$$\mathbf{r}_ö(x_ö, y_ö, z_ö) .$$

that of mass-centre $S_{föt}$ in the co-ordinate system b):

$$\tilde{\rho}_{föt}(\xi_0, s_0 \cos \alpha, -s_0 \sin \alpha), \quad \xi_0 = \text{constant},$$

that of mass-centre $S_{rud\,i}$ in the co-ordinate system b):

$$\tilde{\rho}_{rud\,i}(\xi_i, l_{bi} \sin \beta_i, -r \sin \alpha - l_{ai} \cos \beta_i), \quad \xi_i = \text{constant}, \quad i = 1, \ldots, 6,$$

that of mass-centre S_{di} of the i-th piston at the centre line of the piston pin in the co-ordinate system b):

$$\tilde{\rho}_{di}(\xi_i, 0, \; -r \sin \alpha_i - l \cos \beta_i) .$$

Distance $\overline{SO} = a$.

Common length of the driving rods is r.

Common length of the connecting rods is l.

Eccentricity of mass-centre of the crankshaft is s_0.

Second co-ordinate of the point of the centre line of the i-th connecting rod fitting to the piston pin in co-ordinate system d) is l_{bi}.

$$l_{ai} = l - l_{bi} .$$

3. *Masses*

$M_ö$ mass of the engine case
$M_{föt}$ mass of the crankshaft with balance weights and flywheel
m_{ri} mass of the i-th connecting rod
m_{di} mass of the i-th piston
m mass of other parts of the engine.

4. *Spring constants*

Spring constant of the front suspension:
c_{xe} in x direction,
c_{ye} in y direction,
c_{ze} in z direction.
Of the left rear suspension:
c_{xb} in x direction,
c_{yb} in y direction,
c_{zb} in z direction.
Of the right suspension:
c_{xj} in x direction,
c_{yj} in y direction,
c_{zj} in z direction.

5. *Angles and angular velocities*

Angle of rotation of the crankshaft around its geometric axis is α. The resting and the moving axes are indicated by $\mathbf{e}_2$ and ray $\overrightarrow{OS}_{föt}$, respectively.

Angle of rotation of the i-th connecting rod is α_i (the resting and the moving axes are indicated by the vector $\mathbf{e}_2$ at the point O and the i-th connecting rod, respectively).

The angular velocity of the crankshaft *relative to the engine case* is Ω.

As the crankshaft is taken to be rigid, $\dfrac{d\alpha}{dt} = \dfrac{d\alpha_i}{dt} = \Omega$.

The angle between $\mathbf{e}_3$ and the centre line of the i-th connecting rod is β_i.
The engine is constructed so that:

$$\alpha_k = \alpha_1 + (k-1)\frac{2\pi}{3}, \quad k = 2, \ldots, 6.$$

The angle between vector $\mathbf{e}_2$ at the point O and ray $\overrightarrow{OS}_ö$ is α_0, constant.
Angles of the small rotations of the engine case:
φ around x-axis,
ψ around y-axis,
χ around z-axis.

Hence its angular velocity is approximately

$$\boldsymbol{\omega}_{ö} = \dot{\varphi}\mathbf{i}+\dot{\psi}\mathbf{j}+\dot{\chi}\mathbf{k} \approx \tilde{\boldsymbol{\omega}}_{ö} = \dot{\varphi}\mathbf{e}_1+\dot{\psi}\mathbf{e}_2+\dot{\chi}\mathbf{e}_3 .$$

6. *Inertia matrices of the same parts*

Inertia matrix of the engine case in the co-ordinate system b):

$$\tilde{\mathbf{J}}_{ö} = \begin{bmatrix} J_{11} & -J_{12} & -J_{13} \\ -J_{12} & J_{22} & -J_{23} \\ -J_{13} & -J_{23} & J_{33} \end{bmatrix}.$$

Inertia matrix of the crankshaft with balance in the co-ordinate system c):

$$\overset{c}{\mathbf{J}}_{főt} = \begin{bmatrix} J_{aa} & -J_{ab} & -J_{ac} \\ -J_{ab} & J_{bb} & -J_{bc} \\ -J_{ac} & -J_{bc} & J_{cc} \end{bmatrix}.$$

Inertia matrix of the i-th connecting rod in the co-ordinate system d), as the plane of its motion is the principal inertia plane belonging to the mass-centre of the i-th connecting rod and the momentum of inertia belonging to axis of rod direction in mass-centre is negligible with respect to the others:

$$\overset{d}{\mathbf{J}}_{\text{rud}\, i} = \begin{bmatrix} J_{aai} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & J_{cci} \end{bmatrix}.$$

4. Derivation of equation of the engine motion

In the given case the synthetic method, namely, derivation of linear momentum and angular momentum, seems to be the most advantageous.

For the sake of derivation the linear momentum and the angular momentum have to be expressed with respect to an adequate point of the engine. Both vectors are the sum of terms according to parts a), b), c) and d) specified in the second paragraph.

Writing in the sum of two terms, the respective linear momenta and angular momenta of parts b), c) and d), are likely to be correct. One of the two terms in every formula can be expressed by means of the velocity field of the part in question relative to the engine case (in the co-ordinate system b); and the other by means of the "carrier" velocity field. The source of the latter velocity field is the (carrying) motion of the engine case.

This calculation by means of the "relative" and the "carrier" velocity field is proper, because the entire (absolute) velocity field of every part is calculated by vectorial addition of the carrier and the relative velocity field, and both the linear momentum and the angular momentum are linear functions of the velocity fields.

According to the mentioned linear momentum of the engine (the subscripts "rud i" and "di" referring to the i-th connecting rod and the i-th piston, respectively):

$$\mathbf{I}_{\text{motor}} = \mathbf{I}_{\text{ö}} + \mathbf{I}_{\text{rel főt}} + \mathbf{I}_{\text{carr főt}} + \sum_{i=1}^{6} (\mathbf{I}_{\text{rel rud}\, i} + \mathbf{I}_{\text{carr rud}\, i}) + \sum_{i=1}^{6} (\mathbf{I}_{\text{rel}\, d\, i} + \mathbf{I}_{\text{carr}\, d\, i}), \tag{1}$$

the engine case being denoted by subscript ö, the crankshaft with flywheel and balance weights being denoted by subscript főt.

Angular momentum of the engine with respect to the resting position A of the mass-centre of the engine case:

$$\boldsymbol{\pi}_{\text{motor}\, A} = \boldsymbol{\pi}_{\text{ö}\, A} + \boldsymbol{\pi}_{\text{rel főt}\, A} + \boldsymbol{\pi}_{\text{carr főt}\, A} + \sum_{i=1}^{6} (\boldsymbol{\pi}_{\text{rel rud}\, i\, A} + \boldsymbol{\pi}_{\text{carr rud}\, i\, A}) + \sum_{i=1}^{6} (\boldsymbol{\pi}_{\text{rel}\, d\, i\, A} + \boldsymbol{\pi}_{\text{carr}\, d\, i\, A}). \tag{2}$$

In the following, right sides of both (1) and (2) will be analysed. Calculation will be by matrix calculus and — if it does not cause misunderstanding — no distinction will be made in notation between vector and column matrix.

Some results of transformation theory will be involved. The facts will be formulated by means of co-ordinate systems a) and b), these facts are however valid for any pair of Cartesian right-handed co-ordinate systems.

Denote the direction cosines matrix of axes of the co-ordinate system b) in the co-ordinate system a) by $\mathbf{L}_{ba}$

$$\mathbf{L}_{ba} = \begin{bmatrix} \cos(\mathbf{e}_1, \mathbf{i}) & \cos(\mathbf{e}_1, \mathbf{j}) & \cos(\mathbf{e}_1, \mathbf{k}) \\ \cos(\mathbf{e}_2, \mathbf{i}) & \cos(\mathbf{e}_2, \mathbf{j}) & \cos(\mathbf{e}_2, \mathbf{k}) \\ \cos(\mathbf{e}_3, \mathbf{i}) & \cos(\mathbf{e}_3, \mathbf{j}) & \cos(\mathbf{e}_3, \mathbf{k}) \end{bmatrix}.$$

$\mathbf{L}_{ba}$ is an orthogonal matrix, thus $\mathbf{L}_{ba}^{-1} = \mathbf{L}_{ba}^{*}$.

If the angles φ, ψ, χ are sufficiently small, then approximately

$$\mathbf{L}_{ba} = \begin{bmatrix} 1 & \chi & -\psi \\ -\chi & 1 & \varphi \\ \psi & -\varphi & 1 \end{bmatrix},$$

$$\mathbf{L}_{ba}^{-1} = \mathbf{L}_{ba}^{*} = \begin{bmatrix} 1 & -\chi & \psi \\ \chi & 1 & -\varphi \\ -\psi & \varphi & 1 \end{bmatrix}.$$

An antisymmetric matrix can be derived by means of angular velocity of the co-ordinate system b) relative to the co-ordinate system a), by which multiplying an arbitrary column vector from the left side or computing the cross product of the angular velocity and that vector, yields the same result. This antisymmetric, so-called angular velocity matrix can be expressed in the co-ordinate system a) (denoted by $\mathbf{\Omega}$) and in the co-ordinate system b) (denoted by $\tilde{\mathbf{\Omega}}$), $\mathbf{\Omega}$ and $\tilde{\mathbf{\Omega}}$ can be derived by means of $\mathbf{L}_{ba}$ as follows

$$\mathbf{\Omega} = -\mathbf{L}_{ba}^{*}\dot{\mathbf{L}}_{ba}, \quad \tilde{\mathbf{\Omega}} = \mathbf{L}_{ba}\mathbf{\Omega}\mathbf{L}_{ba}^{*} = -\dot{\mathbf{L}}_{ba}\mathbf{L}_{ba}^{*}. \tag{3}$$

$\mathbf{\Omega}$ equals $\tilde{\mathbf{\Omega}}$ if φ, ψ, χ are sufficiently small, and their matrix is

$$\begin{bmatrix} 0 & -\dot{\chi} & \dot{\psi} \\ \dot{\chi} & 0 & -\dot{\varphi} \\ -\dot{\psi} & \dot{\varphi} & 0 \end{bmatrix}. \tag{4}$$

Moreover, let vector $\mathbf{a}$ correspond to the column matrices $\mathbf{a}$ and $\tilde{\mathbf{a}}$ in the co-ordinate systems a) and b), respectively. It is true that

$$\tilde{\mathbf{a}} = \mathbf{L}_{ba}\,\mathbf{a}\,,$$

and as for (3): $\dot{\mathbf{L}}_{ba} = -\mathbf{L}_{ba}\mathbf{\Omega}$ and $\mathbf{\Omega}^{*} = -\mathbf{\Omega}$:

$$\dot{\mathbf{a}} = (\mathbf{L}_{ba}^{*}\tilde{\mathbf{a}})^{\cdot} = (\mathbf{L}_{ba}^{*})^{\cdot}\,\tilde{\mathbf{a}} + \mathbf{L}_{ba}^{*}\dot{\tilde{\mathbf{a}}} = \dot{\mathbf{L}}_{ba}^{*}\tilde{\mathbf{a}} + \mathbf{L}_{ba}^{*}\dot{\tilde{\mathbf{a}}} = -\mathbf{\Omega}^{*}\mathbf{L}_{ba}^{*}\tilde{\mathbf{a}} + \mathbf{L}_{ba}^{*}\dot{\tilde{\mathbf{a}}} = \mathbf{\Omega}\mathbf{a} + \mathbf{L}_{ba}^{*}\dot{\tilde{\mathbf{a}}}\,. \tag{5}$$

5. Calculation of the linear momentum vector, the angular momentum vector and their derivatives with respect to time

From Fig. 2:

$$\mathbf{I}_{ö} = M_{ö}\,\dot{\mathbf{r}}_{ö} = M_{ö}\begin{bmatrix} x_{ö} \\ y_{ö} \\ z_{ö} \end{bmatrix}^{\cdot}. \tag{6}$$

To calculate $\mathbf{I}_{\text{rel főt}}$, velocity of the point $S_{\text{főt}}$ in the co-ordinate system b) has to be made use of, that is, obviously $\dot{\tilde{\rho}}_{\text{főt}}$:

$$\mathbf{I}_{\text{rel főt}} = \mathbf{L}_{ba}^{*}\tilde{\mathbf{I}}_{\text{rel főt}} = M_{\text{főt}}\,\mathbf{L}_{ba}^{*}\dot{\tilde{\rho}}_{\text{főt}} =$$

$$= M_{\text{főt}}\,\Omega s_{0}\begin{bmatrix} 1 & -\chi & \psi \\ \chi & 1 & -\varphi \\ -\psi & \varphi & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -\sin\alpha \\ -\cos\alpha \end{bmatrix}.$$

Calculation of $\mathbf{I}_{\text{carr föt}}$ needs the carrier velocity of the point $S_{\text{föt}}$; this is the vector of the absolute velocity field of the engine case at point $S_{\text{föt}}$. Making use of Fig. 2 and (5):

$$\mathbf{I}_{\text{carr föt}} = M_{\text{föt}}(\mathbf{r}_{\text{ö}}+\boldsymbol{\rho}_{S_{\text{ö}}S_{\text{föt}}})^{\cdot} = M_{\text{föt}}(\dot{\mathbf{r}}_{\text{ö}}+\boldsymbol{\Omega}\boldsymbol{\rho}_{S_{\text{ö}}S_{\text{föt}}}+\mathbf{L}_{ba}^{*}\tilde{\boldsymbol{\rho}}^{\cdot}_{S_{\text{ö}}S_{\text{föt}}}) =$$

$$= M_{\text{föt}}(\dot{\mathbf{r}}_{\text{ö}}+\boldsymbol{\Omega}\mathbf{L}_{ba}^{*}\tilde{\boldsymbol{\rho}}_{S_{\text{ö}}S_{\text{föt}}}+\mathbf{L}_{ba}^{*}\tilde{\boldsymbol{\rho}}^{\cdot}_{S_{\text{ö}}S_{\text{föt}}}) = M_{\text{föt}}\left\{\begin{bmatrix} x_{\text{ö}} \\ y_{\text{ö}} \\ z_{\text{ö}} \end{bmatrix}^{\cdot} + \right.$$

$$+\begin{bmatrix} -\chi\dot{\chi}-\psi\dot{\psi} & -\dot{\chi}+\varphi\dot{\psi} & \varphi\dot{\chi}+\dot{\psi} \\ \dot{\chi}+\psi\dot{\varphi} & -\chi\dot{\chi}-\varphi\dot{\varphi} & \psi\dot{\chi}-\dot{\varphi} \\ -\dot{\psi}+\chi\dot{\varphi} & \chi\dot{\psi}+\dot{\varphi} & -\psi\dot{\psi}-\varphi\dot{\varphi} \end{bmatrix}\begin{bmatrix} \xi_0 \\ -a\cos\alpha_0+s_0\cos\alpha \\ a\sin\alpha_0-s_0\sin\alpha \end{bmatrix}+ \tag{8}$$

$$\left. +\begin{bmatrix} 1 & -\chi & \psi \\ \chi & 1 & -\varphi \\ -\psi & \varphi & 1 \end{bmatrix}\mathbf{0}\right\}.$$

$\tilde{\boldsymbol{\rho}}^{\cdot}_{S_{\text{ö}}S_{\text{föt}}}$ is zero now, because it is to be calculated by means of the velocity field of the engine case, so the vector $\tilde{\boldsymbol{\rho}}^{\cdot}_{S_{\text{ö}}S_{\text{föt}}}$ must be taken to be fixed to the engine case at the moment.

The further calculation does not cause difficulty according to Fig. 2 and the foregoing. Without going into details:

$$\mathbf{I}_{\text{rel rud } i} = \mathbf{L}_{ba}^{*}\tilde{\mathbf{I}}_{\text{rel rud } i} = m_{ri}\mathbf{L}_{ba}^{*}\tilde{\boldsymbol{\rho}}^{\cdot}_{\text{rud } i} =$$

$$= m_{ri}\mathbf{L}_{ba}^{*}\begin{bmatrix} 0 \\ l_{bi}\dot{\beta}_i\cos\beta_i \\ -r\Omega\cos\alpha_i+l_{ai}\dot{\beta}_i\sin\beta_i \end{bmatrix} \tag{9}$$

and

$$\mathbf{I}_{\text{carr rud } i} = m_{ri}(\mathbf{r}_{A0}+\boldsymbol{\rho}_{\text{rud } i})^{\cdot} = m_{ri}(\mathbf{r}_{\text{ö}}+\boldsymbol{\rho}_{S_{\text{ö}}0}+\boldsymbol{\rho}_{\text{rud } i})^{\cdot} =$$

$$= m_{ri}[\mathbf{r}^{\cdot}_{\text{ö}}+\boldsymbol{\Omega}(\boldsymbol{\rho}_{S_{\text{ö}}0}+\boldsymbol{\rho}_{\text{rud } i})+\mathbf{L}_{ba}^{*}(\tilde{\boldsymbol{\rho}}_{S_{\text{ö}}0}+\tilde{\boldsymbol{\rho}}_{\text{rud } i})] =$$

$$= m_{ri}[\mathbf{r}^{\cdot}_{\text{ö}}+\boldsymbol{\Omega}\mathbf{L}_{ba}^{*}(\tilde{\boldsymbol{\rho}}_{S_{\text{ö}}0}+\tilde{\boldsymbol{\rho}}_{\text{rud } i})+\mathbf{L}_{ba}^{*}(\tilde{\boldsymbol{\rho}}^{\cdot}_{S_{\text{ö}}0}+\tilde{\boldsymbol{\rho}}^{\cdot}_{\text{rud } i})] = \tag{10}$$

$$= m_{ri}\left\{\begin{bmatrix} x_{\text{ö}} \\ y_{\text{ö}} \\ z_{\text{ö}} \end{bmatrix}^{\cdot} + \boldsymbol{\Omega}\mathbf{L}_{ba}^{*}\begin{bmatrix} \xi_i \\ -a\cos\alpha_0+l_{bi}\sin\beta_i \\ a\sin\alpha_0-r\sin\alpha_i-l_{ai}\cos\beta_i \end{bmatrix}+\mathbf{L}_{ba}^{*}\mathbf{0}\right\}.$$

$\tilde{\boldsymbol{\rho}}^{\cdot}_{S_{\text{ö}}0}$ and $\tilde{\boldsymbol{\rho}}^{\cdot}_{\text{rud } i}$ are zero because — as in (8) — velocity field of the engine case has to be made use of here, thus both $\tilde{\boldsymbol{\rho}}_{S_{\text{ö}}0}$ and $\tilde{\boldsymbol{\rho}}_{\text{rud } i}$ must be taken to be fixed to the engine case.

The linear momentum vector of the i-th piston is:

$$\mathbf{I}_{\text{rel}\,di} = \mathbf{L}^*_{ba}\,\tilde{\mathbf{I}}_{\text{rel}\,di} = m_{di}\,\mathbf{L}^*_{ba}\,\tilde{\boldsymbol{\rho}}^{\cdot}_{di} = m_{di}\,\mathbf{L}^*_{ba}\begin{bmatrix} 0 \\ 0 \\ -r\Omega\cos\alpha_i + l\dot{\beta}_i\sin\beta_i \end{bmatrix}, \tag{11}$$

$$\mathbf{I}_{\text{carr}\,di} = m_{di}(\mathbf{r}_{ö}+\boldsymbol{\rho}_{S_ö0}+\boldsymbol{\rho}_{di})^{\cdot} = m_{di}\left[\mathbf{r}^{\cdot}_{ö}+\boldsymbol{\Omega}\mathbf{L}^*_{ba}(\tilde{\boldsymbol{\rho}}_{S_ö0}+\tilde{\boldsymbol{\rho}}_{di})+\mathbf{L}^*_{ba}(\tilde{\boldsymbol{\rho}}^{\cdot}_{S_ö0}+\tilde{\boldsymbol{\rho}}^{\cdot}_{di})\right] =$$

$$= m_{di}\left\{\begin{bmatrix} x_ö \\ y_ö \\ z_ö \end{bmatrix}^{\cdot} + \boldsymbol{\Omega}\,\mathbf{L}^*_{ba}\begin{bmatrix} \xi_i \\ -a\cos\alpha_0 \\ a\sin\alpha_0 - r\sin\alpha_i - l\cos\beta_i \end{bmatrix} + \mathbf{L}^*_{ba}\,\mathbf{0}\right\}. \tag{12}$$

As the angular momentum vectors are to be expressed by matrix calculus, therefore the antisymmetric matrix is to be introduced:

$$\begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix} \tag{13}$$

which corresponds to the cross product with any arbitrary vector $[r_1\; r_2\; r_3]$ from the left side.

To calculate $\boldsymbol{\pi}_{öA}$, the matrix $\mathbf{R}_ö$ has to correspond to the vector $\mathbf{r}_ö$ according to (13):

$$\boldsymbol{\pi}_{öA} = M_ö\,\mathbf{R}_ö\,\mathbf{r}^{\cdot}_ö+\mathbf{L}^*_{ba}\,\tilde{\mathbf{J}}_ö\,\mathbf{L}_{ba}\,\boldsymbol{\omega}_ö\,. \tag{14}$$

To calculate $\boldsymbol{\pi}_{\text{rel}\,\text{főt}A}$, beside $\mathbf{R}_ö$, $\tilde{\mathbf{R}}_{S_öS_{\text{főt}}}$ and $\mathbf{R}_{S_öS_{\text{főt}}}$, corresponding to $\tilde{\boldsymbol{\rho}}_{S_öS_{\text{főt}}}$ and $\boldsymbol{\rho}_{S_öS_{\text{főt}}}$, respectively, have to be introduced. Moreover, there is need of direction cosines matrix of axes of the co-ordinate system c) in the co-ordinate system b):

$$\mathbf{L}_{cb} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}.$$

$$\boldsymbol{\pi}_{\text{rel}\,\text{főt}\,A} = M_{\text{főt}}\,(\mathbf{R}_ö+\mathbf{R}_{S_ö\,S_{\text{főt}}})\,\mathbf{L}^*_{ba}\,\tilde{\boldsymbol{\rho}}^{\cdot}_{\text{főt}} +$$

$$+\,\mathbf{L}^*_{ba}\,\mathbf{L}^*_{cb}\,\overset{c}{\mathbf{J}}_{\text{főt}}\,\mathbf{L}_{cb}\,\mathbf{L}_{ba}\,\mathbf{L}^*_{ba}\begin{bmatrix} \Omega \\ 0 \\ 0 \end{bmatrix} = \tag{15}$$

$$= M_{\text{főt}}\,(\mathbf{R}_ö+\mathbf{L}^*_{ba}\,\mathbf{R}_{S_ö\,S_{\text{főt}}}\,\mathbf{L}_{ba})\,\mathbf{L}^*_{ba}\,\tilde{\boldsymbol{\rho}}^{\cdot}_{\text{főt}}+\mathbf{L}^*_{ba}\,\mathbf{L}^*_{cb}\,\overset{c}{\mathbf{J}}_{\text{főt}}\mathbf{L}_{cb}\begin{bmatrix} \Omega \\ 0 \\ 0 \end{bmatrix}.$$

To calculate $\boldsymbol{\pi}_{\text{carr föt } A}$ — as in (8) — the carrier velocity $(\mathbf{r}_{ö} + \boldsymbol{\rho}_{S_ö S_{föt}})^{\cdot}$ of $S_{föt}$ is needed, that was already calculated. Making use of this and taking into consideration the fact that now the carrier angular velocity — expressed in the co-ordinate system a) — is $\begin{bmatrix}\varphi\\ \psi\\ \chi\end{bmatrix}^{\cdot}$:

$$\boldsymbol{\pi}_{\text{carr föt } A} = M_{föt}\,(\mathbf{R}_{ö}+\mathbf{L}^*_{ba}\,\widetilde{\mathbf{R}}_{S_ö\,S_{föt}}\,\mathbf{L}_{ba})(\mathbf{r}^{\cdot}_{ö}+\boldsymbol{\Omega}\mathbf{L}^*_{ba}\tilde{\boldsymbol{\rho}}_{S_ö\,S_{föt}}) + \tag{16}$$

$$+\,\mathbf{L}^*_{ba}\,\mathbf{L}^*_{cb}\,\overset{c}{\mathbf{J}}_{föt}\,\mathbf{L}_{cb}\,\mathbf{L}_{ba}\begin{bmatrix}\varphi\\ \psi\\ \chi\end{bmatrix}^{\cdot}.$$

For the i-th connecting rod, the direction cosines matrix of axes of the co-ordinate system d) in the co-ordinate system b), moreover $\mathbf{R}_{\text{rud } i}$ and $\widetilde{\mathbf{R}}_{\text{rud } i}$ corresponding to the vectors $\boldsymbol{\rho}_{\text{rud } i}$ and $\tilde{\boldsymbol{\rho}}_{\text{rud } i}$, respectively, are to be introduced:

$$\mathbf{L}_{db} = \begin{bmatrix}1 & 0 & 0\\ 0 & -\sin\beta_i & -\cos\beta_i\\ 0 & \cos\beta_i & -\sin\beta_i\end{bmatrix};$$

$$\boldsymbol{\pi}_{\text{rel rud } iA} = m_{ri}\,[\mathbf{R}_{ö}+\mathbf{L}^*_{ba}(\widetilde{\mathbf{R}}_{S_ö\,O}+\widetilde{\mathbf{R}}^{\cdot}_{\text{rud } i})\,\mathbf{L}_{ba}]\,\mathbf{L}^*_{ba}\,\tilde{\boldsymbol{\rho}}^{\cdot}_{\text{rud } i} + \tag{17}$$

$$+\,\mathbf{L}^*_{da}\,\mathbf{L}^*_{db}\,\overset{d}{\mathbf{J}}_{\text{rud } i}\,\mathbf{L}_{db}\begin{bmatrix}\dot{\beta}_i\\ 0\\ 0\end{bmatrix}.$$

To calculate $\boldsymbol{\pi}_{\text{carr rud } i\,A}$ the carrier velocity is needed, known from (10), of $S_{\text{rud } i}$:

$$\boldsymbol{\pi}_{\text{carr rud } i\,A} = m_{ri}[\mathbf{R}_{ö}+\mathbf{L}^*_{ba}\,(\widetilde{\mathbf{R}}_{S_ö O}+\widetilde{\mathbf{R}}_{\text{rud } i})\mathbf{L}_{ba}][\dot{\mathbf{r}}_{ö}+\boldsymbol{\Omega}\mathbf{L}^*_{ba}(\tilde{\boldsymbol{\rho}}_{S_ö O}+\tilde{\boldsymbol{\rho}}_{\text{rud } i})] + \tag{18}$$

$$+\,\mathbf{L}^*_{ba}\,\mathbf{L}^*_{db}\,\overset{d}{\mathbf{J}}_{\text{rud } i}\,\mathbf{L}_{db}\,\mathbf{L}_{bc}\begin{bmatrix}\varphi\\ \psi\\ \chi\end{bmatrix}^{\cdot}.$$

In calculating angular momentum of the i-th piston, the angular momentum with respect to its own mass-centre can be neglected. Calculation needs $\mathbf{R}_{di}$ and $\widetilde{\mathbf{R}}_{di}$ matrices corresponding to $\boldsymbol{\rho}_{di}$ and $\tilde{\boldsymbol{\rho}}_{di}$, respectively.

$$\boldsymbol{\pi}_{\text{rel } di\,A} = m_{di}\,[\mathbf{R}_{ö}+\mathbf{L}^*_{ba}\,(\widetilde{\mathbf{R}}_{S_ö O}+\widetilde{\mathbf{R}}_{di})\mathbf{L}_{ba}]\,\mathbf{L}^*_{ba}\,\tilde{\boldsymbol{\rho}}^{\cdot}_{di}), \tag{19}$$

$$\boldsymbol{\pi}_{\text{carr } di\,A} = m_{di}\,[\mathbf{R}_{ö}+\mathbf{L}^*_{ba}(\widetilde{\mathbf{R}}_{S_ö O}+\widetilde{\mathbf{R}}_{di})\,\mathbf{L}_{ba}][\dot{\mathbf{r}}_{ö}+\boldsymbol{\Omega}\mathbf{L}^*_{ba}(\tilde{\boldsymbol{\rho}}_{S_ö O}+\tilde{\boldsymbol{\rho}}_{di})]. \tag{20}$$

Differentiation of (6)—(20) with respect to time needs not to be specified here, no difficulties in principle arising in calculation.

6. Linearized motion equation system

Equation system of the engine motion is:

$$\dot{\mathbf{I}}_{\text{motor}} = \mathbf{F}_{\text{motor}} ,$$
$$\dot{\pi}_{\text{motor}A} = \mathbf{M}_{\text{motor}A} , \tag{21}$$

where $\mathbf{F}_{\text{motor}}$ and $\mathbf{M}_{\text{motor A}}$ are the total external force and the total external torque with respect to the point A, respectively; they include the gravity and spring forces R_1, R_2, R_3. Ignoring all but small amplitude vibrations, and in accordance with this, linearizing the formulae, the two vectors in the left sides of the equation system are linear functions of first and second derivatives of the co-ordinates $x_{\ddot{o}}, y_{\ddot{o}}, z_{\ddot{o}}, \varphi, \psi, \chi$ with respect to time, the vectors on the right sides are, however, linear functions of the same and the other co-ordinates of the model. Every β_i and $\dot{\beta}_i$, occurring in the coefficients, can be expressed by means of the well-known relationships for the crank drive, on the basis of Fig. 2, in terms of α_i:

$$\sin \beta_i = \frac{r}{l} \cos \alpha_i, \quad \dot{\beta}_i = - \frac{r\Omega \sin \alpha_i}{l \sqrt{1 - \left(\frac{r}{l}\right)^2 \cos^2 \alpha_i}} .$$

Ω, however, occurs in coefficients with reference not only to β_i and $\dot{\beta}_i$.

If the steady state motion is to be examined for practical demands, then Ω is a periodic function of time and so the coefficients of Eqs (21) are also complicated periodical functions of time.

The other equations of the linearized equation system of the whole model contain constant coefficients.

Thus, the linearized equation system of motion of the whole model is a linear system of differential equations of periodic coefficients in the case of the steady state motion.

Denoting the column matrix containing all co-ordinates of the whole model by $\mathbf{x}$, the equation of motion is

$$\mathbf{A}(t)\,\ddot{\mathbf{x}} + \mathbf{B}(t)\,\dot{\mathbf{x}} + \mathbf{C}(t)\,\mathbf{x} = \mathbf{d}(t) , \tag{22}$$

where all of the coefficient matrices are periodic with a common period. In a given case this period is known; denote it by T. Order of these matrices is n.

7. Solution of equation (22)

Only the periodic solution of Eq. (22) is of interest. Strictly speaking its existence ought to be proved earlier. Empirically, however, we are allowed to assume its existence in this case.

There is no closed solution for Eq. (22) in either the periodic or the general case. An approximation has to be used. The Galiorkin's method seems to be most advantageous.

This method involves a so-called complete system of functions. At the moment, as periodic solution functions are needed, it should be periodic, with the period T. So it is the most natural to employ the well-known complete system of functions consisting of $\frac{1}{\sqrt{2\pi}}$, $\frac{1}{\sqrt{\pi}} \cos k\omega t, \frac{1}{\sqrt{\pi}} \sin k\omega t$, $k = 1, 2, \ldots$ Here $\omega = \frac{2\pi}{T}$. The first $2\sigma + 1$ terms are employed in our approximation.

From the foregoing $\mathbf{x}$ is assumed to be:

$$\hat{\mathbf{x}} = \mathbf{f}_0 + \mathbf{f}_{11} \cos \omega t + \mathbf{f}_{12} \sin \omega t + \mathbf{f}_{21} \cos 2\omega t + \\ + \mathbf{f}_{22} \sin 2\omega t + \ldots + \mathbf{f}_{\sigma 1} \cos \sigma\omega t + \mathbf{f}_{\sigma 2} \sin \sigma\omega t$$

$\mathbf{f}_0, \ldots, \mathbf{f}_{\sigma 2}$ are the wanted column matrices. The unknown elements of them are altogether $(2\sigma + 1)$n.

Inserting $\hat{\mathbf{x}}$ into (22) and in accordance with the principle of weighted error let:

$$\int_0^T [\mathbf{A}(t)\ \ddot{\hat{\mathbf{x}}} + \mathbf{B}(t)\ \dot{\hat{\mathbf{x}}} + \mathbf{C}(t)\ \hat{\mathbf{x}} - \mathbf{d}(t)]\ 1\ dt = \mathbf{0}\,,$$

$$\int_0^T [\mathbf{A}(t)\ \ddot{\hat{\mathbf{x}}} + \mathbf{B}(t)\ \dot{\hat{\mathbf{x}}} + \mathbf{C}(t)\ \hat{\mathbf{x}} - \mathbf{d}(t)] \cos \omega t\ dt = \mathbf{0}\,,$$

$$\int_0^T [\mathbf{A}(t)\ \ddot{\hat{\mathbf{x}}} + \mathbf{B}(t)\ \dot{\hat{\mathbf{x}}} + \mathbf{C}(t)\ \hat{\mathbf{x}} - \mathbf{d}(t)] \sin \sigma\omega t\ dt = \mathbf{0}\,.$$

Detailing the above equations, an inhomogeneous linear equation system arises which consists of just as many equations as needed to determine the column matrices.

8. Optimization tests

Practice does not content itself with the analysis of models of vehicles, as was described earlier. From acoustic aspects, the question emerges how to choose certain parameters of the system so as to optimize the noise field caused by the motion.

Equations are too complicated to allow else than numerical tests, namely, to find solutions for several values of parameters and to compare them. This optimization relies on equations for the noise field caused by motion of the vehicle.

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Summary

The chassis and the engine are usually taken as one rigid body in the mathematical model set up for the examination of vibrations of a vehicle driven by an internal combustion piston engine. This simplification is appropriate for some examinations. To examine the noise field inside and outside the passenger cabin this model is not exact enough. Experience proves that practical solutions are possible by modelling the motion of the driving system of the engine and its elastic suspension. A simple method is presented for the derivation of mechanical equations of the model.

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